Two-Body Problem in Quantum Field Theory

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We study an approach to the relativistic two-body problem that represents a considerable improvement of the Bethe-Salpeter equation. The main advantages are the improved properties of the *simple* ladder approximation. (a) It has the correct static limit and is in this sense equivalent to the sum of all crossed ladders of the Bethe-Salpeter equation, (b) in the case of massless exchange it is possible to solve analytically and obtain the wave functions and the T matrix in closed form, (c) the system is closed, that is, the two-particle T matrix is unitary. (d) The wave functions admit a complete quantum-mechanical interpretation without compromise of relativistic covariance, the current is conserved, and gauge invariance is respected. (e) The evaluation of physical amplitudes is greatly simplified, as illustrated by positronium decay and the Lamb shift. Both are carried out with full relativistic covariance, and the Lamb-shift calculation in particular is greatly simplified and clarified in comparison with other methods. Most of the results are applicable only to the case of spin-0 or spin-$\frac{1}{2}$ particles interacting through the exchange of a single scalar or vector boson. In particular, the calculations of positronium decay and the Lamb shift are carried out with spinless particles. It is shown, however, that the introduction of photon spin presents no problem and that the Lamb shift can be calculated in a fully gauge-invariant manner.

I. INTRODUCTION

ALTHOUGH the relativistic two-body problem has not been solved in the sense of classical mechanics, nor in the sense of quantum mechanics, it is often said to have been treated satisfactorily in quantum field theory—by means of the Bethe-Salpeter equation.[1] Perhaps it has not been sufficiently emphasized that this situation is quite paradoxical: If classical mechanics is a limit of quantum mechanics and if quantum mechanics is a limit of quantum field theory, then a really self-consistent treatment of the problem in quantum field theory would surely provide the answer on every level. The resolution of the paradox is, quite simply, that the Bethe-Salpeter wave function is not susceptible to a self-consistent interpretation.

Recently a new type of relativistic quantum mechanics has developed out of the frustrations of relativistic $SU(6)$ and exact saturations of current algebra. The first "infinite-component field theory" was proposed by Majorana[2] in 1932; the most recent surge of interest dates back to 1966.[3] The motivation sprang from the need to describe infinite multiplets of physical states with nondegenerate mass in a manner consistent with gauge invariance and unitarity. Gauge invariance is almost tantamount to locality, so it is natural that these theories are relevant for current algebra. It was stressed from the beginning, especially by Takabayashi, that many infinite-component theories could be interpreted as composite systems. Later, it was shown that one particular theory, initially chosen in an almost *ad hoc* manner, had a detailed interpretation in terms of two interacting scalar particles.[4] This interpretation was further developed by Fronsdal and Lundberg,[5] who showed that the complete axiomatic formulation of ordinary nonrelativistic quantum mechanics has an exact relativistic parallel in that particular theory.

It is the main purpose of this paper to propose an alternative to the Bethe-Salpeter equation, with the same *a priori* justification as that equation but having important practical and theoretical advantages. Specializing to the case of two scalar particles interacting through a scalar field, we shall show that the new two-body equation is identical to that investigated in Ref. 5, thus demonstrating that the new wave function has a satisfactory interpretation in the sense of relativistic quantum mechanics. It is shown elsewhere that the classical limit is a self-consistent formulation of the classical relativistic two-body problem.[6]

To illustrate the method, we calculate the timelike form factor of a scalar electron (one-photon decay of positronium), the width of the ground state of positronium (two-photon decay) and the Lamb shift of hydrogen. Particle and photon spins are neglected.

II. TWO-BODY EQUATIONS

As long as it is convenient, let the fields $\varphi_1(x)$ and $\varphi_2(x)$ be either Klein-Gordon or Dirac fields. The two-particle Green's function

$$\mathcal{G}=\langle 0|T\varphi_1\varphi_2\bar{\varphi}_1\bar{\varphi}_2|0\rangle \tag{2.1}$$

contains all information that is needed. The on-shell

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[1] H. A. Bethe and E. E. Salpeter, Phys. Rev. **84**, 1232 (1951).

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[3] C. Fronsdal, Phys. Rev. **156**, 1665 (1967); Y. Nambu, *ibid.* **160**, 1171 (1967).

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[5] C. Fronsdal and L.-E. Lundberg, Phys. Rev. D **1**, 3247 (1970).

[6] C. Fronsdal, Phys. Rev. D **3** (to be published).

T matrix ($=i-iS$), as well as its off-shell continuation, is given by the reduction formula

$$i\mathcal{T}=\Delta_{F1}^{-1}\Delta_{F2}^{-1}(\mathcal{G}-\Delta_{F1}\Delta_{F2})\Delta_{F1}^{-1}\Delta_{F2}^{-1}, \quad (2.2)$$

where

$$\Delta_{Fi}=\langle 0|T\varphi_i\bar{\varphi}_i|0\rangle, \quad i=1,2 \quad (2.3)$$

are the dressed one-particle Green's functions or Feynman propagators.

The Bethe-Salpeter equation for $\mathcal{T}$ is

$$\mathcal{T}=-\mathcal{U}-i\mathcal{U}\Delta_{F1}\Delta_{F2}\mathcal{T}. \quad (2.4)$$

Since the perturbation series for Δ_{F1}, Δ_{F2}, and $\mathcal{T}$ are known, that for $\mathcal{U}$ must be considered to be *defined* by (2.4). The first two terms are

$$\mathcal{U}_1=-\mathcal{T}_1, \quad \mathcal{U}_2=-\mathcal{T}_2+i\mathcal{T}_1\Delta_{F1}^0\Delta_{F2}^0\mathcal{T}_1, \quad (2.5)$$

where Δ_{Fi}^0 are the free one-particle Green's functions and $\mathcal{U}_n$ is given for each n by all contributions to $\mathcal{T}$ of order n that are two-particle irreducible, that is, all Feynman graphs without two-particle intermediary states.

Just as (2.4) defines $\mathcal{U}$, so the following equation determines the perturbation series for $\mathcal{W}$:

$$\mathcal{T}=-\mathcal{W}-i\mathcal{W}\hat{\Delta}_1\Delta_{F2}^0\mathcal{T}. \quad (2.6)$$

Here $\Delta_{F1}\Delta_{F2}$ has been replaced by $\hat{\Delta}_1\Delta_{F2}^0$. The "on-shell propagator" $\hat{\Delta}_1$ will be defined presently; for now it is sufficient to specify that it contains a δ function:

$$\hat{\Delta}_1(p_1)=2\pi\delta(p_1^2-m_1^2)\rho(p_1). \quad (2.7)$$

To the lowest orders,

$$\mathcal{W}_1=-\mathcal{T}_1, \quad \mathcal{W}_2=-\mathcal{T}_2+i\mathcal{T}_1\hat{\Delta}_1\Delta_{F2}^0\mathcal{T}_1, \quad (2.8)$$

and $\mathcal{W}_n$ is given for each n by all contributions to $\mathcal{T}$ of order n, except that $\Delta_{F1}\Delta_{F2}$ is replaced by $\Delta_{F1}\Delta_{F2}-\hat{\Delta}_1\Delta_{F2}^0$ in every two-particle segment.

It is clearly possible to write down an infinite set of equations like (2.4) or (2.6), all of them correct since the potential is in each case taken to be defined by the equation itself.

The next step is to replace $\mathcal{U}$ or $\mathcal{W}$ by the first few terms in the perturbation series and solve the resulting equation exactly to obtain nonperturbative approximations for $\mathcal{T}$. Any calculation that substitutes an approximation for $\mathcal{U}$ or $\mathcal{W}$ will be called a ladder approximation[7]; in particular, the approximations $\mathcal{U}\to\mathcal{U}_1$ and $\mathcal{W}\to\mathcal{W}_1$ will be referred to as simple ladder approximations. The basis for choosing between (2.4) and (2.6), and for the choice of $\rho(p_1)$, is mainly a question of optimizing the simple ladder approximation.

At this point it is convenient to specialize and take particle 1 to be spinless. Let the interaction be given by the interaction Lagrangian

$$\mathcal{L}_I=g_1\bar{\varphi}_1\varphi_1A+g_2\bar{\varphi}_2\varphi_2A, \quad (2.9)$$

where A is a real scalar field with mass μ. Particle 2 may be either a Klein-Gordon or a Dirac particle. The Born approximation for the potential is

$$\mathcal{U}_1=\mathcal{W}_1=-\mathcal{T}_1=g_1g_2[(p_1-p_1')^2-\mu^2]^{-1}. \quad (2.10)$$

To optimize the ladder approximation we try to "minimize" $\mathcal{W}_2$. This will be done in detail in Sec. III; the result is that we obtain important cancellations in $\mathcal{W}_2$ in the static limit $m_1\to\infty$ when we take $\rho(p_1)=1$ for $p_{10}>0$. In fact, it may be shown that, if $\rho(p_1)=1$ for positive p_{10}, then all the corrections $\mathcal{W}_n$, $n\geq 2$, become practically negligible when $m_1\to\infty$. In other words, with this choice of $\rho(p_1)$, the simple ladder approximation becomes essentially exact in the static limit.[8] This is in marked contrast with the more conventional ladder approximation based on the Bethe-Salpeter equation; in order to obtain the correct static limit, it is necessary to include an infinite number of diagrams in $\mathcal{U}$, which of course is not feasible. There is a close relationship between the sum of all simple ladders with $\mathcal{W}\to\mathcal{W}_1$ in Eq. (2.6) and the sum of all crossed ladders in the Bethe-Salpeter scheme, and the two become identical in the limit $m_1\to\infty$. We believe that this represents a strong argument in favor of supplanting (2.4) with (2.6) in studies of the two-body problem. *The limit of very large m_1 is more important than any other consideration because this limit is the only case where we know what we are trying to approximate—it is the only means available for testing the ladder approximation.*

Recently it has been shown that the relativistic eikonal approximation can provide another area of confrontation of nonperturbative quantum field theory with experiment. To the extent that this confrontation is successful, one has another criterion with which to compare the merits of the various equations. As far as the Bethe-Salpeter equation is concerned, it fails again: The simple ladder approximation is *not* a good approximation to the eikonal approximation. We plan to investigate our new approach from this point of view; however, preliminary results are very encouraging. First, as may be seen from the work of Tiktopoulos and Treiman,[9] the success of the eikonal approximation depends on precisely the same type of cancellations as are operative in the static limit. Second, it has been shown by Todorov[10] that equations very much related to ours agree with the eikonal approximation in the high-energy limit.

In this paper the technical advantages of the new approach will be further demonstrated by calculating various properties of scalar positronium, as well as the

[7] Different ladder approximations are distinguished by the choice of Green's function and potential. A potential indicated by $\mathcal{W}$ or W with indices implies an equation like (2.6) or (2.12). The meaning of W ladder, W' ladder, etc., is then obvious.

[8] Compare the results of F. Gross, Phys. Rev. **186**, 1448 (1969).

[9] G. Tiktopoulos and S. B. Treiman, Phys. Rev. D **2**, 805 (1970).

[10] I. T. Todorov, Trieste report, 1970 (unpublished) and Ref. 29.

Lamb shift of scalar hydrogen. In both cases the spins of the particles involved will be neglected, although there is no great difficulty in taking into account the spin of the electron and the spin of the photon. However, first we shall emphasize the theoretical advantages of the method.

The matrix products in (2.4) include integrations over momenta, with volume element $(2\pi)^{-4}d^4p_1$. There are no limitations on the integration variables except for the conservation laws; therefore, the complete off-shell $\mathcal{T}$ matrix is involved in (2.4). Similarly, the Bethe-Salpeter wave function is defined for general values of the two momenta, and this is the origin of the difficulties of the interpretation. In (2.6), on the other hand, the factor $\hat{\Delta}_1$ restricts the integration over p_1 to the mass shell and it is therefore possible to restrict both $\mathcal{T}$ and $\mathcal{W}$ to the mass shell $p_1^2=m_1^2$. Of course, it is still necessary to take p_2 off shell. The wave function can also be restricted to the domain $p_1^2=m_1^2$ and it is this reduction in the number of independent variables that makes a complete quantum-mechanical and classical interpretation possible, as will now be shown.

The factor $\hat{\Delta}_1$ in (2.6) may be combined with the volume element $(2\pi)^{-4}d^4p_1$. If we introduce the convention that all matrix products are to be carried out by integration with the volume element

$$[dp_1]=(2\pi)^{-3}d^4p_1\delta(p_1^2-m_1^2)\rho(p_1), \tag{2.11}$$

then (2.6) can be written

$$T=-W+WK_2^{-1}T. \tag{2.12}$$

The new symbols T and W are the same as $\mathcal{T}$ and $\mathcal{W}$, except that the notation for matrix products has been changed. The quantity K_2 is either the Klein-Gordon operator $p_2^2-m_2^2$ or the Dirac operator $p_2\cdot\gamma-m_2$. The equation for the two-particle wave function is now derived in the usual manner. First, in analogy with (2.2), we define the two-particle Green's function G by

$$iT=\Delta_{F2}^{-1}(G-\Delta_{F2})\Delta_{F2}^{-1}. \tag{2.13}$$

From (2.12) it follows immediately that

$$G=i(K_2-W)^{-1}. \tag{2.14}$$

Obviously, this is the Green's function for the wave equation

$$L\psi=0, \quad L\equiv K_2-W, \tag{2.15}$$

where $\psi(p_1,p_2)$ is defined on the mass shell of particle 1 only, so that we are free to take[11]

$$(p_1^2-m_1^2)\psi=0. \tag{2.16}$$

[11] The notation, like that of Ref. 5, is perhaps misleading. To be more precise, we should introduce a function $\psi'=\delta(p^2-m_1^2)\psi$. This is the wave function that satisfies (2.15) and (2.16), whereas ψ is merely irrelevant off shell. Equation (2.16) for ψ is nevertheless not wrong since there is no reason to assume that ψ is continuous off shell. Probably the best interpretation of (2.16) is that it simply defines p_{10} in terms of $\mathbf{p}_1$: $p_{10}^2=\mathbf{p}_1^2+m_1^2$.

Equations (2.15) and (2.16) are identical with the two equations proposed in Ref. 5, except that the latter used the volume element

$$(dp_1)\equiv d^4p_1\delta(p_1^2-m_1^2)\epsilon(p_{10}) \tag{2.17}$$

in place of (2.11). The factor $\epsilon(p_1)$ was included in (2.17) because it gave the computational advantage of making (2.12) and (2.15) exactly soluble in the important special case when μ, the mass of the field that transmits the interaction, is zero. Having already noticed that $\rho(p_1)=1$ for $p_{10}>0$ gives the most accurate ladder approximation, we now adopt the choice

$$\rho(p_1)=\epsilon(p_{10}). \tag{2.18}$$

The remaining discrepancy of $(2\pi)^{-3}$ only means that the potential V of Ref. 5 is related to W_1 by

$$W_1=(2\pi)^3V. \tag{2.19}$$

The static limit of (2.15) was already shown[5] to coincide with the Klein-Gordon or Dirac theories of a particle in a static Yukawa field in the limit $m_1\to\infty$. This constitutes an independent proof of the result that $W-W_1\approx 0$ in the limit $m_1\to\infty$, except that it was necessary to establish the relation (2.19) between the strength of V and the field-theoretic coupling constants g_1 and g_2.

Having thus demonstrated the identity of (2.15) with the quantum mechanics of Ref. 5, it suffices to refer to that paper for a complete discussion of solutions and the interpretation of the equation. In the following we have relegated all calculations that make use of the results of that paper to Appendix A.

A well-known disadvantage of proposing a soluble equation is that defects cannot be overlooked. However, theoretical defects of the ladder approximation—e.g., the existence of spacelike solutions of Eq. (2.15)—are irrelevant as long as it is remembered that the ladders give a first approximation in a scheme that allows higher corrections to be controlled and calculated. On the other hand, it must also be kept in mind that every approximation has a limited domain of validity, and the new ladder approximation must not be used in the neighborhood of $(p_1+p_2)^2=(m_1-m_2)^2$ or below.[12] The same remarks apply to the asymmetrical treatment of the two particles; note, however, that the ladder approximation for T is symmetrical on the mass shell in the case $\mu=0$.[13]

III. FINE STRUCTURE AND RECOIL CORRECTIONS

In preparation for atomic-structure calculations, it is convenient to introduce the dimensionless coupling

[12] By the same token, there is no justification for applying the Bethe-Salpeter ladder approximation in that region.

[13] A proof may be found at the end of the appendix of Ref. 5.

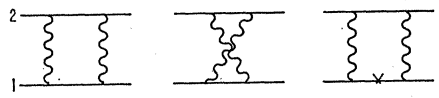

FIG. 1. The first two diagrams are Feynman graphs that contribute to T_2 and hence to $-W_2'$. The third diagram, in which one of the internal lines is marked with a cross to indicate that Δ_{F1} is replaced by $2\pi\delta(p_1^2-m_1^2)\epsilon(p_1)$, is the contribution $W_1K_2^{-1}W_1$ to W_2'. The three terms cancel in the limit $m_1\to\infty$.

constants

$$g_1/2m_1\equiv e_1\equiv Z(4\pi\alpha)^{1/2},\quad g_2/2m_2\equiv e_2\equiv(4\pi\alpha)^{1/2}. \quad (3.1)$$

The static limit is defined by $m_1\to\infty$, with e_1, e_2, and m_2 fixed. Corrections to the ladder approximation (in our sense) will be classified according to the order in which they contribute to the atomic energy levels.

We begin with the W_1-ladder approximation[7] to T, being the exact solution T_L of the equation

$$T_L=-W_1+W_1K_2^{-1}T_L, \quad (3.2)$$

$$W_1=4e_1e_2m_1m_2[(q-q')^2-\mu^2]^{-1}, \quad (3.3)$$

and carry out the necessary corrections in two stages. Let T_L' be the "improved ladder approximation" (or W'-ladder approximation[7]) that consists of all conventional ladder graphs, twisted or not, including photon self-energy insertions. The equation for T_L' is

$$T_L'=-W'+W'K_2^{-1}T_L', \quad (3.4)$$

where W' is given in perturbation theory by a rule that is similar to that given above for W: It consists of all conventional ladder graphs, twisted or not, including photon self-energy insertions, but with $\Delta_{F1}\Delta_{F2}$ replaced by $\Delta_{F1}\Delta_{F2}-\hat{\Delta}_1\Delta_{F2}^0$ in every two-particle segment. This improvement of our ladder approximation does not yet include the main part of the Lamb shift, but since a simple evaluation of the Lamb shift depends crucially on being above to cope with T_L', we investigate this in some detail.

First, to justify the statements made in Sec. II, we calculate the lowest orders of W'. Let

$$W'=W_1'+W_2'+\cdots, \quad (3.5)$$

where the suffix is *the number of dressed photon lines*. Thus W_1' includes W_1 and photon self-energy corrections to W_1. For the moment let us set aside all self-energy corrections to W' and evaluate the remaining contributions to W_2'. The contributing diagrams are shown in Fig. 1, and the explicit formula is

$$W_2'=4m_1m_2e_1e_2\int\sigma(k)d^4k(2\pi)^{-4}\{i[(k-p)^2-m_1^2+i\epsilon]^{-1}+i[(k+p_1-p_2')^2-m_1^2+i\epsilon]^{-1}-2\pi\delta[(p-k)^2-m_1^2]\epsilon(p-k)\}, \quad (3.6)$$

where

$$\sigma(k)=[(p_2-k)^2-\mu^2]^{-1}[(p_2'-k)^2-\mu^2]^{-1}\times[k^2-m_2^2+i\epsilon]^{-1}. \quad (3.7)$$

To lowest order in $1/m_1$, we can approximate in (3.6) as follows (for $p_0>0$):

$$[(k-p)^2-m_1^2+i\epsilon]\to-2m_1(k_0-p_{20}-i\epsilon),$$
$$[(k+p_1-p_2')^2-m_1^2+i\epsilon]\to 2m_1(k_0-p_{20}+i\epsilon),$$
$$2\pi\delta[(p-k)^2-m_1^2]\epsilon(p-k)\to(\pi/m_1)\delta(k_0-p_{20})=(i/2m_1)[(k_0-p_0+i\epsilon)^{-1}-(k_0-p_0-i\epsilon)^{-1}]. \quad (3.8)$$

Thus we see that, because of the choice made for the factor $\rho(p_1)$ [Eq. (2.18)] the three terms in (3.6) cancel to lowest order in $1/m_1$, so that W_2'/W_1 vanishes in the static limit.[14]

The above calculation, which so far ignores all photon self-energy corrections, can easily be extended to all orders, with the result that $W'=W_1+$(corrections of order $1/m_1$)+(photon self-energy corrections). Since this is fairly well known,[8] we leave out the details and turn instead to the problem of the vacuum-polarization graphs. The exact expression for W_1' is given by (3.3) after the replacement of the bare propagator $[(q-q')^2-\mu^2]^{-1}$ by the dressed photon propagator. Similarly, the exact expression for W_2' is obtained from (3.6) by replacing the two photon propagators in $\sigma(k)$ by the dressed propagators. Since this does not alter the convergence properties of the integral (3.6), our approximations (3.8) remain valid and lead to the same conclusion. That is, to lowest order in $1/m_1$, W' (including all photon self-energy corrections) is equal to W_1'. In particular, the contributions of all the diagrams of Fig. 2 (an infinite number) are taken into account to lowest order in $1/m_1$ by evaluating the matrix elements of the single term of Fig. 2(a) between the exact wave functions of the W_1-ladder approximation. This is very useful for evaluating the Lamb shift.

Let us imagine that Eqs. (3.2) and (3.4) have been solved exactly,[15] and let G_L and G_L' be the corresponding Green's functions:

$$G_L=i(K_2-W_1-i\epsilon)^{-1},\quad G_L'=i(K_2-W'-i\epsilon)^{-1}. \quad (3.9)$$

To determine the remaining corrections, including the main part of the Lamb shift, we define a new potential W'' by postulating the following equation for the exact

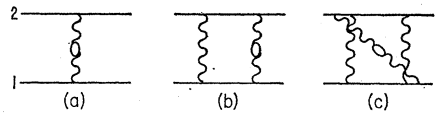

FIG. 2. Photon self-energy insertions.

[14] Note that, as $m_1\to\infty$, $[dp_1]\to d^3p_1/(2\pi)^3(2m_1)$, so that the static limit potentials are $W_1/2m_1$ and $W_2'/2m_1$.

[15] The exact solution of Eq. (3.2)—for vanishing photon mass—was obtained in Ref. 5. The exact wave functions of the W_1 ladder approximation were also given.

T matrix:

$$T-T_L'=-W''-iW''G_L'(T-T_L'). \quad (3.10)$$

Let us expand

$$T-T_L'=T_1''+T_2''+\cdots, \quad (3.11)$$

where T_n'' is the amplitude corresponding to all those Feynman diagrams from which exactly n internal photon lines must be removed in order to obtain a diagram that is included in T_L'. We expand W'' in similar fashion,

$$W''=W_1''+W_2''+\cdots, \quad (3.12)$$

and solve "order by order":

$$W_1''=-T_1'', \quad (3.13)$$

$$W_2''=-T_2''+iW_1''G_L'W_1'', \quad (3.14)$$

and so on.

For the applications it is very important that the improved W'-ladder approximation[7] agrees with the analytically tractable W_1-ladder approximation[15] to a very high order of accuracy. The largest difference between the two ladder approximations is due to the potentials $W_1'-W_1$ and W_2'. The latter has already been shown to go to zero like $1/m_1$ as $m_1\to\infty$; in addition it can easily be shown that $\langle W_2'\rangle\propto(Z\alpha)^5\ln Z\alpha$ for small $Z\alpha$, regardless of the values of the masses. Thus W_2' contributes to the Lamb shift if m_1 is finite, but its effect on the wave functions may be neglected in the evaluation of W_1''; this is true if m_1 is large *or* $Z\alpha$ is small, or both. The effect of $W_1'-W_1$ is not so easily disposed of since this does not vanish as $m_1\to\infty$. However, for small $Z\alpha$ $\langle W_1'-W_1\rangle\propto(Z\alpha)^4\alpha$ (see Sec. V).[16]

In the approximation in which W' is replaced by W_1 it is easy to evaluate W_1''. It consists of all conventional ladder graphs to which have been added a single internal photon line that begins and ends on the same particle—either particle 1 or particle 2. These diagrams are illustrated in Fig. 3. The challenge presented by the evaluation of the Lamb shift is to obtain a good approximation to the sum of all these diagrams. It will now be shown that our method resolves this problem in the same way as the usual static approximation. In view of the applications, we shall ignore the proton (particle 1) fluctuation diagrams.

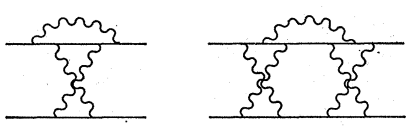

FIG. 3. Typical contributions to W_1''.

[16] The wave functions of the W_1' ladder approximation can be obtained in closed form, but this is hardly useful since the evaluation of the main contribution to the Lamb shift probably cannot be carried to the same degree of accuracy.

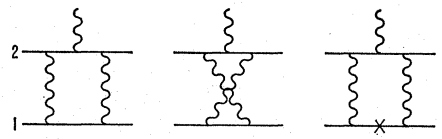

FIG. 4. Contributions to the particle-2 (electron) current. The three graphs are interrelated in the same way as those of Fig. 1 and cancel as $m_1\to\infty$.

Consider the first two graphs of Fig. 4. These are not the only diagrams of this order, but the following argument goes through in the same way for all. These two graphs differ from the first two graphs of Fig. 1 by having an external photon attached to the internal electron line. We now argue in the same way as before, making the approximations (3.8) for the proton propagator, that the sum of the first two graphs of Fig. 4 is equal to the third graph to lowest order in $1/m_1$. Clearly this argument applies to graphs of all orders and allows the conclusion that, to lowest order in $1/m_1$, the insertion of an external photon line with momentum k_μ is accomplished by the same rule as for a free electron:

$$G_L(p)\to iG_L(p)g_2G_L(p-k). \quad (3.15)$$

Next, consider the effect of inserting a virtual photon line, emitted and absorbed by the electron. Obviously the rule, to lowest order in $1/m_1$, is

$$G_L(p)\to -ig_2^2(2\pi)^{-4}\int\frac{d^4k}{k^2}G_L(p)G_L(p-k)G_L(p). \quad (3.16)$$

Thus, to lowest order in $1/m_1$,

$$W_1''=g_2^2(2\pi)^{-4}\int\frac{d^4k}{k^2}G_L(p-k). \quad (3.17)$$

Besides the case of large m_1, there is another circumstance in which (3.17) is a good approximation, namely, when $Z\alpha$ is small, in which case the relative error is of order $Z\alpha$. The main part of the Lamb shift is the expectation value of this quantity between the wave functions of the W_1-ladder approximation. The calculation is carried out in Sec. V.

It may be useful to comment on the extension of the rule (3.15) to the case of the full gauge-invariant treatment of the photon. The position coordinate that is conjugate to the total momentum p is x_2.[5] Hence (3.15) means that the photon couples locally to the electron. Minimal coupling in the sense $p_\mu\to p_\mu-eA_\mu$ thus means minimal coupling to the electron. The rule (3.15) is replaced by

$$G_L(p)\to iG_L(p)e_2I_\mu(p,\,p-k)G_L(p-k), \quad (3.18)$$

where[5]

$$I_\mu(p,p')=(p+p'-2q)_\mu \quad (3.19)$$

is the canonical conserved current operator. The fact

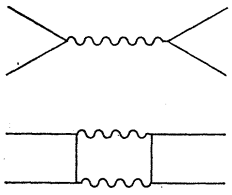

FIG. 5. Graph of the potentials W_{a1} and W_{a2} that are responsible for one- and two-photon decay of positronium.

that I_μ is exactly conserved means that the evaluation of the Lamb shift for the case of vector photons can be carried through in explicitly gauge-invariant fashion.

IV. DECAY OF POSITRONIUM

To order g^2 the T matrix is just the Born approximation $T_1=-W_1$. The potential W_1 was given by (2.10), except that we now introduce an additional term representing the annihilation of particles 1 and 2 into a scalar photon. For future convenience the masses will not be taken equal until the end. The new term in W_1 is

$$W_{a1}=g^2(p^2-\mu^2+i\epsilon)^{-1}. \tag{4.1}$$

This potential has constant matrix elements in q space and is a relativistic version of the ordinary three-dimensional δ-function potential. In first-order perturbation theory it is sufficient to evaluate the expectation value:

$$\langle W_{a1}\rangle\equiv\int[dq]\psi_q{}^*(p)2(p_0-q_0)[W_{a1}\psi(p)]_q. \tag{4.2}$$

Here $p=p_1+p_2$, $q=p_1$, and physical normalization is implied:

$$1=\int[dq]\psi_q{}^*(p)2(p_0-q_0)\psi_q(p). \tag{4.3}$$

The internal coordinate q_μ is written as a suffix, as in Ref. 5. Equation (4.2) may be written

$$\langle W_{a1}\rangle=\bar{A}_1(p^2-\mu^2+i\epsilon)^{-1}A_1, \tag{4.4}$$

with

$$A_1\equiv g\int[dq]\psi_q(p), \tag{4.5a}$$

$$\bar{A}_1\equiv g\int[dq]\psi_q{}^*(p)2(p_0-q_0). \tag{4.5b}$$

The amplitude A_1 describes the coupling of the bound state $\psi_q(p)$ to a state of one off-shell photon, and $\bar{A}_1$ is the (physical) adjoint of A_1.

The numerical value of A_1 is obtained in Appendix A for the case $\mu=0$. The result, for a bound state with quantum numbers n, l, and m, is

$$A_1=g^4 2^{-13/2}\pi^{-2}p_0{}^{-3/2}n^{-3/2}(p^2-m_1{}^2-m_2{}^2)^{-1/2}\delta_{l0}, \tag{4.6a}$$

$$\bar{A}_1=p_0{}^{-1}(p^2-m_1{}^2+m_2{}^2)A_1. \tag{4.6b}$$

The kinematical factor in $\bar{A}_1$ is obtained trivially from the wave equation when the effect of binding is neglected. These formulas are valid on shell, with

$$p^2=m_1{}^2+m_2{}^2+[4m_1{}^2m_2{}^2-(g^2/8\pi n)^2]^{1/2}. \tag{4.7}$$

The decay from a scattering state, which is just the electron form factor in the timelike region, is also calculated in Appendix A:

$$A_1=(2\pi)^{-3}g\Gamma(1-\nu)(2|\mathbf{q}|/\tilde{m})^\nu. \tag{4.8}$$

Here $\tilde{m}$ is an arbitrary function of p^2 with dimension of mass that enters through the treatment of the infrared divergence, ν is the principal quantum number for the scattering state,

$$\nu=i(g^2/8\pi)[(p^2-m_-{}^2)(p^2-m_+{}^2)]^{-1/2}, \tag{4.9}$$

and $\mathbf{q}$ is the c.m. momentum, and $m_\pm\equiv m_1\pm m_2$.

For future reference let us note that Eqs. (4.1), (4.4), and (4.6) give, to lowest order in q,

$$\langle\mathbf{E}\rangle=\delta_{l0}g^6 2^{-13}\pi^{-4}n^{-3}p_0{}^{-4}(p^2-m_1{}^2+m_2{}^2)/(p^2-m_1{}^2-m_2{}^2), \tag{4.10}$$

where $\mathbf{E}$ is the operator whose matrix elements $E_{qq'}$ in momentum space are all equal to unity.

For two-photon decay the relevant potential is that illustrated by Fig. 5 (we now take $m_1=m_2=m$):

$$W_{a2}=ig^4\int d^4k d^4k'(2\pi)^{-4}\delta(p-k-k')\times\{(k^2-\mu^2+i\epsilon)(k'{}^2-\mu^2+i\epsilon)[(q-k)^2-m^2+i\epsilon]\times[(q'-k)^2-m^2+i\epsilon]\}^{-1}. \tag{4.11}$$

Following the procedure of Tavkhelidze,[17] we evaluate the expectation value of the imaginary part, setting $\mu=0$,

$$\langle\mathrm{Im}W_{a2}\rangle=-2^{-7}\pi^{-2}g^4\int d\Omega_k\langle(q\cdot k)^{-1}(q'\cdot k)^{-1}\rangle. \tag{4.12}$$

We exploit the fact that the factor $(q\cdot k)^{-1}$ can be written[5]

$$(q\cdot k)^{-1}=\lim_{k^2\to 0}m(k^2)^{-1/2}\Gamma_4{}^{-1}(q\cdot k)^{-2}=\lim_{k^2\to 0}\pi(k^2)^{-1/2}\Gamma_4{}^{-1}\psi_q(k,1), \tag{4.13}$$

where $\psi_q(k,1)$ is the off-shell ground-state wave function with total momentum k. Thus

$$\langle\mathrm{Im}W_{a2}\rangle=-2^{-7}g^4\int d\Omega_k\bar{A}_2(k)A_2(k), \tag{4.14}$$

with

$$A_2(k)=\lim(k^2)^{-1/2}\psi^\dagger(k,1)\psi(p)(2\pi)^{-3}, \tag{4.15a}$$

$$\bar{A}_2(k)=\lim(k^2)^{-1/2}\psi^\dagger(p)2(p_0-q_0)\psi(k,1)(2\pi)^{-3}, \tag{4.15b}$$

[17] See the lectures given by A. N. Tavkhelidze at the Tata Institute, Bombay, 1963 (unpublished).

which shows that the two-photon state is represented by the ground-state wave function (off shell) in the limit $k^2 \to 0$. Consider now the decay of the ground state, $\psi(p)=\psi(\mathbf{p},1)$; then $\psi^\dagger(k,1)\psi(\mathbf{p},1)$ is the (off-shell) scalar form factor of the ground state,

$$\psi^\dagger(k,1)\psi(\mathbf{p},1)=2[1+2m\,k\cdot p/(k^2)^{1/2}P]^{-1} \times[2p_0(p^2-2m^2)/(2\pi)^3P]^{-1/2}, \quad (4.16)$$

where $P=[(p^2-m_-^2)(m_+^2-p^2)]^{1/2}$. We insert this into (4.15a) and use momentum conservation, $2k\cdot p=p^2$, to get

$$A_2=g^3/2^{11/2}\pi^3 m p^{5/2}(p^2-2m^2)^{1/2}. \quad (4.17)$$

To lowest order in g, we may replace the factor $2(p_0-q_0)$ in (4.15b) by $2m$, to get $\bar{A}_2=2mA_2$, and p by $2m$ in (4.17). This gives

$$\langle \mathrm{Im}W_{a2}\rangle = g^{10}/2^{21}\pi^5 m^8. \quad (4.18)$$

To determine the shift of an energy level in terms of the perturbing potential is not quite so simple as in the Schrödinger theory, but for the case $\mu=0$ the exact formula is (see Appendix A)

$$\Delta E=(1/2E)\langle \mathrm{Im}W_{a2}\rangle(p^2+m_1^2-m_2^2)/(p^2-m_1^2-m_2^2). \quad (4.19)$$

Thus, when $m_1=m_2=m$, and to lowest order in g, we get

$$\Gamma=|\Delta E|=\tfrac{1}{4}m(g^2/16\pi m^2)^5 \quad (4.20)$$

for the width of the ground state of scalar positronium. Except for the special group-theoretical techniques that distinguish our evaluation of the expectation value, the method used is closely related to that of Tavkhelidze.[17]

V. LAMB SHIFT

We wish to calculate the Lamb shift to lowest order in α and $Z\alpha$ without making an expansion in the mass ratio m_2/m_1. To this order in α there are two types of contributions: vacuum-polarization diagrams containing a single electron bubble (such as in Fig. 2) which are part of W', and fluctuation diagrams in which a photon line is attached at both ends to an electron line (as illustrated in Fig. 3, for example) and which are part of W_1''. As discussed in Sec. III, we may approximate W' by W_1' alone for small $Z\alpha$, which here means ignoring all vacuum-polarization diagrams except that of Fig. 2(a). This approximation introduces an error of order $\alpha(Z\alpha)^5\ln(Z\alpha)$ in the Lamb shift, coming from the single-bubble self-energy corrections of W_2'. For the fluctuation diagrams we will use Eq. (3.17). This approximation again introduces an error of order $\alpha(Z\alpha)^5\times\ln Z\alpha$, i.e., a relative error of order $Z\alpha$. In addition, we will not consider further the vacuum-polarization contribution due to nuclear pair creation, or the fluctuation contribution due to photon emission and reabsorption by the nucleus, since these may be easily obtained from the corresponding electron contributions upon multiplication by Z^2 and interchange of the two masses. Then, by using Eq. (1.14) we find that the Lamb shift is given, within a relative error of order $(Z\alpha)^2$, by

$$\Delta E=(1/2m_2)\langle W_{\mathrm{vp}}+W_f\rangle\equiv\Delta E_{\mathrm{vp}}+\Delta E_f, \quad (5.1)$$

where W_{vp} and W_f are the (electron) vacuum-polarization and fluctuation contributions from W_1' and W_1'', respectively.

The Feynman diagram for vacuum polarization gives

$$W_{\mathrm{vp}}(q,q')=ig_1g_2^3[(q-q')^2]^{-2}\int d^4k(2\pi)^{-4} \times(k^2-m_2^2+i\epsilon)^{-1}[(q-q'-k)^2-m_2^2+i\epsilon]^{-1} \quad (5.2)$$

minus mass and wave-function renormalization terms. The evaluation of the integral is straightforward and an expansion in $(q-q')^2/m^2$ yields the convenient form[18]

$$W_{\mathrm{vp}}(q,q')=-(4Z\alpha^2m_1/15m_2) \times[1+(q-q')^2/7m_2^2+\cdots]. \quad (5.3)$$

The wave functions fall off sufficiently rapidly for large momenta that we may neglect all but the first term in Eq. (5.3) with a relative error of order $(Z\alpha)^2$. Thus we write

$$W_{\mathrm{vp}}=-(4Z\alpha^2m_1/15m_2)\mathbf{E}, \quad (5.4)$$

where $\mathbf{E}$ is the operator introduced in Eq. (4.10), with

$$\langle\mathbf{E}\rangle=m_1^2m_2^3(Z\alpha)^3\delta_{L,0}/2\pi n^3m_+^3 \quad (5.5)$$

for the present case to relative order $(Z\alpha)^2$. Inserting these results into Eq. (5.1), we obtain

$$\Delta E_{\mathrm{vp}}=-\frac{\alpha(Z\alpha)^4m_2}{15\pi n^2}\left(\frac{m_1}{m_+}\right)^3. \quad (5.6)$$

The fluctuation diagram gives the operator

$$W_f=g_2^2\int d^4k(2\pi)^{-4}(k^2+i\epsilon)^{-1}G_L(p-k). \quad (5.7)$$

In the Green's function $G_L(p)=i[L(p)+i\epsilon]^{-1}$ and the Lagrangian operator $L(p)=K_2-W_1=(p-q)^2-m_2^2-W_1$, we explicitly indicate the dependence upon the total momentum p of the atom. For the preliminary treatment of Eq. (5.7), we apply the methods of Erickson and Yennie.[19] Since $L(p-k)=L(p)+k^2-2k(p-q)$ for the present p-independent W_1, we combine the denominators by use of the familiar Feynman integration trick

$$W_f=i\alpha m_2^2\pi^{-3}\int_0^1 dz\int d^4k\,D_1^{-2}, \quad (5.8)$$

[18] J. D. Bjorken and S. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1965), pp. 153–158.

[19] G. W. Erickson and D. R. Yennie, Ann. Phys. (N. Y.) **35**, 271 (1965).

where $p_2=p-q$ and

$$D_1=(k-zp_2)^2-z^2m_2^2+z(1-z)L(p)-z^2W_1+i\epsilon. \quad (5.9a)$$

For later convenience we also define

$$D_0\equiv k^2-z^2m_2^2+z(1-z)L(p)-z^2W_1+i\epsilon, \quad (5.9b)$$

$$D_{00}\equiv k^2-z^2m_2^2+z(1-z)L(p)+i\epsilon, \quad (5.9c)$$

$$D_{000}=k^2-z^2m_2^2+i\epsilon \quad (5.9d)$$

and, with a notation consistent with D_1 and D_0,

$$D_\lambda\equiv(k-\lambda zp_2)^2-z^2m_2^2 +z(1-z)L(p)-z^2W_1+i\epsilon. \quad (5.9e)$$

We treat the k-integration shift, by which D_1 and D_0 differ, by writing

$$\frac{1}{D_1^2}-\frac{1}{D_0^2}=\int_0^1 d\lambda\frac{\partial}{\partial\lambda}\frac{1}{D_\lambda^2}\equiv\int_0^1 2\lambda d\lambda\, I_\lambda, \quad (5.10)$$

$$I_\lambda=-\frac{1}{2\lambda}\frac{1}{D_\lambda}\left[\frac{1}{D_\lambda},\frac{\partial D_\lambda}{\partial\lambda}\right]_+\frac{1}{D_\lambda}, =\frac{1}{D_\lambda}\left[\frac{1}{D_\lambda},zp_2(\lambda^{-1}k-zp_2)\right]_+\frac{1}{D_\lambda}. \quad (5.11)$$

But since

$$\frac{\partial}{\partial k_\mu}\left(\frac{1}{D_\lambda}p_{2\mu}\frac{1}{D_\lambda}\right)=\frac{1}{D_\lambda}\left[p_{2\mu}\frac{2}{D_\lambda}(k^\mu-\lambda zp_2^\mu) +(k^\mu-\lambda zp_2^\mu)\frac{2}{D_\lambda}p_{2\mu}\right]\frac{2}{D_\lambda}$$

is a perfect derivative and integrates to zero under $\int d^4k$, this may be used to rewrite I_λ as a double commutator

$$I_\lambda=\frac{z^2}{D_\lambda}\left[p_{2\mu},\left[\frac{1}{D_\lambda},p_2^\mu\right]\right]\frac{1}{D_\lambda} =\frac{z^2}{D_\lambda}\left[q_\mu,\frac{1}{D_\lambda}\left[q^\mu,D_\lambda\right]\frac{1}{D_\lambda}\right]\frac{1}{D_\lambda}. \quad (5.12)$$

This same method may be reapplied to write

$$I_\lambda=I_0+\int_0^\lambda d\eta\frac{\partial}{\partial\eta}I_\eta, \quad (5.13)$$

but since the integral can be shown to give no contributions to ΔE of order lower than $(Z\alpha)^2$ relative to those from I_0, we will use $I_\lambda\to I_0$ in Eq. (5.10). Next we expand in W_1 and note that $D_{00}\doteq D_{000}$ when acting upon a wave function, to obtain[20]

$$I_0\doteq z^3\left\{q_\mu\frac{1}{D_{000}^3D_{00}}[W_1,q^\mu] -[W_1,q^\mu]\frac{1}{D_{000}^3D_{00}}q_\mu\right\}+\cdots. \quad (5.14)$$

[20] We will use the notation $\doteq$ to denote the equivalence of two expressions when they act upon a wave function.

The terms beyond the first may be shown to be of order $(Z\alpha)^2$ relative to the first and will be dropped. Similarly, an expansion in W_1 may be carried out for D_0^{-2}, which then reduces to

$$\frac{1}{D_0^2}\doteq\frac{1}{D_{000}^2}+\frac{2z^2}{D_{000}^3}W_1 +z^4W_1\left(\frac{1}{D_{00}^2D_{000}^2}+\frac{2}{D_{00}D_{000}^3}\right)W_1+\cdots. \quad (5.15)$$

The first term on the right-hand side is removed by mass renormalization and the second by wave function and vertex renormalizations. Terms beyond the third will be dropped and D_{00} replaced by D_{000} in the third term, since the relative error in doing so may be shown to be of order $(Z\alpha)^2$.

The $\int dz\int d^4k$ integrations are now seen to occur in only two forms:

$$J_1\equiv 3i\pi^{-2}\int_0^1 dz\int d^4k\, z^4D_{000}^{-4} =-\frac{1}{2}\int_0^1 dz\, z^4(z^2m_2^2)^{-2}=-\frac{1}{2m_2^4} \quad (5.16)$$

and

$$J_2\equiv i\pi^{-2}\int_0^1 dz\int d^4k\, z^3D_{000}^{-3}D_{00}^{-1} =-\int_0^1 dz\int_0^\infty k^2dk^2z^3(k^2+z^2m_2^2)^{-3} \times[k^2+z^2m_2^2-z(1-z)L(p)-i\epsilon]^{-1}. \quad (5.17)$$

Changing the integration variable from k^2 to

$$y\equiv(k^2+z^2m_2^2)/z(1-z)m_2^2 \quad (5.18)$$

and defining

$$\mathfrak{L}=ym_2^2-L(p)-i\epsilon, \quad (5.19)$$

we obtain

$$J_2=-\frac{1}{m_2^2}\int_0^1 dz\int_{z/(1-z)}^\infty dy\frac{yz(1-z)-z^2}{y^3(1-z)^3\mathfrak{L}} =-\frac{1}{m_2^2}\int_0^\infty dy\int_0^{y/(1+y)} dz\frac{yz(1-z)-z^2}{y^3(1-z)^3\mathfrak{L}} =-\frac{1}{6m_2^2}\int_0^\infty dy\frac{f(y)}{\mathfrak{L}}, \quad (5.20)$$

where

$$f(y)=3y^{-3}[y^2+2y-2(1+y)\ln(1+y)]. \quad (5.21)$$

From Eqs. (5.8) and (5.10), we see that W_f may be written

$$W_f=W_{f1}+W_{f2}, \quad (5.22)$$

where W_{f1} and W_{f2} come from the D_0^{-2} term and the I_λ term, respectively, in Eq. (5.10). Combining Eq. (5.16) with the surviving third term in Eq. (5.15), we obtain

$$W_{f1}=\alpha m_2^2\pi^{-1}W_1J_1W_1 \qquad (5.23)$$
$$=-(\alpha/2\pi m_2^2)W_1W_1.$$

From Eqs. (5.20) and (5.14), we obtain

$$W_{f2}=i\alpha m_2^2\pi^{-3}\int_0^1 \int d^4k\int_0^1 d\lambda\, 2\lambda I_\lambda$$
$$=\alpha m_2^2\pi^{-1}\{q_AJ_2[W_1,q^A]+[q^A,W_1]J_2q_A\}. \qquad (5.24)$$

In Eq. (5.24) we have extended the index summation range from $\mu=0,1,2,3$ to $A=0,1,2,3,4$ by defining the (c-number) operator $q_4\equiv -q^4\equiv m_1$ because the resulting relation $q^Aq_A=0$ simplifies the further development of W_{f2}. Then

$$W_{f2}=-\frac{\alpha}{6\pi}\int f(y)dy\left\{q_A\frac{1}{\mathfrak{L}}[W_1,q^A]+[q^A,W_1]\frac{1}{\mathfrak{L}}q_A\right\}$$
$$=-\frac{\alpha}{6\pi}\int f(y)dy\left\{q_A\frac{1}{\mathfrak{L}}[\mathfrak{L},q^A]+[q^A,\mathfrak{L}]\frac{1}{\mathfrak{L}}q_A\right\}$$
$$=-\frac{\alpha}{6\pi}\int f(y)dy\left\{2q_Aq^A-\left[q_A\frac{1}{\mathfrak{L}}q^A,\mathfrak{L}\right]_+\right\}$$
$$\doteq\frac{\alpha m_2^2}{3\pi}\int_0^\infty yf(y)dy\,q_A\frac{1}{\mathfrak{L}}q^A, \qquad (5.25)$$

since $\mathfrak{L}\doteq ym_2^2$ when acting upon a wave function.

It is worthwhile to digress at this point to compare the static limit of Eq. (5.25) with the conventional evaluation of the Lamb shift. As $m_1\to\infty$, $q_0\approx m_1-\mathbf{q}^2/2m_1$, so that the $A=0$ and $A=4$ terms almost cancel each other. The limit of $L(p)$ is $2m_2(E-H)$. Thus upon multiplication by $1/2m_2$ [cf. Eq. (5.1)], we obtain the limit

$$\frac{1}{2m_2}W_{f2}\to-\frac{\alpha}{3\pi m_2^2}\int_0^\infty f\left(\frac{2k'}{m_2}\right)$$
$$\times\mathbf{q}\cdot\left(\frac{k'}{k'-E+H}-1\right)\mathbf{q}dk'. \qquad (5.26)$$

The second term in the parentheses is what remains from the near cancellation of the $A=0$ and $A=4$ terms of Eq. (5.25) and the new integration variable is $k'\equiv\frac{1}{2}ym_2$. Except for a factor of $\frac{1}{2}$ (due to the distinction between vector and scalar photons), the only difference between this expression and the conventional expression[21] is the appearance of the factor $f(2k'/m_2)$ defined by Eq. (5.21), provided that the integration variable k' is identified with the photon energy in the nonrelativistic calculation. The expansion of $f(2k'/m_2)$ around $k'=0$ is

$$f(2k'/m_2)=1-k'/m_2+\cdots. \qquad (5.27)$$

Therefore, for photon energies that are small compared with m_2, the usual result is obtained. For large k', the conventional expression diverges and it is necessary to devise an elaborate procedure to obtain a finite value[22]: The integration in Eq. (5.26) is cut off at a value λ_c that is large compared to the binding energy but small compared to m_2. This gives an energy shift $E_<$ that depends logarithmically upon λ_c. For photon energies larger than λ_c, one ignores the effect of binding and evaluates an energy shift $E_>$ by standard field-theoretic perturbation theory. Here one encounters an infrared divergence and the integral over the virtual photon mass is cut off below λ_c. Thus $E_>$ also depends logarithmically on λ_c, while this dependence is canceled out in $E_>+E_<$. This procedure is not only complicated and unaesthetic, it also requires careful justification, since λ_c is a photon energy in $E_<$ and a photon mass in $E_>$. Our result, Eq. (5.26), on the other hand, is based on relativistic kinematics from the start. The Green's function $i/L(p)$ reduces to $i/2m_2(E-H)$ in the nonrelativistic limit and to $i(p_2^2-m_2^2)^{-1}$ in the high-energy limit, where the potential is of less importance. Hence Eq. (5.26) interpolates between the two limiting approximations used to calculate $E_<$ and $E_>$, and since

$$f(2k'/m_2)\approx 3m_2/2k' \quad \text{for} \quad k'\gg m_2, \qquad (5.28)$$

the integral is convergent for high as well as low values of k'.

We proceed with the evaluation of Eq. (5.25) by separating $f(y)$ into two parts:

$$f(y)\equiv 1+g(y), \qquad (5.29)$$

and using the identity

$$q_A\left[\frac{ym_2^2}{\mathfrak{L}}\right]q^A=q_A\left[\frac{L(p)}{\mathfrak{L}}\right]q^A \qquad (5.30)$$

in the part of W_{f2} which involves $g(y)$[23]:

$$W_{f2}=\lim_{Y\to\infty}\frac{\alpha}{3\pi}\int_0^Y dy\,q_A\left[\frac{ym_2^2}{\mathfrak{L}}+g(y)\frac{L(p)}{\mathfrak{L}}\right]q^A. \qquad (5.31)$$

Inserting the relations[24]

$$q_A=m_1\Gamma_4^{-1}\Gamma_A, \quad W_1=-2(Z\alpha)m_1m_2\Gamma_4^{-1},$$

and $\langle W_f\rangle=\langle n,L|\Gamma_4W_f|n,L\rangle$ into Eqs. (5.23) and (5.31),

[21] R. W. Huff, Phys. Rev. 186, 1367 (1969).

[22] This criticism does not apply to the treatment of Ref. 19.

[23] The special treatment of the integration limit is necessary because the integrals resulting from the two parts of $f(y)$ are not separately convergent.

[24] These Γ_A are the familiar $SO(4,2)$ generators which are discussed more fully in Refs. 3 and 21. Note that we are using a different expectation value from that of Eq. (4.2). Both are valid for the discrete states and we have chosen the most convenient form for the present context.

we obtain

$$\Delta E_{f1}\equiv\frac{1}{2m_2}\langle W_{f1}\rangle=-\frac{\alpha(Z\alpha)^2m_1^2}{\pi m_2}\langle n,L|\,\Gamma_4^{-1}|n,L\rangle, \quad (5.32)$$

$$\Delta E_{f2}\equiv\frac{1}{2m_2}\langle W_{f2}\rangle$$

$$=\lim_{Y\to\infty}\frac{\alpha m_1^2}{6\pi m_2}\int_0^Y dy\,\langle n,L|\,\Gamma_A[ym_2^2+g(y)\Gamma_4^{-1}\hat{L}]\times\frac{1}{\mathfrak{L}'}\Gamma^A|n,L\rangle, \quad (5.33)$$

where

$$\mathfrak{L}'\equiv ym_2^2\Gamma_4-\hat{L}, \quad (5.34)$$

$$\hat{L}=\Gamma_4^{-1}L(p)=-p^A\Gamma_A+2(Z\alpha)m_1m_2, \quad (5.35)$$

and P^A is given by Eq. (A1). The states $|n,L\rangle$, labeled by the principal quantum number and angular momentum, have the form[25]

$$|n,L\rangle=\mathfrak{N}R|n,L\rangle\rangle, \quad (5.36)$$

where the $|n,L\rangle\rangle$ are the orthonormal eigenvectors of Γ_0 and angular momentum and the $\mathfrak{N}$ is a normalization factor determined by $\langle n,L|\,\Gamma_4|n,L\rangle=1$.[24] R is the $SO(4,2)$ rotation which (for bound states) diagonalizes $\hat{L}$ and is here most conveniently written in the form $R=R''\,R'$, where

$$\bar{R}''[p^\mu\Gamma_\mu]R''=(p^2)^{1/2}\Gamma_0, \quad (5.37a)$$

$$\bar{R}'[2m_1(p^2)^{1/2}\Gamma_0-(p^2+m_1^2-m_2^2)\Gamma_4]R'=p\Gamma_0, \quad (5.37b)$$

with

$$P\equiv(P^AP_A)^{1/2}=[(p^2-m_-^2)(m_+^2-p^2)]^{1/2}=2(Z\alpha)m_1m_2/n. \quad (5.38)$$

Thus

$$\bar{R}\hat{L}R=-[2(Z\alpha)m_1m_2/n](\Gamma_0-n), \quad (5.39a)$$

$$\bar{R}\Gamma_4R=[n(p^2)^{1/2}/m_2(Z\alpha)](\Gamma_0+\Gamma_4-\epsilon\Gamma_0), \quad (5.39b)$$

where

$$\epsilon\equiv\frac{2(Z\alpha)^2m_1m_2^2}{n^2(p^2)^{1/2}[2m_1(p^2)^{1/2}+p^2+m_1^2-m_2^2]}\simeq\frac{(Z\alpha)^2m_2^2}{2n^2m_+^2}. \quad (5.40)$$

From Eq. (5.39b) and the fact that the diagonal matrix elements of Γ_4 in the $|n,L\rangle\rangle$ basis vanish, the normalization factor is found to be

$$\mathfrak{N}^2=(Z\alpha)m_2/n^2(p^2)^{1/2}\simeq(Z\alpha)m_2/n^2m_+. \quad (5.41)$$

With these results, Eqs. (5.32) and (5.33) become, to relative order $(Z\alpha)^2$,

$$\Delta E_{f1}=-\frac{\alpha(Z\alpha)^4m_2}{\pi n^3}\left(\frac{m_1}{m_+}\right)^2M_1, \quad (5.42)$$

$$\Delta E_{f2}=\lim_{Y\to\infty}\frac{2\alpha(Z\alpha)^4m_2}{3\pi n^3}\left(\frac{m_1}{m_+}\right)^3(M_2+M_3), \quad (5.43)$$

where

$$M_1\equiv\langle\langle n,L|\,[\Gamma_0+\Gamma_4-\epsilon\Gamma_0]^{-1}|n,L\rangle\rangle, \quad (5.44a)$$

$$M_2\equiv-\frac{1}{2n^2}\int_0^Y g(y)dy\langle\langle n,L|\,\Gamma_A\frac{1}{\Gamma_0+\Gamma_4-\epsilon\Gamma_0}\times(\Gamma_0-n)\frac{1}{\Lambda_\beta(\epsilon)}\Gamma^A|n,L\rangle\rangle, \quad (5.44b)$$

$$M_3\equiv\frac{1}{4\beta}\int_0^Y ydy\langle\langle n,L|\,\Gamma_A\frac{1}{\Lambda_\beta(\epsilon)}\Gamma^A|n,L\rangle\rangle=\frac{1}{4}\int_0^{Y/\beta}ydy\langle\langle n,L|\,\Gamma_A\frac{1}{\Lambda_1(\epsilon)}\Gamma^A|n,L\rangle\rangle \quad (5.44c)$$

$$\beta\equiv(Z\alpha)^2m_1/(p^2)^{1/2}\simeq(Z\alpha)^2m_1/m_+, \quad (5.45)$$

and

$$\Lambda_\beta(\epsilon)\equiv y(\Gamma_0+\Gamma_4-\epsilon\Gamma_0)+2\beta(\Gamma_0-n)/n^2, \quad (5.46)$$

with Eq. (5.46) including $\Lambda_1(\epsilon)$ as a special case.[26] Note that all $Z\alpha$ dependences are now explicit, with none being hidden within wave functions. This is the principal advantage gained in actually carrying out the rotation R.

Let M_{i0} be the value of M_i when we set $\beta=\epsilon=0$ within the matrix elements (but not in the integration limit in M_3) and write

$$M_i\equiv M_{i0}+\delta M_i. \quad (5.47)$$

We will drop the δM_i terms, since they are clearly of higher order in $Z\alpha$. However, unlike our previous approximations made in evaluating Eq. (5.7), which gave relative errors of order $(Z\alpha)^2$, this neglect of the δM_i may introduce relative errors of order as low as $Z\alpha$. For example,

$$\delta M_1=\epsilon\langle\langle n,L|\,(\Gamma_0+\Gamma_4)^{-1}\Gamma_0(\Gamma_0+\Gamma_4-\epsilon\Gamma_0)|n,L\rangle\rangle. \quad (5.48)$$

Although $\delta M_1\to 0$ as $\epsilon\to 0$, $\epsilon^{-1}\delta M_1$ does not exist in the limit $\epsilon\to 0$ for $L=0$.

We apply Eq. (C6) of Ref. 21 to obtain

$$M_{01}=\langle\langle n,L|\,(\Gamma_0+\Gamma_4)^{-1}|n,L\rangle\rangle=2(2L+1)^{-1}. \quad (5.49)$$

The evaluation of the remaining matrices needed for M_{20} and M_{30} is carried out in Appendix B. With

[25] The notation for the states and eigenvectors follows that of Ref. 21, in which a more complete discussion may be found.

[26] The two expressions for M_3 are related by the change of integration variable $y\to\beta y$.

Eqs. (B6a) and (B7), we find

$$M_{20}=-\frac{1}{2n^2}\int_0^Y dy\,\frac{g(y)}{y}\langle\langle n,L|\Gamma_A\frac{1}{\Gamma_0+\Gamma_4}\times(\Gamma_0-n)\frac{1}{\Gamma_0+\Gamma_4}\Gamma^A|n,L\rangle\rangle$$
$$=\int_0^Y dy\,\frac{g(y)}{y}\delta_{L,0}$$
$$=(\tfrac{5}{6}-\ln Y)\delta_{L,0}. \quad (5.50)$$

Equation (B10) may now be used to express M_{30} in terms of the familiar Bethe logarithm $\gamma(n,L)$:

$$M_{30}=\delta_{L,0}\ln(Y/\beta)-\gamma(n,L). \quad (5.51)$$

When these results are inserted into Eqs. (5.42) and (5.43), and Eq. (5.6) is included, we obtain the complete Lamb shift to lowest order in α and $(Z\alpha)$,

$$\Delta E=\frac{2\alpha(Z\alpha)^4m_2}{3\pi n^3}\left(\frac{m_1}{m_+}\right)^3\left\{\left[\tfrac{5}{6}-\tfrac{1}{10}+\ln\frac{m_+}{(Z\alpha)^2m_1}\right]\delta_{L,0}-\gamma(n,L)-\frac{3m_+}{(2L+1)m_1}\right\}, \quad (5.52)$$

where the term $-\frac{1}{10}$ is the vacuum-polarization contribution. Although terms of higher order in $Z\alpha$ have been neglected, this result gives the exact mass dependence of $(Z\alpha)^4$ term. We have nowhere assumed $m_2\ll m_1$. This also agrees with the mass dependence of the Lamb shift for real hydrogen,[19] but with the magnetic moment term replacing the last term of Eq. (5.52).

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APPENDIX A: EVALUATION OF EXPECTATION VALUES FOR POSITRONIUM DECAY

Evaluation of (4.5)

Let Q, P, and λ be the "five-vectors"[30]

$$Q_A=\{q_\mu,m_1\},\quad p_A=\{2m_1p_\mu,p^2+m_1^2-m_2^2\},\quad \lambda_A=p_A/p \quad (A1)$$

with $P=(P^2)^{1/2}$ and let $\psi_q(p,nlm)$ denote the off-shell wave function for a discrete state. In the limit $p_\mu\to 0$,

$$\lim i\Gamma_4\psi(p,1)=\lim(\lambda\Gamma)\psi(p,1)=\lim\psi(p,1)=m_1\pi^{-1}\lim(\lambda Q)^{-2}=-(m_1\pi)^{-1}. \quad (A2)$$

Thus, with neglect of a phase

$$\int d(q)\,\psi_q(p,nlm)=m_1\pi\lim\int d(q)\,\psi_q{}^*(p,1)\Gamma_4\psi_q(p,nlm)$$
$$=m_1\pi\lim\psi^\dagger(p,1)\psi(p,nlm). \quad (A3)$$

Now, with physical normalization for $\psi(p,nlm)$,

$$\psi^\dagger(p',1)\psi(p,nlm)=-2n^{1/2}\{(-1-\lambda\lambda')^{-n}[1-(\lambda\lambda')^2]^{\frac{1}{2}(n-1)}\}\times[2np_0(p^2-m_1^2-m_2^2)/(2\pi)^3P]^{-1/2}\delta_{l0}. \quad (A4)$$

The limit of the curly brackets as $p_\mu'\to 0$, with neglect of a phase, is $P/2m_1p$. Thus, finally

$$A_1=(2\pi)^{-3}gm_1\pi(2n^{1/2}P/2m_1p_0)\times[2np_0(p^2-m_1^2-m_2^2)/(2\pi)^3P]^{-1/2}\delta_{l0}, \quad (A5)$$

which reduces to (4.6a) on the mass shell $P=g^2/8\pi n$. When $\psi_q(p)$ is a continuum state, we have instead

$$A_1=g(2\pi)^{-3}m_1\pi\lim\psi^\dagger(p',1)\hat{L}(p)^{-1}\psi(\mathbf{p},\mathbf{q}). \quad (A6)$$

The matrix element is the amplitude for photodissociation from the ground state. Using the method that was used in Ref. 5 to obtain the Coulomb and bremsstrahlung amplitudes, we easily obtain

$$\psi^\dagger(p',1)\hat{L}(p)^{-1}\psi(\mathbf{p},\mathbf{q})=\gamma^{-1\nu}!(-\nu)!\lim\sum_{lm}\psi^\dagger(p',1)\times\psi'(p,\nu lm)\psi'(p,\nu lm)^\dagger\psi(\mathbf{p},\mathbf{q}). \quad (A7)$$

In the limit $p'\to 0$ we have, as above,

$$\psi^\dagger(p',1)\psi'(p,\nu lm)\to\nu^{1/2}P(mp)^{-1}\delta_{l0}\delta_{m0}. \quad (A8)$$

Also,

$$\psi'(p,\nu 00)^\dagger\psi(\mathbf{p},\mathbf{q})=2^{-1/2}(\gamma/\pi\tilde{m}\nu!)\tilde{\psi}'(p,\nu)^{A_1\cdots A_{\nu-1}}V_{A_1}\cdots V_{A_{\nu-1}} \quad (A9)$$
$$=(2/\nu)^{1/2}(\gamma/\tilde{m}\nu!)\,|V|^{\nu-1}Y_{\nu 00}(\lambda,V),$$

where $\tilde{m}$ is an arbitrary normalizer with the dimension of mass that comes in through the infrared divergence,

[30] This appendix relies heavily upon the results of Ref. 5.

$V_A=\tilde{m}^{-1}Q_A$, and, in the c.m. system,

$$|V|^{\nu-1}Y_{\nu 00}(\lambda,V)=\pi^{-1}2^{-1/2}|V|^{\nu-1}\sin\nu\varphi/\sin\varphi$$
$$=\pi^{-1}2^{-1/2}(2|\mathbf{q}|/\tilde{m})^{\nu-1}. \quad \text{(A10)}$$

Combining (A6)–(A10), we obtain Eq. (4.8).

The relationship between the energy shift and the expectation value of a potential in first order of perturbation is found as follows. Replace the perturbative potential by its expectation value $\langle W'\rangle$ in the wave equation:

$$[(p-q)^2-m_2^2-W_1-\langle W'\rangle]\psi=0. \quad \text{(A11)}$$

This is solvable exactly if the mass μ of the meson exchange potential W_1 vanishes; for a state with principal quantum number n,

$$4m_1p^2-(p^2+m_1^2-m_2^2-\langle W'\rangle)^2=\text{const}. \quad \text{(A12)}$$

Thus, in the c.m. system,

$$\Delta E=\frac{\partial E}{\partial\langle W'\rangle}\bigg|_{W'=0}\langle W'\rangle$$
$$=\frac{\langle W'\rangle}{2p}\frac{p^2+m_1^2-m_2^2}{p^2-m_1^2-m_2^2}. \quad \text{(A13)}$$

APPENDIX B: EVALUATION OF SPECIAL MATRICES FOR LAMB SHIFT

We wish to evaluate matrices of the form

$$\mathfrak{M}[f(\Gamma_0)]\equiv\Gamma_A(\Gamma_0+\Gamma_4)^{-1}f(\Gamma_0)(\Gamma_0+\Gamma_4)^{-1}\Gamma^A, \quad \text{(B1)}$$

where $f(n)$ is any function which is well defined for the positive integers. Since the ratio of any two Γ_A commutes with any other such ratio,

$$\mathfrak{M}[f(\Gamma_0)]=\Gamma_0(\Gamma_0+\Gamma_4)^{-1}$$
$$\times\bar{\mathfrak{M}}[\Gamma_0^{-2}f(\Gamma_0)](\Gamma_0+\Gamma_4)^{-1}\Gamma_0, \quad \text{(B2)}$$

where

$$\bar{\mathfrak{M}}[f(\Gamma_0)]\equiv\Gamma_A f(\Gamma_0)\Gamma^A. \quad \text{(B3)}$$

Equation (A17) of Ref. 21 may be rewritten to give

$$\bar{\mathfrak{M}}[f(\Gamma_0)]=\Gamma_0^2f(\Gamma_0)-\tfrac{1}{2}\sum_{\pm}(\Gamma_0\pm1)^2f(\Gamma_0\pm1)$$
$$\equiv-\tfrac{1}{2}\Delta[\Gamma_0^2f(\Gamma_0)] \quad \text{(B4)}$$

in terms of the second-difference operator Δ. In applying Eq. (B4), however, it must be remembered that the term $(\Gamma_0-1)^2f(\Gamma_0-1)$ is not present when applied to the eigenvector $|1,0\rangle\rangle$. This is obvious, since the value $n=0$ is not contained in the spectrum of Γ_0. The coefficient $(\Gamma_0-1)^2$ of $f(\Gamma_0-1)$ usually ensures that this restriction is satisfied automatically, but for functions for which $\lim_{n\to0}[n^2f(n)]\neq0$ special attention is needed. Thus, although the second difference operator applied to any linear function yields zero, we obtain

$$\bar{\mathfrak{M}}[\Gamma_0^{-1}]=0, \quad \bar{\mathfrak{M}}[\Gamma_0^{-2}]=\tfrac{1}{2}\delta_{1,\Gamma_0}, \quad \text{(B5)}$$

where the operator Kronecker δ is just the projection operator for the state $|1,0\rangle\rangle$. For $\mathfrak{M}[f(\Gamma_0)]$ these two special cases become

$$\Gamma_A(\Gamma_0+\Gamma_4)^{-1}\Gamma_0(\Gamma_0+\Gamma_4)^{-1}\Gamma^A=0 \quad \text{(B6a)}$$

and

$$\Gamma_A(\Gamma_0+\Gamma_4)^{-2}\Gamma^A=\tfrac{1}{2}\Gamma_0(\Gamma_0+\Gamma_4)^{-1}|1,0\rangle\rangle$$
$$\times\langle\langle1,0|(\Gamma_0+\Gamma_4)^{-1}\Gamma_0. \quad \text{(B6b)}$$

Applying Eq. (C6) of Ref. 21, and noting that $\Gamma_0+\Gamma_4$ is diagonal in L, we obtain the matrix elements

$$\langle\langle n',L'|\Gamma_A(\Gamma_0+\Gamma_4)^{-2}\Gamma^A|n,L\rangle\rangle=2(nn')^{1/2}\delta_{L,0}\delta_{L',0}. \quad \text{(B7)}$$

A comparison with Eq. (C15) of Ref. 21 shows that $\Gamma_A(\Gamma_0+\Gamma_4)^{-2}\Gamma_A$ is a representation of the coordinate-space δ-function operator.

The familiar Bethe logarithm may be expressed in a more convenient form for our purposes by combining Eqs. (2.8), (2.9), (2.11), (2.13), and (2.17) of Ref. 21 and Eq. (B4) above:

$$\gamma(n,L)=\lim_{\lambda\to\infty}\left[\delta_{L,0}\ln\lambda-\int_0^\lambda d\hat{k}\,\frac{n\nu\hat{k}}{8}\right.$$
$$\left.\times\langle\langle n,L|R^{-1}\Gamma_A\frac{1}{\Gamma_0-\nu}\Gamma^AR|n,L\rangle\rangle\right], \quad \text{(B8)}$$

where $\nu^{-2}=\hat{k}+n^{-2}$, and R is an $SO(4,2)$ rotation operator satisfying

$$R^{-1}\Gamma_0R=\frac{1}{2}\left(\frac{n}{\nu}+\frac{\nu}{n}\right)\Gamma_0+\frac{1}{2}\left(\frac{n}{\nu}-\frac{\nu}{n}\right)\Gamma_4$$
$$=\tfrac{1}{2}n\nu\Lambda_1(0)+\nu. \quad \text{(B9)}$$

Carrying out this rotation in Eq. (B8), we obtain

$$\gamma(n,L)=\lim_{\lambda\to\infty}\left[\delta_{L,0}\ln\lambda-\frac{1}{4}\int_0^\lambda\hat{k}d\hat{k}\right.$$
$$\left.\times\langle\langle n,L|\Gamma_A\frac{1}{\Lambda_1(0)}\Gamma^A|n\rangle\rangle\right]. \quad \text{(B10)}$$