Behavior of Commutator Matrix Elements at Small Distances. II. Equal-Time Limits of Charge Moments and Time Derivatives

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From general principles of quantum field theory (especially locality and Poincaré invariance, but without use of the spectrum condition), it is shown that the equal-time limits of current-density commutators exist if the limits for the commutators between one current density and one generalized charge exist. If the equal-time limits of the current-density commutators containing at least one zeroth component exist, then the limits of the space-space components exist also. If the equal-time limits between one current density and the nth time derivative of the generalized charges exist, then also the limits of all time derivatives up to order n of the density commutators exist. Explicit expressions for the first time derivative of current-density commutators in terms of the Gell-Mann Σ and meson commutators are derived.

I. INTRODUCTION

THE concept of equal-time commutation relations between electromagnetic and weak hadron currents $j_\alpha{}^\mu(x)_a$ as well as their charges[1,2] combines assumptions on the behavior of current-commutator matrix elements in small regions of space and time with algebraic structures related to an underlying symmetry group. Within the framework of general quantum field theory the rigorous formulation of the various equal-time commutation relations is[3–5]

(a) charge-charge relations (CCR):

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,1)_a,j_\beta{}^0(0,1)_b]|\Phi\rangle^T =ic^{\alpha\beta\gamma}\langle\Psi|j_\gamma{}^0(0,1)_c|\Phi\rangle, \quad (1)$$

(b) charge-density relations (CDR):

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,1)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T =ic^{\alpha\beta\gamma}\langle\Psi|j_\gamma{}^\nu(0)_c|\Phi\rangle, \quad (2)$$

(c) density-density relations (DDR):

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,\mathbf{h})_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T =\{ic^{\alpha\beta\gamma}[\delta^{\mu0}\langle\Psi|j_\gamma{}^\nu(0)_c|\Phi\rangle+\delta^{\mu k}\delta^{\nu 0}\langle\Psi|j_\gamma{}^k(0)_c|\Phi\rangle] +\delta^{\mu k}\delta^{\nu r}\langle\Psi|A_{\alpha,\beta}{}^{kr}(0)_{ab}|\Phi\rangle\}h(\mathbf{0}) \quad (3)$$

for all infinitely often differentiable (C^∞) functions $h(\mathbf{x})$ and Ψ,Φ from a certain domain of state vectors in a Hilbert space H.

φ_ϵ is an arbitrary element from the following class of functions of type[6] δ:

$$\varphi_\epsilon(x^0)=:(1/\epsilon)\varphi(x^0/\epsilon), \quad (4)$$

where $\varphi(x^0)$ is again a C^∞ function with its support concentrated in the interval $[-a,a]$. The equality sign with the colon means "is by definition." Furthermore $\varphi(x^0)$ is normalized:

$$\int dx^0\varphi(x^0)=1. \quad (5)$$

The smeared currents and generalized charges are defined by[7]

$$j^\mu(\varphi_\epsilon,h)=:\int d^4x\, j^\mu(x)\varphi_\epsilon(x^0)h(\mathbf{x}),$$
$$j^\mu(\varphi_\epsilon,1)=:\int d^4x\, j^\mu(x)\varphi_\epsilon(x^0). \quad (6)$$

T denotes subtraction of the vacuum expectation value before the limit is performed:

$$\langle\Psi|[j^\mu(\varphi_\epsilon,h),j^\nu(0)]|\Phi\rangle^T =\langle\Psi|[j^\mu(\varphi_\epsilon,h),j^\nu(0)]|\Phi\rangle -\langle 0|[j^\mu(\varphi_\epsilon,h);j^\nu(0)]|0\rangle. \quad (7)$$

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[1] B. Renner, *Current Algebras and their Applications* (Pergamon, New York, 1968).

[2] S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[3] B. Schroer and P. Stichel, Commun. Math. Phys. **3**, 258 (1966).

[4] A. H. Völkel, Phys. Rev. D **1**, 3377 (1970).

[5] The distribution and use of indices at the currents throughout this paper are as follows: (i) Upper Greek indices $\mu, \nu, \lambda, \ldots = 0, 1, 2, 3$ to the left of the argument(s) indicate tensor properties with respect to the Lorentz group, the corresponding Latin indices $k, l, r, \ldots = 1, 2, 3$ their restriction to the space parts. (ii) Lower Latin indices $a, b, c = V, A$ to the right of the argument(s) differentiate between vectors (V) and axial vectors (A). In the commutation relations we have the connection $a\neq b\rightarrow c=A$ and $a=b\rightarrow c=V$. (iii) Lower Greek indices α, β, γ to the left of the argument(s) refer to internal (broken) symmetry groups. The usual summation convention for double indices is used.

[6] F. Treves, *Topological Vector Spaces, Distributions and Kernels* (Academic, New York, 1967), Sec. 28, especially exercise 28.2.

[7] In the charge-charge relations (1), the two space integrals have also to be interpreted as limits of currents integrated with suitable test functions (Ref. 3). We are not concerned with these relations in the present article. Since the commutator $\langle\Psi|[j^\nu(x),j^\mu(0)]|\Phi\rangle^T$ vanishes for $x^2=x^{02}-\mathbf{x}^2<0$, we can admit arbitrary C^∞ functions $h(\mathbf{x})$ in the commutators.

This removes all c-number gradient terms from the right-hand side of the commutation relations.

Finally $A_{\alpha\beta}{}^{kr}$ in DDR depends on the specific model under consideration. The most important cases are the algebra of fields[8] with $A_{\alpha\beta}{}^{rk}=0$ and the quark model where $A_{\alpha\beta}{}^{rk}$ is again linear in the currents.[2,9]

Besides the algebraic structures, represented in the occurrence of the structure constants $c^{\alpha\beta\gamma}$ of a (broken) symmetry group, the equal-time commutation relations contain strong assumptions on the good behavior of the commutator matrix elements at small distances. The existence of the limits as well as the occurrence of only the δ function in (3) is by no means obvious. Moreover, even if the charge-density limits (CDR) exist, this may not be true any longer for the density-density limits (DDR). The existence of the limits for the special test function $h(\mathbf{x})\equiv 1$ does not imply their existence for all C^∞ functions $h(\mathbf{x})$.

In a recent publication,[4] hereafter referred to as V.I, we have shown the following by means of microcausality and temperedness of current matrix elements:

(i) There always exist finite positive numbers $N(\mu,\nu)$ (the order of the commutator matrix elements[10,11]) and $m\leq N$ (with $m=0$ if the spectrum condition holds) such that

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T=0 \tag{8}$$

for all C^∞ functions $h(\mathbf{x})$ with the property

$$\lim_{\epsilon\to 0}\epsilon^{-2N-m+1}h(\epsilon\mathbf{x})=0. \tag{9}$$

In other words, the equal-time limits are zero for all test functions $h(\mathbf{x})$ which vanish more strongly than $|\mathbf{x}|^{2N+m}$. This is for instance true for all generalized charge moments of order $r>2N+m$ which are obtained by the special choice of test functions

$$h(\mathbf{x})\equiv f^{i,j,r-i-j}(\mathbf{x})=(x^1)^i(x^2)^j(x^3)^{r-i-j}. \tag{10}$$

(ii) The equal-time limits of the first $2N+m-1$ charge moments

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,\mathbf{f}^{i,j,r-i-j})_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$

exist *if and only if* the limits of the corresponding density commutators exist and are given by

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon;h)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$
$$=\sum_{r=0}^{2N+m-1}\sum_{i=0}^{r}\sum_{j=0}^{r-i}\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{i,j,r-i-j})_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$
$$\times\frac{1}{i!j!(r-i-j)!}\frac{\partial^r}{\partial(z^1)^i\partial(z^2)^j\partial(z^3)^{r-i-j}}h(\mathbf{z})\bigg|_{\mathbf{z}=0}. \tag{11}$$

This means that the equal-time limit is completely known [for all C^∞ functions $h(\mathbf{x})$] if it is known for the generalized charges and their first $2N+m-1$ moments.

By means of these results the Gell-Mann commutation relations for the current densities (3) are equivalent to the following finite set of relations for the generalized charges and their first $2N+m-1$ moments:

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon;1)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$
$$=\{ic^{\alpha\beta\gamma}[\delta^{\mu 0}\langle\Psi|j_\gamma{}^\nu(0)_c|\Phi\rangle+\delta^{\mu k}\delta^{\nu 0}\langle\Psi|j_\gamma{}^k(0)_c|\Phi\rangle]$$
$$+\delta^{\mu k}\delta^{\nu s}\langle\Psi|A_{\alpha\beta}{}^{ks}(0)_{ab}|\Phi\rangle\}, \tag{12a}$$

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{i,j,r-i-j})_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T=0 \tag{12b}$$

for all r with $1\leq r\leq 2N+m-1$.

The next problem is whether the existence of the equal-time limits for the charge moments follows already from their existence for the generalized charges. Since in the equal-time limit the support of the commutators in the space variables due to microcausality shrinks to the point $\mathbf{x}=0$, one would expect that the integration with a power in $\mathbf{x}$ neutralizes terms which can be troublesome in the case of charges. In other words, it is not quite unrealistic to believe that the existence of the equal-time limits of all generalized charge moments already follows from that of the generalized charges.

On the other hand, if the commutator matrix elements in a neighborhood of the origin behave like

$$\langle\Psi|[j_\alpha{}^\mu(x),j_\beta{}^\nu(0)]|\Phi\rangle^T$$
$$\approx B_{\alpha\beta}{}^{\mu\nu}\delta(\mathbf{x})+C_{\alpha\beta}{}^{\mu\nu;r}\frac{\partial}{\partial x^r}\delta(\mathbf{x})\delta(x^0),$$

then the limit $x^0\to 0$ exists for the generalized charges but not for their moments:

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,1),j_\beta{}^\nu(0)]|\Phi\rangle^T=B_{\alpha\beta}{}^{\mu\nu},$$

$$\lim_{\epsilon\to 0}\int d^4x\langle\Psi|[j_\alpha{}^\mu(x),j_\beta{}^\nu(0)]|\Phi\rangle^T\varphi_\epsilon(x^0)x^s$$
$$=C_{\alpha\beta}{}^{\mu\nu;s}\varphi(0)\lim_{\epsilon\to 0}(1/\epsilon)=\infty.$$

In Sec. II we prove that the last case cannot occur if Poincaré invariance holds. The equal-time commutators between a density and all moments of generalized charges, and thereby also those between two densities, exist if and only if the equal-time commutators between a density and the generalized charges themselves exist. In other words, the step from the equal-time commutators containing one generalized charge to that of two densities does not involve further existence problems. The only real sharpening of the assumption lies in the specification of the structure of the gradient—or so-called Schwinger—terms; in the case of the Gell-Mann relations in the assumption of their vanishing.

[8] N. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

[9] C. A. Orzalesi, University of Maryland Technical Report No. 833, 1968 (unpublished).

[10] Every tempered distribution is a derivative of finite degree of a continuous function. The degree of this derivative is called the order of the distribution (Ref. 11).

[11] I. M. Gelfand and G. E. Schilow, *Verallgemeinerte Funktionen (Distributionen)* (Deutscher Verlag der Wissenschaften, Berlin, 1962), Vol. II.

Can one go one step further? Is it possible, starting from the equal-time limit for the (0-ν) components (containing a proper charge)

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon;1),j_\beta{}^\nu(0)]|\Phi\rangle^T,$$

to derive the existence of the limit for the (k-ν) components (containing the space charges):

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^k(\varphi_\epsilon;1),j_\beta{}^\nu(0)]|\Phi\rangle^T.$$

In Sec. II we construct a class of counterexamples, which demonstrates that this is not possible. Even for the existence of the equal-time density commutators for the (0-0) component, we need the existence of the limit for all four generalized charges—at least if the order $N(0,0)$ of the commutator is larger than or equal to 2.

In connection with the Bjorken (high-energy) limit of Green's functions,[12] the time derivatives of equal-time commutators and their possible ambiguities[13] have gained growing interest. In Sec. III we show that the equal-time limits of all time derivatives up to order n for the density commutators exist if the limits of the commutators between one density and the nth time derivative of the generalized charges exist. Especially, we derive an explicit expression for the first time derivative in terms of the right-hand sides of the Gell-Mann Σ and meson commutators or the time derivative of the charge-density commutator. This expression shows that the first time derivative has nonzero q-number gradient terms given by the Gell-Mann relations for the currents themselves.

Moreover, the ambiguities discussed by Brandt and Sucher[13] can only show up in the time derivative of the commutator containing a generalized charge, but not in their moments, if the usual commutation relations (without ambiguities) hold for the currents themselves. Of course another possibility is that these ambiguities occur already in one of the usual generalized charge-density commutators. In the case of a conserved current, this is only possible for the (k-ν) component containing a generalized space charge.

II. EQUAL-TIME COMMUTATORS OF CHARGE MOMENTS

We begin this section with the specification of the assumptions we need. We consider the currents $j_\alpha{}^\mu$ to be members of a polynomial algebra of fields which satisfy the usual postulates of Wightman fields, with the possible exception of the spectrum condition.[14–17] In detail we require

(I) the fields $j_\alpha{}^\mu(f)_a=\int d^4x\, j_\alpha{}^\mu(x)_a f(x)$, smeared with test functions $f(x)$ from S_4,[11] are operators with a dense domain in a Hilbert space H;

(II) local commutativity (microcausality):

$$[j_\alpha{}^\mu(x);j_{\beta}{}^\nu(y)]|\Psi\rangle=0 \quad \text{for } (x-y)^2<0;\qquad (13)$$

(III) Poincaré invariance: (i) existence of a unitary representation $U(\Lambda,a)$ of the Poincaré group in H, (ii) the currents are covariant under $U(\Lambda,a)$,

$$U(\Lambda,a)j_\alpha{}^\mu(f)U(\Lambda,a)^{-1}=(\Lambda^{-1})^\mu{}_\nu j_\alpha{}^\nu(f_{\Lambda,a}),\qquad (14)$$

with $f_{\Lambda,a}(x)=:f(\Lambda^{-1}(x-a))$.

Beyond these general assumptions we need some further assumptions on the equal-time commutators containing a generalized charge. We assume[18]

$$\text{(A.I)}\quad \begin{aligned}&\text{(a)}\ \lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,1)_a,j_\beta{}^\mu(0)_b]|\Phi\rangle^T\\ &\qquad\qquad =ic^{\alpha\beta\gamma}\langle\Psi|j_\gamma{}^\mu(0)|\Phi\rangle,\\ &\text{(b)}\ \lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^k(\varphi_\epsilon,1)_a,j_\beta{}^\mu(0)_b]|\Phi\rangle^T \text{ exists.}\end{aligned}$$

Both limits are assumed to exist for all Ψ,Φ from a domain D in H which is stable under Poincaré transformations.

This additional assumption is equivalent to the following one:

$$\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,1)_a,j_\beta{}^\mu(0)_b]|\Phi\rangle^T =ic^{\alpha\beta\gamma}\langle\Psi|j_\gamma{}^\mu(0)_c|\Phi\rangle+R_\epsilon{}^{0\mu}(\Psi,\Phi),\quad (15)$$

$$\langle\Psi|[j_\alpha{}^k(\varphi_\epsilon,1)_a,j_\beta{}^\mu(0)_b]|\Phi\rangle^T =L^{k\mu}(\Psi,\Phi)+R_\epsilon{}^{k\mu}(\Psi,\Phi),\quad (16)$$

with

$$\lim_{\epsilon\to 0}R_\epsilon{}^{\mu\nu}(\Psi,\Phi)=0\quad\text{for all }\Psi,\Phi\in D.\qquad (17)$$

Implicitly this last formulation of equal-time limits is always understood in the following considerations. Rewriting all our following relations for the limits into this second form, the critical reader can convince himself that nowhere have we interchanged Poincaré transformations and equal-time limits.

The key to all our results besides Poincaré invariance is the following lemma by Łojasiewicz[19] on the existence of the primitive function for limits of distributions.

Lemma I. If $T(x)$ is a distribution for which

$$\lim_{\epsilon\to 0}\partial_0 T(\varphi_\epsilon)=-\lim_{\epsilon\to 0}\int dx^0 T(x^0)\frac{d}{dx^0}\varphi_\epsilon(x^0)$$

[12] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[13] R. A. Brandt and J. Sucher, Phys. Rev. **177**, 2218 (1969).

[14] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

[15] L. Gårding and A. S. Wightman, Arkiv Fysik **28**, 129 (1964).

[16] R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, 1965).

[17] K. Hepp, in *Axiomatic Field Theory and Particle Symmetries* (Gordon and Breach, New York, 1965), Vol. 1.

[18] We assume that the limits exist independent of the δ sequences from the class specified in the Introduction. Then the statement "existence" in all our results also means independent of the δ sequence. The results of Sec. II and the first part of Sec. III are independent of the right-hand side of (A.I). Only the existence of the limits independent of the δ sequence is necessary. Possibly most of our results remain valid if we admit from the beginning a dependence of the limits on the δ sequences.

[19] S. Łojasiewicz, *Studia Mathematica* T. XVII, 1 (1958), Sections 4.3–4.5.

exists, then also $\lim_{\epsilon\to 0} T(\varphi_\epsilon)$ exists for every φ_ϵ specified in Eqs. (4) and (5).

We give here a "handwaving" proof of this lemma. The rigorous proof can be found in the paper of Łojasiewicz.[19] From the existence of

$$\lim_{\epsilon\to 0}\int dx^0 T(x^0)\frac{d}{dx^0}\varphi_\epsilon(x^0)$$

it follows that

$$\lim_{\epsilon\to 0}\int dx^0 x^0\left(\frac{d}{dx^0}T(x^0)\right)\varphi_\epsilon(x^0)=0$$

and

$$\lim_{\epsilon\to 0}\int dx^0 x^0 T(x^0)\frac{d}{dx^0}\varphi_\epsilon(x^0)\ \text{exists,} \tag{18}$$

since in both cases the multiplication by x^0 means smoothing of the distribution in the limiting point. Equation (18) means

$$\lim_{\epsilon\to 0}\left[\int dx^0 T(x^0)\varphi_\epsilon(x^0)+\int dx^0 x^0 T(x^0)\frac{d}{dx^0}\varphi_\epsilon(x^0)\right]=0.$$

Since the limit of the second term exists, our lemma is proved.

Let us consider infinitesimal pure Lorentz transformations

$$\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu,$$

with

$$\omega^\mu{}_\nu=-\omega_\nu{}^\mu,\quad \omega^r{}_k=0\quad \text{(no rotations).}$$

From (14) we get, with

$$f_\Lambda(x)=f(x)-\omega^0{}_k M^k{}_0 f(x),$$
$$M^k{}_0=:x^k\frac{\partial}{\partial x^0}+x^0\frac{\partial}{\partial x^k}, \tag{19}$$

for the transformation law of the currents

$$U(\Lambda)j_\alpha{}^\mu(f)U(\Lambda)^{-1}=j_\alpha{}^\mu(f)-\omega^0{}_k[j_\alpha{}^\mu(M^k{}_0f)+g^{\mu 0}j_\alpha{}^k(f)-g^{\mu k}j_\alpha{}^0(f)] \tag{20}$$

and furthermore, for the commutator sandwiched between the states $\{\hat\Psi;\hat\Phi\}=U(\Lambda)^{-1}\{\Psi;\Phi\}$ with Ψ,Φ arbitrary from D,

$$\begin{aligned}\langle\hat\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h);j_\beta{}^\nu(0)]|\hat\Phi\rangle^T
&=\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h);j_\beta{}^\nu(0)]|\Phi\rangle^T\\
&\quad-\omega^0{}_k\{\langle\Psi|[j_\alpha{}^\mu(M^k{}_0(\varphi_\epsilon,h));j_\beta{}^\nu(0)]|\Phi\rangle^T\\
&\quad+g^{\mu 0}\langle\Psi|[j_\alpha{}^k(\varphi_\epsilon,h),j_\beta{}^\nu(0)]|\Phi\rangle^T\\
&\quad+g^{\nu 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h),j_\beta{}^k(0)]|\Phi\rangle^T\\
&\quad-g^{\mu k}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,h),j_\beta{}^\nu(0)]|\Phi\rangle^T\\
&\quad-g^{\nu k}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h),j_\beta{}^0(0)]|\Phi\rangle^T\}.\end{aligned} \tag{21}$$

From this relation it is very easy to prove our first result.

Theorem 1. Under the assumption (A.I) (the existence of commutators containing one generalized charge), the equal-time commutators between one density and all charge-moments,

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,\mathbf{f}^{k,l,m})_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T,$$

with

$$f^{k,l,m}(\mathbf{x})=(x^1)^k(x^2)^l(x^3)^m,$$

exist.

Proof. We prove this theorem by complete induction. From Eq. (21) taken for $h(\mathbf{x})=f^{0,0,0}(\mathbf{x})\equiv 1$ and assumption (A.I), it follows that

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(M^k{}_0(\varphi_\epsilon,h))_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T
=-\lim_{\epsilon\to 0}\int d^4x\langle\Psi|\left[\frac{\partial}{\partial x^0}j_\alpha{}^\mu(x)_a,j_\beta{}^\nu(0)_b\right]|\Phi\rangle^T\times\varphi_\epsilon(x^0)x^k \tag{22}$$

exists for all $\Psi,\Phi\in D$. Therefore by Łojasiewicz's Lemma I, the equal-time limits of all first moments exist. Now assuming that the equal-time limits for the charge moments of order n,

$$[h(\mathbf{x})=f^{k,l,n-k-l}(\mathbf{x}),\quad k,l=0,\ldots,n],$$

exist, it follows in the same way as in the first step from (21) and Lemma I that the equal-time limits for all charge moments of order $n+1$ exist. This proves our theorem.

Combining Theorem 1 with the results of V.I, we immediately obtain our main result.

Theorem 2. If the equal-time commutators between one current density and one generalized charge

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,1)a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$

exist for Ψ,Φ from a certain domain $D\in H$ stable under the Lorentz group, then the equal-time limits of two current densities also exist and are given by

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T
=\sum_{n=0}^{2N+m-1}\sum_{i=0}^{n}\sum_{j=0}^{n-i}\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,f^{i,j,n-i-j})_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T
\times\frac{1}{i!j!(n-i-j)!}\frac{\partial^n}{\partial(z^1)^i\partial(z^2)^j\partial(z^3)^{n-i-j}}h(\mathbf{z})\bigg|_{\mathbf{z}=0} \tag{23}$$

for all C^∞ functions $h(\mathbf{x})$ and some fixed $m\le N$. If in addition the spectrum condition holds, then m is equal to zero.

As already mentioned in the Introduction, $N=N(\mu,\nu)$ in Eq. (23) is the (always finite) order of the commutator matrix elements (before the limit is performed).

From the proof of Theorem 1 it is obvious that the existence of the equal-time limits of the moments with degree $n\geq 2$ for every commutator component (μ,ν) depends on the limits of the commutators for *all* four generalized charges. Only the first moment of the (0-0) component is independent of the space-space components of one charge and one density. Since we furthermore know from V.I that the equal-time limits of the commutators between one density and all charge moments of degree $n\geq 2N+m$ vanish, we arrive at the corollary to Theorem 2.

Corollary 2.1. If the spectrum condition holds $(m=0)$, if the order $N(0,0)$ is at most 1, and if the equal-time limits

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,1)_a,j_\beta{}^\mu(0)_b]|\Phi\rangle^T=ic^{\alpha\beta\gamma}\langle\Psi|j_\gamma{}^\mu(0)_c|\Phi\rangle$$

and

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^k(\varphi_\epsilon;1)_a,j_\beta{}^0(0)_b]|\Phi\rangle^T$$

exist, then the equal-time limit of the (0,0) component of the density commutator exists and is given by

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,h)_a,j_\beta{}^0(0)_b]|\Phi\rangle^T=ic^{\alpha\beta\gamma}\langle\Psi|j_\gamma{}^0(0)_c|\Phi\rangle+\sum_{k=1}^{3}\lim_{\epsilon\to 0}\langle\Psi|[Q_\alpha{}^k(\varphi_\epsilon)_a,j_\beta{}^0(0)_b]|\Phi\rangle^T\frac{\partial}{\partial_z{}^k}h(\mathbf{z})\bigg|_{\mathbf{z}=0}\tag{24}$$

for all C^∞ functions $h(\mathbf{z})$ and $Q_\alpha{}^k$ defined by

$$Q_\alpha{}^k(\varphi_\epsilon)_a=:\int d^4x\, j_\alpha{}^0(x)_a x^k\varphi_\epsilon(x^0).$$

At a first glance the results obtained up to now seem to be unsatisfactory, since we have to assume the existence of the limits for the commutators between all four generalized charges $j_\alpha{}^\mu(\varphi_\epsilon,1)_a$ and one density. The question is, can one go one step further and prove the existence of the space parts

$$\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^k(\varphi_\epsilon,1),j_\beta{}^\mu(0)]|\Phi\rangle^T,\quad k=1,2,3$$

from that of the zeroth component? This would mean for instance that the equal-time density commutators containing one conserved density always exist.

We are going to construct a class of counterexamples which show that this cannot be done. Our assumptions (A.I) are really minimal for the existence of the density limits.

For simplicity we restrict ourselves to the commutator of a conserved current $j_\alpha{}^\mu(x)$ and a Lorentz scalar $A(y)$ taken between states of sharp momenta p_1, p_2 and equal mass M:

$$\langle p_1M|[j^\mu(x),A(0)]|Mp_2\rangle^T=e^{i\Delta x}F^\mu(x),\tag{25}$$

$$\Delta=\tfrac{1}{2}(p_1-p_2),$$

$$\tilde{F}^\mu(q)=\frac{1}{(2\pi)^{5/2}}\int d^4x\, e^{iqx}F^\mu(x).\tag{26}$$

Furthermore, we assume the mass spectrum of possible intermediate states in the commutator to be symmetric. That means that if m is the lowest intermediate mass, then in the rest frame of $p=\frac{1}{2}(p_1+p_2)$, $\tilde{F}^\mu(q)$ vanishes for $p^0-(m^2+\mathbf{q}^2)^{1/2}\leq q^0\leq -p^0+(m^2+\mathbf{q}^2)^{1/2}$. As a consequence of this assumption, we may represent the commutator matrix elements by a generalized Jost-Lehmann representation containing locality, the spectrum condition, and current conservation.[20] The case of an unsymmetric spectrum as well as the matrix elements of two conserved or unconserved currents can be discussed in the same way by means of the corresponding Dyson representations.[21,22]

For a given commutator matrix element $\tilde{F}^\mu(q)$ of a conserved current there exists a unique set of tempered distributions

$$Z^r(\mathbf{u},s),\quad \psi^r(\mathbf{u},s),\quad E(\mathbf{u},s)$$

with the following properties.

(a) The support of all of them is contained in

$$|\mathbf{u}|\leq p^0,\quad s\geq s_0(|\mathbf{u}|)=\max\{0,m-[(p^0)^2-|\mathbf{u}|^2]^{1/2}\},$$

(b) $Z^r(\mathbf{u},s)$, $\psi^r(\mathbf{u},s)$ are vectors and $E(\mathbf{u},s)$ is a scalar with respect to rotations,

(c) $E(\mathbf{u},s)$ is the extension in the sense of Schwartz[23] of a distribution on the regular surface $s+(\mathbf{u}-\mathbf{\Delta})^2=0$ such that in the rest frame of p, $\tilde{F}^\mu(q)$ is given by

$$\tilde{F}^0(q)=\epsilon(q^0)q^0\int d\mathbf{u}ds\,\delta((q^0)^2-(\mathbf{q}-\mathbf{u})^2-s)\times\left\{\delta(s+(\mathbf{u}-\mathbf{\Delta})^2)E(\mathbf{u},s)+(\mathbf{q}-\mathbf{\Delta})\left[\boldsymbol{\psi}(\mathbf{u},s)-\frac{2}{s+(\mathbf{u}-\mathbf{\Delta})^2}\frac{\partial}{\partial s}\mathbf{Z}(\mathbf{u},s)\right]\right\},\tag{27}$$

$$\tilde{F}^r(q)=\epsilon(q^0)\int d\mathbf{u}ds\,\delta((q^0)^2-(\mathbf{q}-\mathbf{u})^2-s)\times\left[q^0\psi^r(\mathbf{u},s)-2\frac{\partial}{\partial s}Z^r(\mathbf{u},s)+\frac{2(q+\Delta-2u)^r}{s+(\mathbf{u}-\mathbf{\Delta})^2}(\mathbf{q}-\mathbf{\Delta})\frac{\partial}{\partial s}\mathbf{Z}(\mathbf{u},s)-(q+\Delta-2u)^r\delta(s+(\mathbf{u}-\mathbf{\Delta})^2)E(\mathbf{u},s)\right].\tag{28}$$

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[20] A. H. Völkel, Commun. Math. Phys. **5**, 57 (1967).

[21] Uta Völkel and A. H. Völkel, Commun. Math. Phys. **7**, 261 (1968).

[22] Uta Völkel and A. H. Völkel, Nuovo Cimento **63A**, 203 (1969).

[23] L. Schwartz, *Théorie des distributions I/II* (Hermann, Paris, 1957/59).

The terms containing $E(\mathbf{u},s)$ vanish if m is larger than M.[24]

From these representations we obtain for the equal-time limits of the generalized charges and one density

$$\lim_{\epsilon\to 0}\langle p_1 M|[j^0(\varphi_\epsilon,1),A(0)]|Mp_2\rangle^T=\lim_{\epsilon\to 0}(2\pi)^2\left\{\int dq^0\tilde{\varphi}_\epsilon(q^0)\tilde{F}^0(q)\right\}\Bigg|_{\mathbf{q}=\mathbf{\Delta}}$$

$$=(2\pi)^2\int d^3u\, E(\mathbf{u};-(\mathbf{u}-\mathbf{\Delta})^2) \tag{29}$$

and

$$\lim_{\epsilon\to 0}\langle p_1 M|[j^r(\varphi_\epsilon,1),A(0)]|Mp_2\rangle^T=\lim_{\epsilon\to 0}\frac{(2\pi)^2}{2}\int d^3u ds\Bigg\{\psi^r(\mathbf{u},s)\{\tilde{\varphi}_\epsilon([s+(\mathbf{u}-\mathbf{\Delta})^2]^{1/2})+\tilde{\varphi}_\epsilon(-[s+(\mathbf{u}-\mathbf{\Delta})^2]^{1/2})\}$$

$$-\frac{2}{[s+(\mathbf{u}-\mathbf{\Delta})^2]^{1/2}}\frac{\partial}{\partial s}Z^r(\mathbf{u},s)\{\tilde{\varphi}_\epsilon([s+(\mathbf{u}-\mathbf{\Delta})^2]^{1/2})-\tilde{\varphi}_\epsilon(-[s+(\mathbf{u}-\mathbf{\Delta})^2]^{1/2})\}\Bigg\}$$

$$-(2\pi)^2\int d^3u(\Delta-u)^r E(\mathbf{u},-(\mathbf{u}-\mathbf{\Delta})^2). \tag{30}$$

Owing to the bounded support in $|\mathbf{u}|$, the limit of the zeroth component (29) and the last term in (30) always exist. However, it is easy to find spectral functions ψ^r and Z^r such that the first term in (30) does not exist. Take for instance

$$Z^r(\mathbf{u},s)\equiv 0,$$
$$\psi^r(\mathbf{u},s)=\Theta(p^0-|\mathbf{u}|)\Theta(s-s_0(\mathbf{u}))\times[u^r g_1(\mathbf{u}^2,\mathbf{\Delta u},s)+\Delta^r g_2(\mathbf{u}^2,\mathbf{\Delta u},s)], \tag{31}$$

where the g_i are C^∞ functions in all arguments and grow like a polynomial for $s\to\infty$.

A further interesting result follows immediately from Eq. (21), taking $\mu=0$ and $\nu=r$:

Theorem 3. If the equal-time limits between two current densities containing at least one zeroth component exist, then the limits for two space components also exist, and have at most as many gradient terms as the former limits.

III. LIMITS FOR TIME DERIVATIVES OF COMMUTATORS

The results of Sec. II are independent of any assumptions on the structure of the commutators containing a generalized charge. Only the existence of these limits, independent of the δ sequences, has been exploited to prove the existence of the corresponding density limits.

In this section we first derive some general results on the existence of equal-time limits for the time derivatives of commutators, similar to Sec. II. We start with proving, from the assumption (A.I) only, the existence of the equal-time commutators between one density and all moments of degree $r\geqslant n$ of the nth time derivative of the generalized charges. Furthermore it will be shown that the existence of the equal-time commutators between the nth time derivative of the generalized charges and one density again implies the existence of the limits for all the corresponding charge moments and thereby also that of the densities.

In the second part of this section we derive an explicit expression for the first time derivative of the density commutators from the structure (3) or equivalently (12) of the equal-time commutation relations for the currents themselves. Finally we make some predictions on the structure of the Σ and meson commutators.

The keys to our considerations are the results of (V.I) and the following simple property of distributions.

Lemma II. If $\lim_{\epsilon\to 0}T(\varphi_\epsilon)$ exists independent of the sequence, then

$$\lim_{\epsilon\to 0}\int dx^0 x^0 T(x^0)\frac{d}{dx^0}\varphi_\epsilon(x^0)$$

exists, and furthermore

$$\lim_{\epsilon\to 0}\int dx^0\varphi_\epsilon(x^0)x^0\frac{d}{dx^0}T(x^0)=0. \tag{32}$$

Proof. We prove only the second part of this lemma, since the first part is then obvious:

$$\lim_{\epsilon\to 0}\int dx^0\varphi_\epsilon(x^0)x^0\frac{d}{dx^0}T(x^0)=-\lim_{\epsilon\to 0}T(\varphi_\epsilon)$$
$$-\lim_{\epsilon\to 0}\int dx_0 T(x^0)\frac{1}{\epsilon}\left[y^0\frac{d}{dy^0}\varphi(y^0)\right]_{y^0=x^0/\epsilon}. \tag{33}$$

However, with $\varphi(x^0)$, also $\psi(x^0)=:-x^0\varphi'(x^0)$ is a sequence of type δ, which proves Lemma II.

[24] Uta Völkel, B. Schroer, and A. H. Völkel, Commun. Math. Phys. 10, 69 (1968).

Next we derive the analog of Theorem 1 and (V.I) Theorem I for the time derivatives of current commutators.

Theorem 4. Under the assumptions (A.I),[18] the equal-time limits of the time derivatives

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0{}^n j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{k,l,r-k-l})_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T, \quad 0\le k,l\le r$$

exist for all $r\ge n$, and furthermore

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0{}^n j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{k,l,r-k-l})_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T=0 \quad (34)$$

for all $r\ge 2(N+n)+m$ [m fixed, $0\le m\le N$, and $m=0$ if the spectrum condition holds].

This theorem states that if the equal-time commutators between generalized charges and a density exist, then the equal-time limits of the nth time derivative of commutators between all charge moments of degree higher than $n-1$ and one density exist also. If the degree of the charge moments is higher than $2(N+n)+m$, then the limits vanish.

Proof. The proof of the first part is again done by complete induction with respect to the order n of the time derivative.

(a) $n=1$: From Theorem 1 and Eq. (21) we know that

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0 j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{k,l,r-k-l})_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$

exists for all $r\ge 1$.

(b) Assume that our lemma is correct for the first n derivatives, that is,

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0{}^n j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{k,l,r-k-l})_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$

exists for all $r\ge n$; $0\le k,\ l\le r$.

Taking

$$\varphi_\epsilon(x^0)=\frac{d^n}{d(x^0)^n}\hat{\varphi}_\epsilon(x^0) \quad\text{and}\quad h(\mathbf{x})=\mathbf{f}^{k,l,r-k-l}(\mathbf{x}),$$

with $\hat{\varphi}_\epsilon$ a sequence of type δ and $r\ge n$, we deduce from (21) that the equal-time limit

$$\begin{aligned}
&\lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(M_0{}^i(\partial_0{}^n\hat{\varphi}_\epsilon;\mathbf{f}^{k,l,r-k-l}))_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T\\
&=\lim_{\epsilon\to 0}\int d^4x\Big\{(-1)^{n+1}\langle\Psi|[\partial_0{}^{n+1}j_\alpha{}^\mu(x)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T x^i\mathbf{f}^{k,l,r-k-l}(\mathbf{x})\\
&\qquad+(-1)^n\langle\Psi|[\partial_0{}^{n-1}j_\alpha{}^\mu(x)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T\frac{\partial}{\partial x^i}\mathbf{f}^{k,l,r-k-l}(\mathbf{x})\\
&\qquad+(-1)^n x^0\langle\Psi|[\partial_0{}^n j_\alpha{}^\mu(x)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T\frac{\partial}{\partial x^i}\mathbf{f}^{k,l,r-k-l}(\mathbf{x})\Big\}\hat{\varphi}_\epsilon(x^0) \qquad (35)
\end{aligned}$$

exists for all $r\geqslant n$ and $i=1,2,3$. However, the second term of Eq. (35) exists for all $r\geqslant n$ due to the induction assumption. The third term exists according to Lemma II. Therefore also the first term of (35) in which the occurring charge moment is of degree $r+1$ exists. This proves the first part of the theorem.

The second part of Theorem 4 follows immediately from Theorem I of (V.I) since the nth derivative of a distribution of order N is of the order $N+n$.

Furthermore, from Eqs. (21) and (35) and Lemma I, it follows in exactly the same way as in the proof of Theorem 1 (induction with respect to the degree of the moments) that the existence of the equal-time commutators between the nth derivative of the generalized charges and one density implies the existence of the equal-time limits for all the higher moments. Combining this fact with Theorem II of (V.I), we arrive at the following generalization of Theorem 2.

Theorem 5. If the equal-time limits

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0{}^n j_\alpha{}^\mu(\varphi_\epsilon;1)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T$$

exist for some n, then the equal-time limits of all time derivatives up to order n of the density commutators exist and are given by

$$\begin{aligned}
&\lim_{\epsilon\to 0}\langle\Psi|[\partial_0{}^{\bar n} j_\alpha{}^\mu(\varphi_\epsilon;h)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T\\
&=\sum_{r=0}^{2(N+\bar n)+m-1}\sum_{i=0}^{r}\sum_{j=0}^{r-i}\lim_{\epsilon\to 0}\langle\Psi|[\partial_0{}^{\bar n} j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{i,j,r-i-j})_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T\frac{1}{i!j!(r-i-j)!}\frac{\partial^r}{\partial(z^1)^i\partial(z^2)^j\partial(z^3)^{r-i-j}}h(\mathbf{z})\Big|_{\mathbf{z}=0} \qquad (36)
\end{aligned}$$

for all $\bar n$ with $0\le\bar n\le n$ and all C^∞ functions $h(\mathbf{x})$.

Of course the assumption in this theorem may turn out to be a very critical one. In nontrivial cases of quantum field theory these limits will possibly exist for at most the first few time derivatives ($n\le 2$). For the first derivatives of the $(0,\nu)$ and $(\mu,0)$ components as well as for the second derivative of the (0,0) component, the existence follows from that of the Σ commutator[1,2,25] and the meson commutator, respectively, as we show below. Even more, in these cases one can derive explicit expressions for these limits in terms of the Gell-Mann Σ and meson commutation relations.

For this we sharpen our assumptions. We assume with Gell-Mann that the equal-time limits for the commutators between one density and the charge moments of degree $1\leq r\leq 2N+m-1$ vanish (no gradient terms)[26]:

$$\text{(A.II)}\quad \lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon;\mathbf{f}^{k,l,r-k-l})_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T=0$$
$$\text{for } 1\leq r\leq 2N+m-1;\ k,l\geq 0;\ 0\leq k+l\leq r. \quad (37)$$

Together with (A.I) this is equivalent to the Gell-Mann commutation relations for the densities:

$$\text{(GM)}\quad \lim_{\epsilon\to 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon;h)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T =\langle\Psi|A_{\alpha\beta}{}^{\mu\nu}(0)_{ab}|\Phi\rangle h(\mathbf{0}) \quad (38)$$

for all C^∞ functions $h(\mathbf{x})$. Furthermore, the space-time components are

$$A_{\alpha\beta}{}^{0\nu}(0)_{ab}=A_{\alpha\beta}{}^{\nu 0}(0)_{ab}=ic^{\alpha\beta\gamma}j_\gamma{}^\nu(0)_c. \quad (39)$$

The space-space components $A_{\alpha\beta}{}^{rs}(0)_c$ are model dependent. We assume here that they are linear in the currents:

$$A_{\alpha\beta}{}^{rs}(x)_{ab}=T^{rs}{}_\nu(\alpha a,\beta b,\gamma c)j_\gamma{}^\nu(x)_c. \quad (40)$$

This ansatz covers the most important models such as

(a) algebra of fields[8]: $T^{rs}{}_\nu=0$;
(b) quark model[1,2,9]:

$$T^{rs0}(\alpha B;\beta B;\gamma V)=-if^{\alpha\beta\gamma}g^{rs},$$
$$T^{rsk}(\alpha B;\beta B;\gamma A)=id^{\alpha\beta\gamma}\epsilon^{rsk},$$
$$B=A;V,$$
$$T^{rs0}(\alpha B;\beta E;\gamma A)=-if^{\alpha\beta\gamma}g^{rs},$$
$$T^{rsk}(\alpha B;\beta E;\gamma V)=id^{\alpha\beta\gamma}\epsilon^{rsk},$$
$$B\neq E;\quad B=A,V;\quad E=A,V.$$

All the remaining $T^{rs\nu}$ are zero.

From the additional assumptions (37)–(40) on the structure of the current commutators, we now derive the following result on the *existence and structure* of their first time derivative on the submanifold of C^∞ functions $h(\mathbf{x})$ which vanish linearly in the origin.[27] For the sake of clarity we formulate our results again as a theorem.

Theorem 6. Under the assumptions (A.I) and (A.II) [or equivalently (GM)], it follows for all C^∞ functions $h(\mathbf{x})$ and $k=1,2,3$ that

$$\lim_{\epsilon\to 0}\int d^4x\ \varphi_\epsilon(x^0)x^kh(\mathbf{x})\langle\Psi|[\partial_0j_\alpha{}^0(x)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T =\langle\Psi|A_{\alpha\beta}{}^{k\nu}(0)_{ab}|\Phi\rangle h(\mathbf{0}), \quad (41)$$

$$\lim_{\epsilon\to 0}\int d^4x\ \varphi_\epsilon(x^0)x^kh(\mathbf{x})\langle\Psi|[\partial_0j_\alpha{}^r(x)_a,j_\beta{}^0(0)_b]|\Phi\rangle^T =\langle\Psi|A_{\alpha\beta}{}^{rk}(0)_{ab}|\Phi\rangle h(\mathbf{0}), \quad (42)$$

$$\lim_{\epsilon\to 0}\int d^4x\ \varphi_\epsilon(x^0)x^kh(\mathbf{x})\langle\Psi|[\partial_0j_\alpha{}^r(x)_a,j_\beta{}^s(0)_b]|\Phi\rangle^T$$
$$=\{ic^{\alpha\beta\gamma}[g^{rk}\langle\Psi|j_\gamma{}^s(0)_c|\Phi\rangle+g^{sk}\langle\Psi|j_\gamma{}^r(0)_c|\Phi\rangle]$$
$$+T^{rs0}(\alpha a;\beta b;\gamma c)\langle\Psi|j_\gamma{}^k(0)_c|\Phi\rangle$$
$$-T^{rsk}(\alpha a;\beta b;\gamma c)\langle\Psi|j_\gamma{}^0(0)_c|\Phi\rangle\}h(\mathbf{0}), \quad (43)$$

$$\lim_{\epsilon\to 0}\int d^4x\ \varphi_\epsilon(x^0)x^kh(\mathbf{x})\langle\Psi|[\partial_\mu j_\alpha{}^\mu(x)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T=0. \quad (44)$$

Moreover, if the equal-time limit between a divergence of a current and a (proper) charge $j_\beta{}^0(\varphi_\epsilon;\mathbf{1})_b$ exists and is a scalar with respect to pure Lorentz transformations, then one can establish a result similar to (44) also for the meson commutator (equal-time commutator between two current divergences).

Corollary 6.1. If in addition to (A.I) and (A.II) the limit

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_\mu j_\alpha{}^\mu(0)_a;j_\beta{}^0(\varphi_\epsilon;\mathbf{1})_b]|\Phi\rangle^T$$

exists and is a scalar with respect to proper Lorentz transformations, then we have for all C^∞ functions $h(\mathbf{x})$

$$\lim_{\epsilon\to 0}\int d^4x\ \varphi_\epsilon(x^0)x^kh(\mathbf{x})\langle\Psi|[\partial_\mu j_\alpha{}^\mu(x)_a,\partial_\nu j_\beta{}^\nu(0)_b]|\Phi\rangle^T=0. \quad (45)$$

Before we prove Theorem 6 and Corollary 6.1, let us insert here several comments concerning the results.

(1) Theorem 6 states that if the Gell-Mann commutation relations for the current densities hold, then the equal-time limits of their first time derivatives exists

[25] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[26] This structure assumption is made for simplicity. If gradient terms occur in the Gell-Mann relations one can perform the same considerations with similar results.

[27] Equation (44) of Theorem 5 as well as Eq. (58) for the structure of the unitary spin antisymmetric part of the Σ commutator have been derived formally first by Kuo and Sugawara [T. K. Kuo and M. Sugawara, Phys. Rev. **163**, 1716 (1967)]. However, we do *not* agree with the general results of these authors. Neither their connection between Schwinger terms of the Gell-Mann algebra and gradient terms of the corresponding Σ commutator nor their proof of the symmetry of the Schwinger term seems to us to be correct. Their results are in contradiction to the equal-time commutator between one total charge and one density (2), which fixes the "scaling of the Schwinger terms." With the proper scaling, a first-order Schwinger term in the Gell-Mann relations induces *a first- and second-order* gradient term in the Σ commutator if and only if the Schwinger term is antisymmetric in the unitary spin variables. This follows along the same lines as applied in Sec. III of the present article. The "proof" of the symmetry by the criticized authors is based on the implicit assumption that no second-order gradient term occurs in the Σ commutator. However, then by Lorentz invariance also no first-order term can occur.

for the subclass of all C^∞ functions which vanish at least linearly in the origin. For this subclass of smearing functions in the space variables $\mathbf{x}$, the limits are independent of the δ sequence. For instance, all generalized charge moments belong to this subclass, but not the generalized charges themselves.

In order to extend the limit to *all* C^∞ functions $h(\mathbf{x})$ (in the customary language this means the existence of the limit for the densities), we have to make sure that the limits of the corresponding charges,

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0 j_\alpha{}^\mu(\varphi_\epsilon;1)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T,$$

exist. This does not follow from the Gell-Mann relations and the general principles alone. On the other hand, this offers the possibility for the occurrence of ambiguities, or the dependence of the limits on the δ sequences in the first time derivative, as discussed by Brandt and Sucher[12] in connection with finite electromagnetic mass shifts, without disturbing the Gell-Mann relations for the current commutators themselves.

However, introducing the relation

$$h(\mathbf{0})=\left[\frac{\partial}{\partial x^k}[x^k h(x)]\right]_{\mathbf{x}=0} \tag{46}$$

into (41)–(44), we see that in any case, with or without the above ambiguities, the equal-time limits of the first time derivative have, in general, finite first-order gradient (Schwinger) terms, explicitly given by the right-hand side of the Gell-Mann commutation relations. These gradient terms do not contain any ambiguities if the Gell-Mann relations hold for the currents themselves.

(2) The second part of Theorem 6 and Corollary 6.1 states that the Σ and meson commutators do not have any gradient terms if the commutation relations for the currents themselves have the structure proposed by Gell-Mann. In other words, if these equal-time commutators exist at all, then they have the form[28]

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_\mu j_\alpha{}^\mu(\varphi_\epsilon;h)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T =\langle\Psi|\Sigma_{\alpha\beta}{}^\nu(0)_{ab}|\Phi\rangle h(\mathbf{0}), \tag{47}$$

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_\mu j_\alpha{}^\mu(\varphi_\epsilon;h)_a, \partial_\nu j_\beta{}^\nu(0)_b]|\Phi\rangle^T =\langle\Psi|M_{\alpha\beta}(0)_{ab}|\Phi\rangle h(\mathbf{0}) \tag{48}$$

for *all* C^∞ functions $h(\mathbf{x})$.

For the existence of these limits, we must again assure their existence for $h(\mathbf{x})\equiv 1$ only. If $j_\beta{}^\nu$ is a conserved current, then of course the zeroth component of the Σ commutator exists with

$$\Sigma_{\alpha\beta}{}^0(0)_{aV}=ic^{\alpha\beta\gamma}\partial_\mu j_\gamma{}^\mu(0)_c. \tag{49}$$

[28] The index ν in Σ^ν does *not* describe a vector with respect to Lorentz transformations. It simply counts the four possible components.

From the equal-time commutator between a conserved charge and a divergence of a current, the existence of (47) for $h(\mathbf{x})\equiv 1$ and Eq. (49) easily follow by means of translation invariance and Eq. (44). Moreover it is easy to show that the part of the Σ commutator $\Sigma_{\alpha\beta}{}^0(0)_{ab}$ which is antisymmetric in $(\alpha,a)\leftrightarrow(\beta,b)$ always exists.

Since the Gell-Mann relations hold for all C^∞ functions $h(\mathbf{x})$, they are also true for the translated ones:

$$h_{\mathbf{y}}(\mathbf{x})=:h(\mathbf{y}+\mathbf{x}).$$

If we take for Ψ,Φ the translated states

$$\{\Psi,\Phi\}=U(1;y)^{-1}\{\hat\Psi,\hat\Phi\}$$

with $\hat\Psi,\hat\Phi$ arbitrary states from D, then we get by means of translational invariance from (38)

$$\lim_{\epsilon\to 0}\int d^4x\langle\hat\Psi|[j_\alpha{}^\mu(x)_a, j_\beta{}^\nu(y)_b]|\hat\Phi\rangle^T\varphi_\epsilon(x^0-y^0)h(\mathbf{x}) =\langle\hat\Psi|A_{\alpha\beta}{}^{\mu\nu}(y)_{ab}|\hat\Phi\rangle h(\mathbf{y}). \tag{50}$$

Restricting $h(\mathbf{x})$ to the class O_M, that is, to C^∞ functions growing at most like a polynomial at infinity, we can multiply this equation with an arbitrary function $f(y)$ from S_4,[11,14,16,23] and integrate over y. For our purposes it is enough to take the products $f(y)=u(y^0)g(\mathbf{y})$. Equation (50) then reads

$$\lim_{\epsilon\to 0}\langle\hat\Psi|[j_\alpha{}^\mu(\varphi_\epsilon;h)_a\otimes j_\beta{}^\nu(u,g)_b]|\hat\Phi\rangle^T =\langle\hat\Psi|A_{\alpha\beta}{}^{\mu\nu}(u;hg)_{ab}|\hat\Phi\rangle. \tag{51}$$

Here we have introduced on the left-hand side the notation

$$\langle\Psi|[j^\mu(\varphi,h)\otimes j^\nu(u,g)]|\Phi\rangle^T =:\int d^4x d^4y\langle\Psi|[j^\mu(x);j^\nu(y)]|\Phi\rangle^T \times\varphi(x^0-y^0)u(y^0)h(\mathbf{x})g(\mathbf{y}). \tag{52}$$

Taking $\mu=\nu=0$ and $u(x^0)=\partial\hat u(x^0)/\partial x^0$ with $\hat u$ arbitrary from S_1, it follows from (51) by partial integration that

$$\lim_{\epsilon\to 0}\{\langle\hat\Psi|[\partial_\mu j_\alpha{}^\mu(\varphi_\epsilon;h)_a\otimes j_\beta{}^0(\hat u;g)_b]|\hat\Phi\rangle^T +\langle\hat\Psi|[j_\alpha{}^0(\varphi_\epsilon;h)_a\otimes\partial_\mu j_\beta{}^\mu(\hat u;g)_b]|\hat\Phi\rangle^T\} =ic^{\alpha\beta\gamma}\langle\hat\Psi|\partial_\mu j_\gamma{}^\mu(\hat u;hg)_c|\hat\Phi\rangle. \tag{53}$$

Since the right-hand side of this equation is independent of φ_ϵ, we can restrict ourselves without loss of generality to symmetric sequences φ_ϵ.

The space-time translated version of (44) reads, in analogy to (51),

$$\lim_{\epsilon\to 0}\int d^4x d^4y\langle\hat\Psi|[\partial_\mu j_\alpha{}^\mu(x)_a, j_\beta{}^\nu(y)_b]|\hat\Phi\rangle^T \times\varphi_\epsilon(x^0-y^0)u(y^0)(x-y)^k h(\mathbf{x})g(\mathbf{y})=0. \tag{54}$$

Applying an infinitesimal Lorentz transformation to this equation, we obtain at once in the standard way

$$\lim_{\epsilon\to 0}\int d^4x d^4y \langle\hat{\Psi}|[\partial_\mu j_\alpha{}^\mu(x)_a, j_\beta{}^0(y)_b]|\hat{\Phi}\rangle^T \times(x^0-y^0)\varphi_\epsilon(x^0-y^0)u(y^0)h(\mathbf{x})g(\mathbf{y})=0. \quad (55)$$

From the last two equations it follows that

$$\langle\hat{\Psi}|[j_\alpha{}^0(\varphi_\epsilon;h)_a\otimes\partial_\mu j_\beta{}^\mu(u,g)_b]|\hat{\Phi}\rangle^T = -\langle\hat{\Psi}|[\partial_\mu j_\beta{}^\mu(\varphi_\epsilon;h)_b\otimes j_\alpha{}^0(u,g)_a]|\hat{\Phi}\rangle^T + r_\epsilon{}^{\alpha\beta}(\hat{\Psi};\hat{\Phi})_{ab}, \quad (56)$$

with

$$\lim_{\epsilon\to 0} r_\epsilon{}^{\alpha\beta}(\hat{\Psi};\hat{\Phi})_{ab}=0. \quad (57)$$

If we insert (56) into (53) and drop the smearing over y, we get

$$\lim_{\epsilon\to 0}\{\langle\hat{\Psi}|[\partial_\mu j_\alpha{}^\mu(\varphi_\epsilon;h)_a, j_\beta{}^0(0)_b]|\hat{\Phi}\rangle^T-[(\alpha,a)\leftrightarrow(\beta,b)]\} = ic^{\alpha\beta\gamma}\langle\hat{\Psi}|\partial_\mu j^\mu(0)|\hat{\Phi}\rangle h(\mathbf{0}). \quad (58)$$

This relation holds for all O_M functions $h(\mathbf{x})$. However, owing to microcausality, the left-hand side is independent of the asymptotic behavior of $h(\mathbf{x})$. Therefore it is true for all C^∞ functions. In other words, the antisymmetric part of the zeroth component of the Σ commutator always exists and we have

$$\Sigma_{\alpha\beta}{}^0(0)_{ab}=\begin{cases} ic^{\alpha\beta\gamma}\partial_\mu j_\gamma{}^\mu(0)_c & \text{if } j_\beta{}^\nu \text{ is conserved}\\ \tfrac{1}{2}ic^{\alpha\beta\gamma}\partial_\mu j_\gamma{}^\mu(0)_c+\hat{\Sigma}_{\alpha\beta}(0)_{ab} & \text{if } j_\beta{}^\nu \text{ is not conserved,}\end{cases} \quad (59)$$

with

$$\hat{\Sigma}_{\alpha\beta}(0)_{ab}=\hat{\Sigma}_{\beta\alpha}(0)_{ba}. \quad (60)$$

(3) If we abandon in Corollary 6.1, the assumption that Σ^0, or equivalently $\hat{\Sigma}$, is a scalar with respect to proper Lorentz transformations, we get instead of (45)

$$\lim_{\epsilon\to 0}\int d^4x\langle\Psi|[\partial_\mu j_\alpha{}^\mu(x)_a,\partial_\nu j_\beta{}^\nu(0)_b]|\Phi\rangle^T \times x^k x^l h(\mathbf{x})\varphi_\epsilon(x^0)=0. \quad (61)$$

In this case the meson commutator can have a finite first-order gradient term which is forbidden if $\hat{\Sigma}$ is a (pseudo-)scalar.

After these remarks on the content of Theorem 6 and Corollary 6.1, we come back to their proofs.

Proof of Theorem 6. Inserting Eqs. (20) and (21) into the Gell-Mann commutation relations (39) and (40), we obtain the following equation:

$$\begin{aligned}\lim_{\epsilon\to 0}\{&\langle\Psi|[j_\alpha{}^\mu(M^k{}_0(\varphi_\epsilon,h))_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T\\ &+g^{\mu 0}\langle\Psi|[j_\alpha{}^k(\varphi_\epsilon,h)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T+g^{\nu 0}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h)_a,j_\beta{}^k(0)_b]|\Phi\rangle^T\\ &-g^{\mu k}\langle\Psi|[j_\alpha{}^0(\varphi_\epsilon,h)_a,j_\beta{}^\nu(0)_b]|\Phi\rangle^T-g^{\nu k}\langle\Psi|[j_\alpha{}^\mu(\varphi_\epsilon,h)_a,j_\beta{}^0(0)_b]|\Phi\rangle^T\}\\ =&\,ic^{\alpha\beta\gamma}\langle\Psi|\{g^{\mu 0}g^{\nu 0}j_\gamma{}^k(0)_c-(g^{0\mu}g^{\nu k}+g^{\nu 0}g^{\mu k})j_\gamma{}^0(0)_c\}|\Phi\rangle\\ &+g^{\mu r}g^{\nu s}\{T^{rs0}(\alpha a,\beta b,\gamma c)\langle\Psi|j_\gamma{}^k(0)_c|\Phi\rangle-T^{rsk}(\alpha a,\beta b,\gamma c)\langle\Psi|j_\gamma{}^0(0)_c|\Phi\rangle\},\end{aligned} \quad (62)$$

$$M^k{}_0 f(x)=\left(x^k\frac{\partial}{\partial x^0}+x^0\frac{\partial}{\partial x^k}\right)f(x).$$

In the last four terms on the right-hand side of Eq. (62) we can use the Gell-Mann commutation relations (39) once more. Equations (41)–(44) of Theorem 6 then follow by means of Lemma II in a straightforward way.

Proof of Corollary 6.1. The additional assumption of Corollary 6.1 is, according to our discussion above, equivalent to

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_\mu j_\alpha{}^\mu(\varphi_\epsilon,h)_a\otimes j_\beta{}^0(u,g)_b]|\Phi\rangle^T=\langle\Psi|\Sigma_{\alpha\beta}{}^0(0)_{ab}|\Phi\rangle, \quad (63)$$

with

$$U(\Lambda)\Sigma_{\alpha\beta}{}^0(f)_{ab}U(\Lambda)^{-1}=\Sigma_{\alpha\beta}{}^0(f_\Lambda)_{ab}, \quad f_\Lambda(x)=:f(\Lambda^{-1}x). \quad (64)$$

Applying a Lorentz transformation in (63), we obtain the condition

$$\lim_{\epsilon\to 0}\left\{\langle\Psi|[\partial_\kappa j_\alpha{}^\kappa(\varphi_\epsilon,h)_a\otimes j_\beta{}^k(u,g)_b]|\Phi\rangle^T + \int d^4x d^4y\langle\Psi|\left[\partial_\kappa j_\alpha{}^\kappa(x)_a,\frac{\partial}{\partial y^0}j_\beta{}^0(y)_b\right]|\Phi\rangle^T(x-y)^k\varphi_\epsilon(x^0-y^0)u(y^0)h(\mathbf{x})g(\mathbf{y})\right\}=0. \quad (65)$$

On the other hand, we get from (54) by the special choice of test functions $\partial g(\mathbf{x})/\partial x^r$ with $g(\mathbf{x})\in S_3$

$$\lim_{\epsilon\to 0}\int d^4x d^4y \langle\Psi|[\partial_\mu j_\alpha{}^\mu(x)_a, j_\beta{}^r(y)_b]|\Phi\rangle^T \frac{\partial}{\partial y^r} g(\mathbf{y})h(\mathbf{x})(x-y)^k\varphi_\epsilon(x^0-y^0)u(y^0)$$

$$\equiv \lim_{\epsilon\to 0}\bigg\{\langle\Psi|[\partial_\mu j_\alpha{}^\mu(\varphi_\epsilon,h)_a\otimes j_\beta{}^k(u,g)_b]|\Phi\rangle^T + \int d^4x d^4y\bigg[\langle\Psi|\bigg(\partial_\mu j_\alpha{}^\mu(x)_a, \frac{\partial}{\partial y^0} j_\beta{}^0(y)_b\bigg)|\Phi\rangle^T$$

$$-\langle\Psi|(\partial_\mu j_\alpha{}^\mu(x)_a, \partial_\lambda j_\beta{}^\lambda(y)_b)|\Phi\rangle^T\bigg](x-y)^k\varphi_\epsilon(x^0-y^0)u(y^0)h(\mathbf{x})g(\mathbf{y})\bigg\} = 0. \quad (66)$$

From the combination of the last two equations, the validity of Corollary 6.1 follows for all O_M functions $h(\mathbf{x})$. Since, owing to microcausality, Eq. (45) is independent of the asymptotic behavior of $h(\mathbf{x})$, it is true for all C^∞ functions $h(\mathbf{x})$.

If the equal-time commutator between the time derivative of a charge and a current density, and furthermore the Σ and meson commutators are given, then it is very easy to write down the complete equal-time limits for the first time derivatives of all current commutators and the second time derivative of their (0,0) components. From the Gell-Mann commutation relations (38)–(40), the Σ and meson commutators (47) and (48), and Poincaré invariance, we get by lengthy but straightforward calculations the following corollary.

Corollary 6.2. If the Gell-Mann relations hold and if the limits

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_\mu j_\alpha{}^\mu(0)_a, j_\beta{}^\nu(\varphi_\epsilon,1)_b]|\Phi\rangle^T = \langle\Psi|\Sigma_{\alpha\beta}{}^\nu(0)_{ab}|\Phi\rangle,$$

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0 j_\alpha{}^k(\varphi_\epsilon,1)_a, j_\beta{}^r(0)_b]|\Phi\rangle^T$$

are given, then we have for all C^∞ functions $h(\mathbf{x})$ and all $\Psi,\Phi\in D$

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0 j_\alpha{}^0(\varphi_\epsilon,h)_a, j_\beta{}^\nu(0)_b]|\Phi\rangle^T = \langle\Psi|\Sigma_{\alpha\beta}{}^\nu(0)_{ab}|\Phi\rangle h(\mathbf{0}) + \langle\Psi|A_{\alpha\beta}{}^{r\nu}(0)_{ab}|\Phi\rangle \frac{\partial}{\partial x^r}h(\mathbf{x})\bigg|_{\mathbf{x}=0}, \quad (67)$$

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0 j_\alpha{}^k(\varphi_\epsilon,k)_a, j_\beta{}^0(0)_b]|\Phi\rangle^T$$

$$= \{\langle\Psi|\Sigma_{\alpha\beta}{}^k(0)_{ab}|\Phi\rangle + \langle\Psi|\partial_\mu A_{\alpha\beta}{}^{k\mu}(0)_{ab}|\Phi\rangle\}h(\mathbf{0}) + \langle\Psi|A_{\alpha\beta}{}^{ks}(0)_{ab}|\Phi\rangle \frac{\partial}{\partial x^s}h(\mathbf{x})\bigg|_{\mathbf{x}=0}, \quad (68)$$

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0 j_\alpha{}^k(\varphi_\epsilon;h)_a, j_\beta{}^r(0)_b]|\Phi\rangle^T$$

$$= \lim_{\epsilon\to 0}\langle\Psi|[\partial_0 j_\alpha{}^k(\varphi_\epsilon;1)_a, j_\beta{}^r(0)_b]|\Phi\rangle^T h(\mathbf{0}) + \{ic^{\alpha\beta\gamma}[g^{ks}\langle\Psi|j_\gamma{}^r(0)_c|\Phi\rangle + g^{rs}\langle\Psi|j_\gamma{}^k(0)_c|\Phi\rangle]$$

$$+T^{kr0}(\alpha a;\beta b;\gamma c)\langle\Psi|j_\gamma{}^s(0)_c|\Phi\rangle - T^{krs}(\alpha a;\beta b;\gamma c)\langle\Psi|j_\gamma{}^0(0)_c|\Phi\rangle\} \frac{\partial}{\partial x^s}h(\mathbf{x})\bigg|_{\mathbf{x}=0}. \quad (69)$$

If in addition the meson commutator is given, then it follows furthermore that

$$\lim_{\epsilon\to 0}\langle\Psi|[\partial_0{}^2 j_\alpha{}^0(\varphi_\epsilon;h)_a, j_\beta{}^0(0)_b]|\Phi\rangle^T$$

$$= \{\langle\Psi|\partial_\mu\Sigma_{\alpha\beta}{}^\mu(0)_{ab}|\Phi\rangle - \langle\Psi|M_{\alpha\beta}(0)_{ab}|\Phi\rangle\}h(\mathbf{0})$$

$$+\{\langle\Psi|\partial_\mu A_{\alpha\beta}{}^{r\mu}(0)_{ab}|\Phi\rangle + \langle\Psi|[\Sigma_{\alpha\beta}{}^r(0)_{ab} + (\alpha a)\leftrightarrow(\beta b)]|\Phi\rangle\} \frac{\partial}{\partial x^r}h(\mathbf{x})\bigg|_{\mathbf{x}=0}$$

$$+\langle\Psi|A_{\alpha\beta}{}^{rs}(0)_{ab}|\Phi\rangle \frac{\partial^2}{\partial x^r\partial x^s}h(\mathbf{x})\bigg|_{\mathbf{x}=0}. \quad (70)$$

IV. FINAL REMARK

Finally we want to make a remark on the dependence of our results on the class of admitted δ sequences. It is essential for Łojasiewicz's Lemma I, and thereby for Theorems 1, 2 and 5, that assumption (A.I) hold for the whole class of sequences specified in the Introduction.[29] If we weaken our assumption (A.I) and admit only the subclass of symmetric sequences $\varphi_\epsilon(x) = \varphi_\epsilon(-x)$, then Lemma I and therefore also Theorems

[29] Or equivalently, assumptions on the structure of the equal-time commutators between one density and a finite number of generalized charge moments.

1, 2, and 5 are not true any more. One can construct examples which show that an analog to Łojasiewicz's lemma does not hold in this case. The reason is that one cannot draw any conclusion from the symmetric part of the first derivative of a function on the symmetric part of its primitive function.

On the other hand, Theorems 3, 4, and 6 as well as their corollaries remain true under this weakened assumption.

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Causality and Analyticity in Formal Scattering Theory

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The connection between causality and analyticity in scattering theory is formulated in terms of Hilbert-space concepts. The usual rules of nonrelativistic quantum mechanics are assumed to hold for the "in" and "out" states of the scattering system. We show that there is a (physically verifiable) causality condition which implies that each diagonal S-matrix element $S_{\alpha\alpha}(E)$ must be the limit of an analytic function of the energy E, regular in $\mathrm{Im}E>0$. The implications for partial-wave amplitudes and for the forward scattering amplitude in elastic two-body collisions are discussed.

I. INTRODUCTION

ANALYTICITY properties of scattering amplitudes are an essential ingredient of S-matrix theory. It is important to know which of these properties can be deduced from causality conditions that are physically verifiable. This question has been the subject of a number of publications.[1–7] Direct proofs of analyticity from causality are available for nonrelativistic elastic scattering by spherically symmetric interactions that vanish beyond a finite radius.[1,6]

In attempting to generalize these proofs in the context of formal scattering theory, two types of difficulties are encountered. First, because scattering states cannot contain negative-frequency Fourier components, events in a scattering experiment cannot be localized in time with arbitrary sharpness.[2] However, Screaton[4] has shown that for a simple linear system described by a (scalar) equation

$$O(t)=\int_{-\infty}^{\infty} dt'F(t-t')I(t'),$$

a causality condition can be formulated which implies that the Fourier transform of F is analytic in a half-plane, even though O and I have only non-negative frequency components. The second difficulty is connected with the use of monochromatic states $|E,\alpha)$ of the free Hamiltonian H_0 in defining the S matrix. In the mathematical theory of Hilbert space, the diagonalization of a self-adjoint operator is expressed in terms of projection operators corresponding to

$$P(E)=\sum_\alpha \int_0^E dE'|E',\alpha)(E',\alpha|,$$

whose properties can be established under very general conditions.[8] The properties of the mapping $(E,\alpha|\psi\rangle$ depend on "representation theorems"[9] which are limited in scope. Very little can be said about matrix elements of the form $(E,\alpha|T|E',\alpha')$ if T is an arbitrary Hilbert-space operator. For example, if we know that $(E,\alpha|\psi\rangle$ is continuous in E for any normalizable $|\psi\rangle$, we cannot state that $(E,\alpha|T|E',\alpha')$ is continuous in E, E' unless we impose strong restrictions on the operator T, e.g., the condition that T be a compact operator.[10–12]

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