1, 2, and 5 are not true any more. One can construct examples which show that an analog to Łojasiewicz's lemma does not hold in this case. The reason is that one cannot draw any conclusion from the symmetric part of the first derivative of a function on the symmetric part of its primitive function.

On the other hand, Theorems 3, 4, and 6 as well as their corollaries remain true under this weakened assumption.

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Causality and Analyticity in Formal Scattering Theory

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The connection between causality and analyticity in scattering theory is formulated in terms of Hilbert-space concepts. The usual rules of nonrelativistic quantum mechanics are assumed to hold for the "in" and "out" states of the scattering system. We show that there is a (physically verifiable) causality condition which implies that each diagonal S-matrix element $S_{\alpha\alpha}(E)$ must be the limit of an analytic function of the energy E, regular in $\mathrm{Im}E>0$. The implications for partial-wave amplitudes and for the forward scattering amplitude in elastic two-body collisions are discussed.

I. INTRODUCTION

ANALYTICITY properties of scattering amplitudes are an essential ingredient of S-matrix theory. It is important to know which of these properties can be deduced from causality conditions that are physically verifiable. This question has been the subject of a number of publications.[1–7] Direct proofs of analyticity from causality are available for nonrelativistic elastic scattering by spherically symmetric interactions that vanish beyond a finite radius.[1,6]

In attempting to generalize these proofs in the context of formal scattering theory, two types of difficulties are encountered. First, because scattering states cannot contain negative-frequency Fourier components, events in a scattering experiment cannot be localized in time with arbitrary sharpness.[2] However, Screaton[4] has shown that for a simple linear system described by a (scalar) equation

$$O(t)=\int_{-\infty}^{\infty} dt' F(t-t')I(t'),$$

a causality condition can be formulated which implies that the Fourier transform of F is analytic in a half-plane, even though O and I have only non-negative frequency components. The second difficulty is connected with the use of monochromatic states $|E,\alpha)$ of the free Hamiltonian H_0 in defining the S matrix. In the mathematical theory of Hilbert space, the diagonalization of a self-adjoint operator is expressed in terms of projection operators corresponding to

$$P(E)=\sum_\alpha \int_0^E dE'\,|E',\alpha)(E',\alpha|\,,$$

whose properties can be established under very general conditions.[8] The properties of the mapping $(E,\alpha|\psi\rangle$ depend on "representation theorems"[9] which are limited in scope. Very little can be said about matrix elements of the form $(E,\alpha|T|E',\alpha')$ if T is an arbitrary Hilbert-space operator. For example, if we know that $(E,\alpha|\psi\rangle$ is continuous in E for any normalizable $|\psi\rangle$, we cannot state that $(E,\alpha|T|E',\alpha')$ is continuous in E, E' unless we impose strong restrictions on the operator T, e.g., the condition that T be a compact operator.[10–12]

1 N. G. van Kampen, Phys. Rev. **91**, 1267 (1953).
2 R. J. Eden and P. V. Landshoff, Ann. Phys. (N. Y.) **31**, 370 (1965).
3 A. Peres, Ann. Phys. (N. Y.) **37**, 179 (1966).
4 G. R. Screaton, Phys. Rev. **165**, 1610 (1968); **182**, 1415 (1969).
5 D. Iagolnitzer and H. P. Stapp, Commun. Math. Phys. **14**, 15 (1969).
6 H. M. Nussenzveig, Phys. Rev. **177**, 1848 (1969).
7 A more complete bibliography list is available in Refs. 3, 5, and 6.
8 N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Ungar, New York, 1961), Sec. 61.
9 N. Dunford and J. T. Schwartz, *Linear Operators* (Interscience, New York, 1963), Sec. XII.3.
10 Reference 9, p. 516.
11 It is worth remarking that in the standard proof (Ref. 12) of the analyticity of the forward scattering amplitude in potential scattering, the assumptions about the potential that are introduced are needed to show that the operator $T(E)$ is compact. The analyticity of the forward scattering amplitude is then obtained without further assumptions about the potential.
12 N. N. Khuri, Phys. Rev. **107**, 1148 (1957).

In order to discuss the implications of causality in a general context, it seems necessary to restrict our attention to those properties of the scattering which can be expressed in terms of projection operators without explicit reference to the representation $|E,\alpha)$. The diagonal elements $S_{\alpha\alpha}(E)$ of the S matrix fall into this category. In Sec. II, using standard theorems about the spectral representation of operators, we obtain information about the functions $S_{\alpha\alpha}(E)$ which is needed to apply Screaton's result.[4] In Sec. III we show that there is a causality condition which implies that each $S_{\alpha\alpha}(E)$ is the limit of an analytic function of E, holomorphic in $\mathrm{Im}E>0$. Our causality condition is physically verifiable, at least in an idealized thought experiment. The argument is quite general in that the rules of quantum mechanics are assumed to hold only for the asymptotic states of the system at $t=\pm\infty$. The existence of a Hamiltonian H which governs the time development of the system at finite times (or of an interpolating wave function Ψ) is not necessary. In Sec. IV, we discuss the special problems encountered in the application of this method to (i) partial-wave amplitudes for a spherically symmetric interaction and (ii) the forward direction scattering amplitude, for elastic scattering of two particles by a spin-independent interaction.

II. DIAGONAL S-MATRIX ELEMENTS AS SPECTRAL FUNCTIONS

At the risk of repeating well-known facts, we shall list a number of assumptions about the scattering system which are needed for our main result in Sec. III. Because of assumptions B and C, we are excluding from consideration the case of multichannel scattering where rearrangement channels are present.

A. The state of the system is describable asymptotically as $t\rightarrow\pm\infty$ by vectors ψ in a (separable) Hilbert space $\mathcal{H}$.

B. There exists a unitary[13] operator S in $\mathcal{H}$ such that if the system is in the state $\psi_{\text{in}}(t)$ at $t=-\infty$, then at $t=+\infty$ it will be in the state

$$\psi_{\text{out}}(t)=S\psi_{\text{in}}(t). \tag{2.1}$$

C. The states $\psi_{\text{in}}(t)$, $\psi_{\text{out}}(t)$ evolve in time according to a Hamiltonian H_0,

$$\begin{aligned}\psi_{\text{in}}(t)&=e^{-iH_0t}\psi_{\text{in}}(0),\\ \psi_{\text{out}}(t)&=e^{-iH_0t}\psi_{\text{out}}(0),\end{aligned} \tag{2.2}$$

where H_0 is a positive self-adjoint operator[13] in $\mathcal{H}$ having a continuous spectrum only.

This condition on H_0 ensures that there is a resolution of the identity corresponding to H_0 with the following properties.[14]

(i) $P(E)$ is a projection operator[13] (hence bounded and self-adjoint) which reduces H_0 for any $E\in[0,\infty)$.

(ii) $P(0)=0$, $P(\infty)=1$; for $0\leqslant E\leqslant E'<\infty$, $P(E)\times P(E')=P(E)$.

For any $E\in[0,\infty)$:

(iii) $P(E)$ is strongly continuous in E.

(iv) For each $\varphi\in\mathcal{H}$, $(\varphi,P(E)\varphi)$ is a non-negative, nondecreasing continuous function of E.

These projectors make it possible to define functions of operators

$$f(H_0)=\int_0^\infty f(E)dP(E) \tag{2.3}$$

(for certain classes of functions f), and to develop an operational calculus for these functions,[15] which provides a rigorous justification of statements such as

$$G_0(z)=(z-H_0)^{-1}=-i\int_0^\infty dt\, e^{-iH_0t+izt},\quad \mathrm{Im}z>0. \tag{2.4}$$

In particular we shall need the following representation of $P(E)$:

$$\begin{aligned}P(E)&=-\frac{1}{2\pi i}\lim_{\epsilon\to0+}\int_0^E dE'[G_0(E'+i\epsilon)-G_0(E'-i\epsilon)]\\ &=\frac{1}{2\pi}\lim_{\epsilon\to0+}\int_0^E dE'\int_{-\infty}^\infty dt\, e^{-iH_0t+iEt-\epsilon|t|},\end{aligned} \tag{2.5}$$

which is easily established by using standard results[16] of the theory. The limiting processes implied by Eqs. (2.4) and (2.5) are all strong operator limits.

D. There is a finite number of self-adjoint operators A_i in $\mathcal{H}$ $(i=1,\ldots,n)$ with projectors[17] P_{α_i} such that H_0, A_1, ..., A_n commute and form a complete set of operators.

The product of these operators will be denoted P_α, where α stands for the ordered set $\alpha=(\alpha_1,\ldots,\alpha_n)$. In physicists' notation, we have then

$$\begin{aligned}P(E)&=\int_0^E dE'\sum_\alpha|E',\alpha)(E',\alpha|,\\ P_\alpha&=\int_0^\infty dE\,|E,\alpha)(E,\alpha|.\end{aligned} \tag{2.6}$$

Consider now the restriction to the subspace $\mathcal{H}_\alpha=P_\alpha\mathcal{H}$ of the operators H_0 and S. From (2.1) and (2.2), H_0 and S commute; hence $P_\alpha H_0=H_0P_\alpha$ and $P_\alpha SP_\alpha$ also commute. Clearly $P_\alpha H_0$ is a self-adjoint operator in $\mathcal{H}_\alpha$.

[13] We use these terms as defined in the mathematical literature; see Ref. 8, Vol. 1, pp. 63, 72, and 85.

[14] Ref. 8, Secs. 61, 66, and 68.

[15] M. H. Stone, *Linear Transformations in Hilbert Space* (American Mathematical Society, New York, 1932), Chap. 6.

[16] Reference 9, Theorems X.1.1 and XII.2.3.

[17] If some of the operators A_i have continuous spectra, one should use the corresponding differential projectors $dP(\alpha_i)/d\alpha_i$. However, we shall not observe this distinction as it would require a cumbersome notation and would not in any case alter the argument.

Assumption D implies that $P_\alpha H_0$ has a simple (i.e., nondegenerate) spectrum. Since S is a unitary operator, it is bounded. Hence $P_\alpha S P_\alpha$ must also be bounded. We now invoke a theorem[18] of Hilbert-space theory which states that if a bounded operator Q commutes with a self-adjoint operator A which has a simple spectrum, then Q is a function of A. This theorem gives us the integral representation

$$P_\alpha S P_\alpha = \int_0^\infty S_{\alpha\alpha}(E) P_\alpha dP(E) \tag{2.7}$$

and the following information about $S_{\alpha\alpha}(E)$: (a) $S_{\alpha\alpha}(E)$ is measurable on every compact subset of the interval $[0,\infty)$,[19] (b) $S_{\alpha\alpha}(E)$ belongs to $L_\sigma^2[0,\infty)$ with respect to the measure $\sigma=(\varphi,P(E)\varphi)$ for any $\varphi\in\mathcal{H}_\alpha$,[19] (c) $S_{\alpha\alpha}(E)$ is a bounded function on $E\in[0,\infty)$.[18] As a distribution, $S_{\alpha\alpha}(E)$ therefore belongs to the class[20] L^∞ whose test functions are the elements of $L^1[0,\infty)$.

III. CAUSALITY CONDITION

Consider the scattering of states ψ_{in} which are in a particular subspace $\mathcal{H}_\alpha$. From (2.1) and (2.7) we have

$$\begin{aligned} \psi_{\text{in}}(t) &= P_\alpha \psi_{\text{in}}(t)\,, \\ P_\alpha \psi_{\text{out}}(t) &= \int_0^\infty S_{\alpha\alpha}(E) dP(E) \psi_{\text{in}}(t)\,. \end{aligned} \tag{3.1}$$

For the detection process, we select a state χ (in the same subspace $\mathcal{H}_\alpha$) which is independent of E and t. Thus

$$\begin{aligned} \chi &= P_\alpha \chi\,, \\ (\chi,\psi_{\text{out}}(t)) &= \int_0^\infty S_{\alpha\alpha}(E) d(\chi, P(E)\psi_{\text{in}}(t))\,. \end{aligned} \tag{3.2}$$

We define

$$\begin{aligned} I(t) &= (\chi,\psi_{\text{in}}(t))\,, \\ O(t) &= (\chi,\psi_{\text{out}}(t))\,, \end{aligned} \tag{3.3}$$

$$I(E) = \frac{1}{2\pi} \lim_{\epsilon\to 0+} \int_{-\infty}^\infty dt\, e^{iEt-\epsilon|t|} I(t)\,. \tag{3.4}$$

Equation (2.5) now gives[21]

$$\begin{aligned} (\chi,P(E)\psi_{\text{in}}(t)) &= \frac{1}{2\pi} \lim_{\epsilon\to 0+} \int_0^E dE' \int_{-\infty}^\infty dt' \\ &\quad \times e^{iE't'-\epsilon|t'|} (\chi,\psi_{\text{in}}(t+t')) \\ &= \int_0^E dE' e^{-iE't} I(E') \end{aligned} \tag{3.5}$$

and enables us to write

$$O(t) = \int_0^\infty dE\, e^{-iEt} S_{\alpha\alpha}(E) I(E)\,. \tag{3.6}$$

Screaton's theorem is directly applicable to the pair of equations (3.4) and (3.6). The condition of causality will be formulated as follows.

E. There exists a number M independent of χ, $\psi_{\text{in}}(0)$, and t such that

$$|O(t)| \le M \sup_{-\infty<t'\le t} |I(t')|\,. \tag{3.7}$$

In Ref. 4 Screaton shows that if $S_{\alpha\alpha}(E)$ is known to be a tempered distribution, then condition (3.7) leads to the following conclusions.

F. There is an analytic function $S_{\alpha\alpha}(z)$ regular in $\text{Im}z>0$ which approaches $S_{\alpha\alpha}(E)$ on $E\in[0,\infty)$ as $z\to E+i0$. Furthermore, $S_{\alpha\alpha}(E)$ is the Fourier transform of a causal tempered distribution

$$S_{\alpha\alpha}(E) = \frac{1}{2\pi} \int_{-\infty}^\infty dt\, e^{iEt} S_{\alpha\alpha}(t)\,, \tag{3.8}$$

where the support of $S_{\alpha\alpha}(t)$ is $[0,\infty)$. Also, $S_{\alpha\alpha}(E)$ satisfies a twice-subtracted dispersion relation.

Since our $S_{\alpha\alpha}(E)$ belongs to L^∞ it is certainly a tempered distribution.[20] From Screaton's proof and the uniqueness of distributions, we can conclude[22] that $S_{\alpha\alpha}(z)$ approaches its boundary value pointwise almost everywhere as $z\to E+i0$.

With our choice (3.3), the functions $|O(t)|^2$, $|I(t)|^2$ are certainly observable quantities according to the rules of quantum mechanics. In his proof Screaton uses a specific form of test function $I_s(E)$ of very fast decrease. It is important to check that any choice of function $I(E)$ of this class is physically realizable. First we note that the choice

$$(\psi_{\text{in}}(0), P(E)\psi_{\text{in}}(0)) = \int_0^E dE' |I_s(E')| \tag{3.9}$$

is physically realizable since the quantity on the left-hand side represents the (integrated) energy probability density in the state ψ_{in}, which can be chosen at will. Secondly we are allowed to choose

$$\chi = \int_0^\infty \exp[-i\arg I_s(E)] dP(E) \psi_{\text{in}}(0)\,, \tag{3.10}$$

since this operation is equivalent to a unitary operator in $\mathcal{H}$. With this choice of χ and ψ_{in}, it is easy to verify from (3.5) that our function $I(E)$ will reproduce the test function $I_s(E)$.

[18] Reference 8, Sec. 71, Theorem 1.

[19] Reference 15, Theorems 7.16 and 8.1.

[20] E. J. Beltrami and M. R. Wohlers, *Distributions and the Boundary Values of Analytic Functions* (Academic, New York, 1966), Sec. 1.1.

[21] The interchange of limits in this step can be justified by elementary calculus. Note that from property C (iii), $(\chi,P(E)\psi_{\text{in}}(t))$ must be a continuous function of E.

[22] Reference 20, Theorem 3.19.

If we could show that it is always possible to find a set of projectors P_α which simultaneously reduce S and H_0, then our result could be extended to every S-matrix element. A general theorem of this nature does not seem to be available in the mathematical literature. In some scattering systems, because of the symmetry of the interaction, this simultaneous diagonalization can be carried out explicitly. In this case we must have[23] $|S_{\alpha\alpha}(E)|=1$, so that

$$S_{\alpha\alpha}(E)=e^{2i\delta_\alpha(E)},$$

with a real $\delta_\alpha(E)$, and the conclusion F applies to every S-matrix element.

The above discussion does not apply to rearrangement collisions because they violate[24] conditions B and C. However, with minor modifications the argument can be extended to inelastic scattering in systems having conventional channels,[24] i.e., corresponding to a finite number of excitational levels for each particle. Operators P_c projecting onto the channel subspaces would have to be added to the set P_{α_i}. Conclusion F could then be established for matrix elements $S_{c\alpha,c\alpha}(E)$ corresponding to the same initial and final channels.

IV. PARTIAL-WAVE AMPLITUDES AND FORWARD SCATTERING AMPLITUDE

We consider first the case of elastic scattering of two particles by a spherically symmetric interaction. The S-matrix is diagonalized as usual by decomposition into partial waves. Equation (3.1) reads (with $E=k^2$)

$$\psi_{\text{out}}(l,r,t)=\int_0^\infty dk\, kr^{1/2}J_{l+1/2}(kr)S_l(k) \times\int_0^\infty dr'\, r'^{1/2}J_{l+1/2}(kr')\psi_{\text{in}}(l,r',t), \quad (4.1)$$

where J is the standard Bessel function. It is not evident at first glance that this S_l agrees with the partial-wave S-matrix element $e^{2i\delta_l}$ which is customarily defined through the asymptotic behavior of the function $\Psi^{(+)}(l,k,r)$. According to the Hankel transform theorem, we can state that if

$$\psi_{\text{in}}(l,r,t)=\int_0^\infty dk\, a(k)r^{1/2}J_{l+1/2}(kr)e^{-ik^2t}, \quad (4.2)$$

then[25]

$$\psi_{\text{out}}(l,r,t)=\int_0^\infty dk\, S_l(k)a(k)r^{1/2}J_{l+1/2}(kr)e^{-ik^2t}. \quad (4.3)$$

Suppose that $a(k)$ is a real function with a single peak at $k=k_0$. From the asymptotic behavior of J,

$$r^{1/2}J_{l+1/2}(kr) \underset{r\to\infty}{\sim} \left(\frac{2}{\pi k}\right)^{1/2}\frac{1}{2i}\left[e^{ikr-il\pi/2}-e^{-ikr+il\pi/2}\right], \quad (4.4)$$

and the usual stationary phase argument it is clear that, at large negative t, ψ_{in} will have an incoming peak at $r=-2k_0t$ of amplitude $c(k_0)e^{il\pi/2}$ and an unobservable image peak at $r=2k_0t$. At large positive t, ψ_{out} will have an image peak in $r<0$ and an observable outgoing peak at $r=2k_0t-\delta'(k)$ of amplitude $c(k_0)S_l(k_0)e^{-il\pi/2}$. Thus we can make the identification $S_l(k)=e^{2i\delta_l(k)}$.

In order to realize a given test function $I_s(E)$ in the application of Screaton's theorem, one can simply choose

$$\chi(l,r)=\int_0^\infty dk\, a^*(k)r^{1/2}J_{l+1/2}(kr) \quad (4.5)$$

and set $\frac{1}{2}[a(k)/k]^2$ equal to $I_s(E)$. The argument of Sec. III is thus directly applicable to this case and leads to conclusion F for the function $S_l(k)$.

For the case of elastic scattering of two particles by a spin-independent interaction, the method yields a proof of analyticity for the scattering amplitude in the forward direction. We choose

$$\psi_{\text{in}}(\mathbf{r},t)=\frac{1}{(2\pi)^{1/2}}\int_0^\infty dk\, a(k)e^{ikz-ik^2t}, \quad (4.6)$$

$$\chi(\mathbf{r})=\frac{1}{(2\pi)^{1/2}}\int_0^\infty dk\, b(k)e^{ikz}. \quad (4.7)$$

Then, in accordance with the usual three-dimensional formalism, we have[26]

$$\psi_{\text{out}}(\mathbf{r},t)=\psi_{\text{in}}(\mathbf{r},t)+\frac{i}{4\pi^2}\int d^3k\,\frac{1}{k}A(\mathbf{k},k\hat{\mathbf{z}})e^{i\mathbf{k}\cdot\mathbf{r}} \times\int_{-\infty}^\infty dz'\, e^{-ikz'}\psi_{\text{in}}(\mathbf{r}',t). \quad (4.8)$$

We make the following choice for the functions $I(t)$ and $O(t)$:

$$I(t)=\int_{-\infty}^\infty dz\, \chi^*(\mathbf{r})\psi_{\text{in}}(\mathbf{r},t)=\int_0^\infty dk\, b^*(k)a(k)e^{-ik^2t}, \quad (4.9)$$

$$O(t)=(\chi,\psi_{\text{out}}(t)-\psi_{\text{in}}(t)) =2\pi i\int_0^\infty dk\,\frac{1}{k}A(k\hat{\mathbf{z}},k\hat{\mathbf{z}})b^*(k)a(k)e^{-ik^2t}. \quad (4.10)$$

From these equations one can construct a direct proof that $A(k\hat{\mathbf{z}},k\hat{\mathbf{z}})$ must satisfy condition F.

[23] Reference 8, Sec. 74.
[24] M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), Chap. 5.
[25] Functions which are equal amost everywhere are considered to be equivalent.
[26] The amplitude A has the standard normalization $d\sigma/d\Omega=|A(\mathbf{k},\mathbf{k}')|^2$ (Ref. 24, Chap. 6).

V. DISCUSSION

Consider a system governed by a Hamiltonian $H=H_0+H'$. Let $\Psi^{(+)}(t)$ be the state which develops in time in accordance with H and which approaches $\psi_{\rm in}(t)$ as $t\to-\infty$. Most previous formulations of causality in nonrelativistic quantum mechanics[1,6,7] involved a comparison between $\Psi^{(+)}(t)$ and $\psi_{\rm in}(t)$. It was always necessary to assume that the interaction vanishes beyond a finite radius. These formulations of causality were not decisive in that similar analyticity properties could also be deduced from an assumption of conservation of probability,[1] or of the completeness of the states of H outside the interaction region.[27]

A similar situation may be observed if we replace the function $O(t)$ in (3.7) by $U(t)=(\chi,\Psi^{(+)}(t))$. $\Psi^{(+)}(t)$ and $\psi_{\rm in}(t)$ are related as follows[28]:

$$\Psi^{(+)}(t)=\psi_{\rm in}(t)+\frac{1}{2\pi}\int_0^\infty dE\, e^{-iEt}G^{(+)}(E)H' \times\int_{-\infty}^{\infty} dt' e^{iEt'}\psi_{\rm in}(t')\,, \quad (5.1)$$

where

$$G^+(E)=\lim_{\epsilon\to0+}[E-H+i\epsilon]^{-1}.$$

From a causality condition of this type one could, at best, deduce the analyticity in $\operatorname{Im}z>0$ of matrix elements $(\varphi_1,G(z)H'\varphi_2)$ for any $\varphi_1,\ \varphi_2\in\mathcal{H}$. From the point of view of Hilbert-space theory, the analyticity of the resolvent $G(z)=(z-H)^{-1}$ is a very weak condition. Either one of the following conditions is sufficient to ensure that $G(z)$ is a bounded analytic operator in $\operatorname{Im}z\neq0$: (i) H is a self-adjoint operator in $\mathcal{H}$, or (ii) H is Hermitian and there exists a resolution of the identity corresponding to H.[29] However, unless the interaction has zero range, there is no possibility of deducing the analyticity of scattering amplitudes from this information. Thus the imposition of a causality condition between $\Psi^{(+)}(t)$ and $\psi_{\rm in}(t)$ is not likely to lead to analyticity properties which are not already guaranteed by other very basic assumptions of the theory.

The same qualitative conclusion may be obtained from the time-dependent perturbation expansion which, subject to convergence, yields the following integral representations for $\Psi^{(+)}(t)$ and $\psi_{\rm out}(t)$[28]:

$$\Psi^{(+)}(t)=\psi_{\rm in}(t)+\int_{-\infty}^{\infty} dt' G^+(t-t')H'\psi_{\rm in}(t')\,, \quad (5.2)$$

$$\psi_{\rm out}(t)=\psi_{\rm in}(t)-i\int_{-\infty}^{\infty} dt' e^{-iH_0(t-t')}H'\psi_{\rm in}(t') -i\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt'' e^{-iH_0(t-t')} \times H'G(t'-t'')H'\psi_{\rm in}(t'')\,, \quad (5.3)$$

where

$$G^+(t)=\begin{cases}-ie^{-iHt}, & t>0\\ 0, & t<0.\end{cases} \quad (5.4)$$

The kernel is *a priori* causal in Eq. (5.2) but not in Eq. (5.3).

It may be objected that, unlike $|U(t)|$, the function $|O(t)|$ cannot be measured in the laboratory. We would like to show that for a scattering system which has Møller wave operators $\Omega^{(\pm)}$, the quantity $|O(t)|$ can in fact be measured during the course of a scattering experiment. Since[28] $S=\Omega^{(-)\dagger}\Omega^{(+)}$, we have

$$\Psi^{(+)}(t)=\Omega^{(+)}\psi_{\rm in}(t)\,,\qquad \psi_{\rm out}(t)=\Omega^{(-)\dagger}\Psi^{(+)}(t)\,. \quad (5.5)$$

We introduce a modified detector state $\Xi=\Omega^{(-)}\chi$ and use the fact that

$$(\Xi,\Psi^{(+)}(t))=(\Omega^{(-)}\chi,\Omega^{(+)}\psi_{\rm in}(t)) =(\chi,\psi_{\rm out}(t))=O(t)\,. \quad (5.6)$$

Since the probability $|(\Xi,\Psi^{(+)}(t))|^2$ can be measured during the course of the experiment, it is thus possible in principle to make a direct comparison between $|O(t)|$ and $|I(t)|$.[30]

[27] I. Saavedra, Nucl. Phys. **29**, 137 (1962).
[28] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), Chap. 6.
[29] Reference 8, Sec. 65, Appendix I.3.

[30] The time dependence of $|O|$ cannot of course be obtained by consecutive measurements on the same system. It must be interpreted as the result of measurements at different times on a set of identical systems which evolve from the same state $\psi_{\rm in}$.