Linear Dependences and the Multiloop Veneziano Amplitude*

MICHIO KAKU
Department of Physics, University of California, Berkeley, California 94720
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By using the identities of Brower and Weis, several apparent inconsistencies associated with twisting the Sciuto three-Reggeon vertex are clarified. By using these results, all linear dependences are eliminated from planar, nonplanar, and overlapping n-loop amplitudes and shown to have only a minor effect. The Thorn projection operator is reduced to a hypergeometric function in the number operator and finally to a rational function. The equivalence of these techniques with the projected propagator of Gross and Schwarz and Ida is demonstrated.

I. INTRODUCTION

THE unitarization program[1] for the Veneziano amplitude has been hampered by the existence of certain linear dependences[2] among factorized residues, which have been interpreted as "spurious" particles which do not couple to scalar particles. The elimination of these spurious states has been accomplished by using Ward identities[3] and the spurious-particle projection operator of Thorn.[4] Because the correction due to the linear dependences in the single-loop amplitude is essentially trivial, there has been speculation that if one naively constructs the n-loop amplitude from three-Reggeon vertex functions,[5] the correction will again be minor. By exploiting the powerful identities of Brower and Weis,[6] this conjecture is shown to be valid for planar, nonplanar, and overlapping n-loop amplitudes. (In the case of the planar n-loop amplitude,[7] the corrections are found to occur almost exclusively in the Jacobian, which generates a factor $1-y$ for each loop, where y corresponds to the multiplier of each of the n Koba-Nielsen projective transformations.)

The recent works of Gross and Schwarz and Ida[8] in constructing the projected propagator are certainly more general than the methods presented here, but it unnecessarily complicates the evaluation of the traces over harmonic-oscillator states, since the linear dependence correction is minor.

In Sec. II, the identities of Brower and Weis[6] are used to clarify several inconsistencies concerning the Sciuto three-Reggeon vertex function. In Sec. III, all corrections due to linear dependences in the planar, nonplanar, and overlapping n-loop amplitudes are shown to be minor. And finally, in Sec. IV, the equivalence of these techniques, at least for a certain class of diagrams, is shown to be consistent with the methods of Gross and Schwarz and Ida.

The multiloop amplitudes are constructed by inserting loops into a multiperipheral tree. The spurion operators placed along the base line of diagram are easily removed. The spurion operators located in each loop, however, convert into hypergeometric functions, which can be reduced finally into simple rational functions. We summarize the linear dependence correction into two rules:

Rule I: For every planar loop appearing along the base line, replace the standard beta function situated between the two Sciuto vertices with

$$D\rightarrow\int_0^1 t^{R-\alpha-1}(1-t)^{-c}\left[\frac{1-t}{1-t(1-x)}\right]^{R-\alpha}dt,$$

where x is the product of all propagator variables in the upper portion of the loop.

Rule II: For every nonplanar or overlapping loop configuration, merely add the c-number expression

$$\left[1-\frac{z_iz_jx}{(1-z_i)(1-z_j)}\right]^{-c},$$

where x is again the product of all propagator variables along the upper portion of the loop, and i and j refer to the propagators adjacent to the position of the two Sciuto vertices (see Figs. 9 and 10).

II. LINEAR DEPENDENCES AND THREE-REGGEON VERTEX FUNCTION

The original Sciuto three-Reggeon vertex function has peculiar inconsistencies associated with it which may be eliminated by carefully eliminating spurious states. In particular, the curious $(1-z)^R$ factor is shown to be intimately related to the Ward identities of Chiu, Matsuda, and Rebbi.[3]

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[2] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969); S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).

[3] C. B. Chiu, S. Matsuda, and C. Rebbi, Phys. Rev. Letters **23**, 1526 (1969).

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The convention of using the propagator with spurious states subtracted out shall be adopted throughout this paper[4]:

$$D \to [1-P^\dagger(-\pi)]D[1-P(\pi)], \qquad (2.1)$$

where

$$P(\pi)=[A^\dagger(-\pi)+\tfrac{1}{2}m^2]\frac{1}{A(\pi)[A^\dagger(-\pi)+\frac{1}{2}m^2]}A(\pi)$$

and

$$A(\pi)=\pi(a_1-\tfrac{1}{2}\pi)+\sum_{n=1}^{\infty} n a_n{}^\dagger a_n-\sum_{n=1}^{\infty}[n(n+1)]^{1/2}a_n{}^\dagger a_{n+1}.$$

This prescription is ambiguous when the "b" leg of the Sciuto vertex (Fig. 1) is examined because the propagator is not the usual beta function:

$$\begin{aligned}
&D_a(k)\hat{D}_b(p)D_c'(q)W_{abc}|0\rangle_{abc}\\
&\equiv\int_0^1 dz_a\int_0^1 dz_b\int_0^1 dz_c(1-z_a)^{-c}(1-z_b)^{-c}(1-z_c)^{-c}\\
&\qquad\times z_a{}^{R_a-\alpha(k^2)-1}z_b{}^{R_b-\alpha(p^2)-1}z_c{}^{R_c-\alpha(q^2)-1}\\
&\qquad\times(1-z_b)^{R_c-\alpha(q^2)}\exp[pa^\dagger+qb^\dagger+pc^\dagger\\
&\qquad\quad+(a^\dagger,b^\dagger)_-+(b^\dagger,c^\dagger)_++(c^\dagger,a^\dagger)_0]|0\rangle_{abc}.
\end{aligned} \qquad (2.2)$$

Because the propagator on the "b" leg is not the traditional beta function, the projection operator $1-P$ must be attached to the propagator, away from the vertex function. Considerable confusion arises when the four-Reggeon vertex function is created by linking the "b" legs of two Sciuto vertices together (Fig. 2).

Fortunately, only a minor change is necessary to correct this situation, since a simple change of variables will change $(1-z_b)^{R_c}$ into $(1-z_c)^{R_b}$:

$$\begin{aligned}
&\int_0^1 dz_a\int_0^1 dz_b\int_0^1 dz_c(1-z_a)^{-c}(1-z_b)^{-c}(1-z_c)^{-c}\\
&\quad\times z_a{}^{R_a-\alpha(k^2)-1}z_b{}^{R_b-\alpha(p^2)-1}z_c{}^{R_c-\alpha(q^2)-1}\\
&\quad\times(1-z_b)^{R_c-\alpha(q^2)}W_{abc}|0\rangle_{abc}=\int_0^1 d\bar{z}_a\int_0^1 d\bar{z}_b\int_0^1 d\bar{z}_c\\
&\quad\times\bar{z}_a{}^{R_a-\alpha(k^2)-1}\bar{z}_b{}^{R_b-\alpha(p^2)-1}\bar{z}_c{}^{R_c-\alpha(q^2)-1}(1-\bar{z}_c)^{R_b-\alpha(p^2)}\\
&\quad\times(1-\bar{z}_a)^{-c}(1-\bar{z}_b)^{-c}(1-\bar{z}_c)^{-c}W_{abc}|0\rangle_{abc}
\end{aligned} \qquad (2.3)$$

[where $z_c(1-z_b)=\bar{z}_c$, $z_b=(1-\bar{z}_c)\bar{z}_b$, $z_a=\bar{z}_a$]

or (2.4)

$$D_a\hat{D}_bD_c'W_{abc}|0\rangle_{abc}=D_aD_b'\hat{D}_cW_{abc}|0\rangle_{abc}.$$

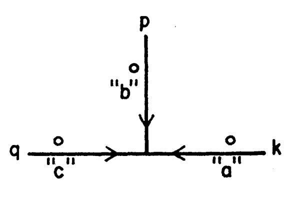

FIG. 1. Sciuto three-Reggeon vertex function.

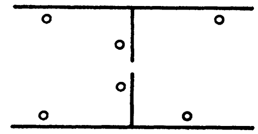

FIG. 2. Four-Reggeon vertex function formed by linking the "b" legs of two Sciuto vertices.

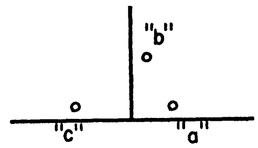

FIG. 3. Vertex function with a twisted "b" leg.

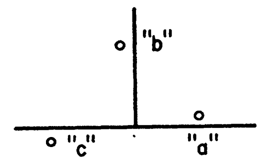

FIG. 4. Vertex function with a twisted "a" leg.

Because the "b" propagator is now a beta function, (2.4) makes possible a consistent interpretation of the four-Reggeon vertex function composed of two Sciuto vertices linked by their "b" legs. Equation (2.4) shall also be used to demonstrate the equivalence of the method of Gross and Schwarz and Ida[8] with the methods presented here for planar, nonplanar, and overlapping loops.

The inconsistencies mentioned earlier arise when attempts are made to perform arbitrary twists on the legs of the Sciuto vertex. For example, applying the twist operator of Caneschi, Schwimmer, and Veneziano[9] directly to the vertex function successfully reverses the position of the "dots," but fails to make the corresponding changes in the propagators:

$$\begin{aligned}
D_a\hat{D}_bD_c'\Omega_bW_{abc}|0\rangle_{abc}&=D_a\hat{D}_bD_c'W_{abc}'|0\rangle_{abc}\\
&\neq D_a'\hat{D}_bD_cW_{abc}'|0\rangle_{abc}
\end{aligned} \qquad (2.5)$$

and

$$\begin{aligned}
D_a\hat{D}_bD_c'\Omega_aW_{abc}|0\rangle_{abc}&=D_a\hat{D}_bD_c'W_{abc}''|0\rangle_{abc}\\
&\neq D_aD_b'\hat{D}_cW_{abc}''|0\rangle_{abc},
\end{aligned} \qquad (2.6)$$

where

$$W_{abc}'=\Omega_bW_{abc},\quad W_{abc}''=\Omega_aW_{abc}.$$

[Equations (2.5) and (2.6) are represented by Figs. 3 and 4, respectively.] The apparent inconsistency in (2.6) is resolved once we know (2.4), which allows us to flip the position of $(1-z)^R$. Equation (2.5), however, remains unexplained. To make matters worse, if we now decide to twist *away* from the vertex function and then apply the Ward identities, we capture unwanted factors of S:

$$\begin{aligned}
&(1-P^\dagger)_b\Omega_b{}^\dagger D_a\hat{D}_bD_c'W_{abc}|0\rangle_{abc}\\
&\quad=(1-P^\dagger)_bD_a\hat{D}_bD_c'S^{-1}{}_bW_{abc}'|0\rangle_{abc}\\
&\quad\neq(1-P^\dagger)_bD_a'\hat{D}_bD_cW_{abc}'|0\rangle_{abc},
\end{aligned} \qquad (2.7)$$

[9] L. Caneschi, A. Schwimmer, and G. Veneziano, Phys. Letters **30B**, 356 (1969).

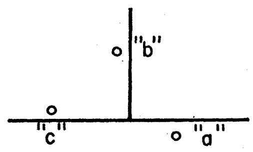

FIG. 5. Vertex function with a twisted "c" leg (the symmetric vertex of Caneschi, Schwimmer, and Veneziano).

where

$$S_b^{-1} \equiv (1-z_b)^{-A_b}.$$

A resolution of the difficulties in twisting toward and away from the vertex function lies in a study of linear dependences. Especially useful are the identities of Brower and Weis,[6] which will be referred to throughout the paper:

$$(A_a + A_b{}^\dagger - R_c + \tfrac{1}{2}q^2) V_{abc}|0\rangle_{abc} = 0, \quad (2.8)$$

$$(A_b + A_c{}^\dagger - R_a + \tfrac{1}{2}k^2) V_{abc}|0\rangle_{abc} = 0, \quad (2.9)$$

$$(A_c + A_a{}^\dagger - R_b + \tfrac{1}{2}p^2) V_{abc}|0\rangle_{abc} = 0, \quad (2.10)$$

where

$$V_{abc} \equiv \exp[pa^\dagger + qb^\dagger + kc^\dagger + (a^\dagger, b^\dagger)_- + (b^\dagger, c^\dagger)_- + (c^\dagger, a^\dagger)_-]$$

and

$$(A_a + A_b{}^\dagger - A_c{}^\dagger) W_{abc}|0\rangle_{abc} = 0, \quad (2.11)$$

$$(-R_a + \tfrac{1}{2}k^2 + A_b + R_c - \tfrac{1}{2}q^2) W_{abc}|0\rangle_{abc} = 0, \quad (2.12)$$

$$(A_a{}^\dagger - R_b + \tfrac{1}{2}p^2 - A_c) W_{abc}|0\rangle_{abc} = 0. \quad (2.13)$$

V_{abc} is the symmetric vertex given in Fig. 5. One set can be derived from the other by multiplying by Ω_c.

With these identities, we can now state the rule for twisting the legs of the vertex function: Always twist *away* from the vertex, apply the Ward identity, and then use the formulas of Brower and Weis to eliminate all S's. (Since the twist operator was derived from its action upon scalar trees, we expect trouble when it is allowed to act indiscriminately on three-Reggeon functions. The motivation for twisting away from the vertex function is that scalar trees presumably are then being twisted.)

The proper way in which to twist on each of the various legs is as follows:

$$\begin{aligned}(1-P^\dagger)_b \Omega_b{}^\dagger D_a \hat{D}_b D_c{}' W_{abc}|0\rangle_{abc} &= (1-P^\dagger)_b D_a \hat{D}_b D_c{}' (1-z_b)^{-R_c+\alpha(q^2)} \\ &\quad \times (1-z_b)^{R_a-\alpha(k^2)} W_{abc}{}'|0\rangle_{abc} \\ &= (1-P^\dagger)_b D_a{}' \hat{D}_b D_c W_{abc}{}'|0\rangle_{abc}.\end{aligned} \quad (2.14)$$

We have used (2.12).

$$\begin{aligned}(1-P^\dagger)_a(1-P^\dagger)_b(1-P^\dagger)_c \Omega_a{}^\dagger D_a \hat{D}_b D_c{}' W_{abc}|0\rangle_{abc} &= (1-P^\dagger)_a(1-P^\dagger)_b(1-P^\dagger)_c D_a \hat{D}_b D_c{}' S_b{}^\dagger (S_c{}^\dagger)^{-1} \\ &\quad \times W_{abc}{}''|0\rangle_{abc} \\ &= (1-P^\dagger)_a(1-P^\dagger)_b(1-P^\dagger)_c D_a D_b{}' \hat{D}_c W_{abc}{}''|0\rangle_{abc}.\end{aligned} \quad (2.15)$$

We have used (2.11) and (2.4).

The last step in the identity is possible because the spurion operator A can pass through a modified propagator until it annihilates on $1-P$. [See (3.1) for an example of this.]

$$\begin{aligned}(1-P^\dagger)_a(1-P^\dagger)_c \Omega_c{}^\dagger D_a \hat{D}_b D_c{}' W_{abc}|0\rangle_{abc} &= (1-P^\dagger)_a(1-P^\dagger)_c D_a D_b{}' \hat{D}_c (1-z_c)^{A_a{}^\dagger + \frac{1}{2}m^2} \\ &\quad \times (1-z_c)^{-R_b+\alpha(p^2)} V_{abc}|0\rangle_{abc} \\ &= (1-P^\dagger)_a(1-P^\dagger)_c D_a D_b D_c V_{abc}|0\rangle_{abc}.\end{aligned} \quad (2.16)$$

We have used (2.4) and (2.10).

In the original paper by Caneschi and Schwimmer, the troublesome $(1-z)^R$ factor disappears through a commutation past Ω:

$$\begin{aligned}(1-P)_c \Omega_c (1-z_b)^{R_c-\alpha} W_{abc}|0\rangle_{abc} &= (1-P)_c (1-z_b)^{A_c{}^\dagger + \frac{1}{2}m^2} \Omega_c W_{abc}|0\rangle_{abc} \\ &= (1-P)_c \Omega_c W_{abc}|0\rangle_{abc}.\end{aligned}$$

In this case, twisting toward the vertex is allowed because the three-Reggeon vertex function, looking from the "a" or "c" legs, is actually a tree in disguise. Confusion arises, however, if one tries to twist toward the symmetric vertex to recover the Sciuto vertex; the vertex itself reappears, but the $(1-z)^R$ does not. This difficulty is resolved if we twist away from the symmetric vertex:

$$\begin{aligned}(1-P^\dagger)_a \Omega_c{}^\dagger D_a D_b D_c V_{abc}|0\rangle_{abc} &= (1-P^\dagger)_a D_a D_b D_c (1-z_c)^{R_b-\alpha(p^2)} (1-z_c)^{-A_a{}^\dagger - \frac{1}{2}m^2} \\ &\quad \times W_{abc}|0\rangle_{abc} \\ &= (1-P^\dagger)_a D_a \hat{D}_b D_c{}' W_{abc}|0\rangle_{abc}.\end{aligned} \quad (2.17)$$

We have used (2.4) and (2.13).

The previous calculations reveal that the $(1-z)^R$ factor is necessary to absorb all gauge terms $(1-z)^A$ arising from the Ward identities, which are transformed by the Brower-Weis identities.

III. LINEAR DEPENDENCES AND N-LOOP AMPLITUDE

The elimination of all linear dependences from n-loop diagrams requires that the identities presented earlier be generalized to include the commutation of A past propagators which are no longer simple beta functions:

$$(A_a + \tfrac{1}{2}m^2) N_{abc}|0\rangle_{abc} = [(1-z_b) A_c{}^\dagger - z_b A_b{}^\dagger] N_{abc}|0\rangle_{abc}, \quad (3.1)$$

$$(A_c + \tfrac{1}{2}m^2) N_{abc}|0\rangle_{abc} = [(1-z_b) A_a{}^\dagger] N_{abc}|0\rangle_{abc}, \quad (3.2)$$

$$(A_b + \tfrac{1}{2}m^2) N_{abc}|0\rangle_{abc} = [z_b (R_a - \alpha(k^2))] N_{abc}|0\rangle_{abc}, \quad (3.3)$$

where

$$\begin{aligned}N_{abc}|0\rangle_{abc} \equiv \int_0^1 dz_a \int_0^1 dz_b \int_0^1 dz_c\, z_a{}^{R_a-\alpha(k^2)-1} &\times z_b{}^{R_b-\alpha(p^2)-1} z_c{}^{R_c-\alpha(q^2)-1} (1-z_b)^{R_c-\alpha(q^2)} \\ &\times W_{abc}|0\rangle_{abc} (1-z_a)^{-c} (1-z_b)^{-c} (1-z_c)^{-c}.\end{aligned}$$

(It is understood that all z's occur inside the integrals.) The plan for removing all spurious states from n-loop diagrams is surprising simple: First, notice that projection operators occurring along the base line of Fig. 6 are removed by (3.1) and (3.2), much like in the case of a multiperipheral tree; second, notice that projection operators left remaining in each loop do *not* vanish because all spurion operators A and $A^\dagger$ get converted into R's by (3.3); third, because the projection operator is a hypergeometric function in A and $A^\dagger$,[6] we expect hypergeometric functions in R to accumulate along the base line of Fig. 6; and lastly, these hypergeometric functions reduce to trivial rational functions of R. The only critical step in the whole procedure is to verify that hypergeometric functions of R do, in fact, appear along the base line.

Toward this goal, we first present the projected vertex function as a hypergeometric function[6]:

$$\hat{V}=(1-P)V(1-P^\dagger)=V+\frac{(A^\dagger+\frac{1}{2}m^2)V(A+\frac{1}{2}m^2)}{c\,1!}+\frac{(A^\dagger+\frac{1}{2}m^2)(A^\dagger+\frac{1}{2}m^2+1)V(A+\frac{1}{2}m^2+1)(A+\frac{1}{2}m^2)}{c(c+1)2!}+\cdots \quad (3.4)$$

and

$$(A+n)(A+n-1)\cdots(A+1)A\prod_{m=1}^{r}(V_mD_m{}^0)=\prod_{m=1}^{r}(V_mD_m{}^{n+1})(A+n)(A+n-1)\cdots(A+1)A\,, \quad (3.5)$$

where

$$D_m{}^{n+1}\equiv\int_0^1 x^{R-\alpha_m+n}(1-x)^{-c}dx.$$

We see that in much the same way as A picks up a factor of x by commuting past vertices and propagators, $(A+n)\cdots A$ picks up a factor of x^{n+1} (where x is the product of all propagator variables). Immediately we see that a projected vertex placed anywhere along the upper section of each loop in Fig. 7 decomposes because of (3.5) until all A's accumulate on the "b" legs of the Sciuto vertices. It is now a simple matter to generalize (3.3), in much the same way as (3.5) is a generalization of $AVD^0=VD^1A$. As in the derivation of (3.3) itself, we require that extraneous terms vanish because of certain total derivatives:

$$M_{abc}|0\rangle_{abc}\equiv\int_0^1 dz_b z_b{}^{R_b-\alpha(p^2)-1}(1-z_b)^{R_c-\alpha(q^2)-c}W_{abc}|0\rangle_{abc}.$$

It follows that

$$(A_b+\tfrac{1}{2}m^2)M_{abc}|0\rangle_{abc}=\{z_b[R_a-\alpha(k^2)]-z_b(R_c-\tfrac{1}{2}q^2)+(1-z_b)[R_b-\alpha(p^2)]\}M_{abc}|0\rangle_{abc}=z_b[R_a-\alpha(k^2)]M_{abc}|0\rangle_{abc}. \quad (3.6)$$

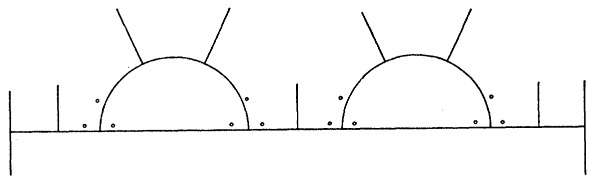

FIG. 6. N-loop amplitude formed by inserting Sciuto vertices to a multiperipheral tree.

We have used the following fact:

$$\frac{d}{dz_b}[z_b{}^{R_b-\alpha(p^2)}(1-z_b)^{R_c-\alpha(q^2)-c+1}]=\{-z_b(R_c-\tfrac{1}{2}q^2)+[R_b-\alpha(p^2)](1-z_b)\}\times[z_b{}^{R_b-\alpha(p^2)-1}(1-z_b)^{R_c-\alpha(q^2)-c}]. \quad (3.7)$$

In general, we find that

$$(A_b+\tfrac{1}{2}m^2+n)(A_b+\tfrac{1}{2}m^2+n-1)\cdots(A_b+\tfrac{1}{2}m^2)\times M_{abc}|0\rangle_{abc}=z_b{}^{n+1}(R_a-\alpha+n)\times(R_a-\alpha+n-1)\cdots(R_a-\alpha)M_{abc}|0\rangle_{abc}. \quad (3.8)$$

Consider the planar n-loop amplitude (Fig. 6). The original projected vertex situated somewhere along the upper portion of each loop decomposes by sending A's to the left and right. These A's, in turn, collect on the "b" legs of Sciuto vertices, which in turn converts them into R's. The two halves of the projected vertex are finally reunited; instead of a hypergeometric function in A and $A^\dagger$, it is now a hypergeometric function in R (Fig. 7). It is not hard to reduce this function, which lies next to the original beta function situated between Sciuto vertices:

$$(R-\alpha)D=(R-\alpha)\frac{\Gamma(1-c)\Gamma(R-\alpha)}{\Gamma(1-c+R-\alpha)}=\frac{-c\Gamma(-c)\Gamma(R-\alpha+1)}{\Gamma(1-c+R-\alpha)}=\int_0^1 t^{R-\alpha-1}(1-t)^{-c}\left(\frac{-ct}{1-t}\right)dt. \quad (3.9)$$

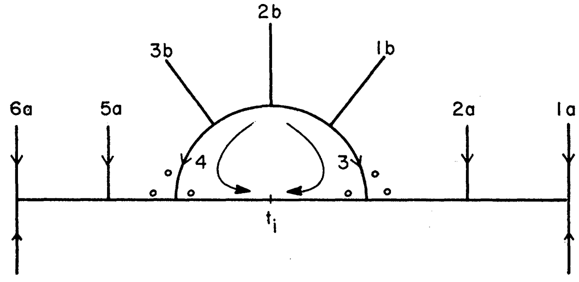

FIG. 7. Decomposing and recombination of the hypergeometric function into a modified propagator.

In general

$$
\begin{aligned}
{}_2F_1(R-\alpha, R-\alpha; c; x)D
&=\sum_{n=0}^{\infty}\int_0^1 t^{R-\alpha-1}(1-t)^{-c} \\
&\qquad\times\left[(-1)^n\binom{R-\alpha+n-1}{n}\left(\frac{xt}{1-t}\right)^n\right] \\
&=\int_0^1 t^{R-\alpha-1}(1-t)^{-c}\left(\frac{1-t}{1-t(1-x)}\right)^{R-\alpha}. \qquad (3.10)
\end{aligned}
$$

This is one of the main results of this paper: The effect of eliminating *all* linear dependences from a planar n-loop amplitude is essentially a trivial one. The prescription is merely to modify the original beta function situated between a pair of Sciuto vertices by a factor of

$$\left(\frac{1-t_i}{1-t_i(1-x_i)}\right)^{R-\alpha},$$

where x_i is the product of all propagator variables in the upper part of the ith loop, and t_i is the propagator variable of the lower loop.

To illustrate the entire procedure, we shall eliminate linear dependences from the single planar loop of Fig. 7. We will use the following definitions:

$$
\begin{aligned}
W_{aba}' &\equiv \exp[p_1a^\dagger+(a^\dagger,b^\dagger)_+] \\
&\qquad\times\exp[p_1a+(a,b^\dagger)_-]\exp(p_2b^\dagger), \\
W_{aba} &\equiv \exp[-p_1a^\dagger+(a^\dagger,b)_-] \\
&\qquad\times\exp[-p_1a+(a,b)_+]\exp(p_3b).
\end{aligned}
$$

Now we begin:

$$
\begin{aligned}
A=\langle 0|{}_{ab}V_1{}^a(1-P^\dagger)^a D_1{}^a V_2{}^a D_2{}^{a'} W_{aba}'(1-P)^a \\
\times D_3{}^a[\hat{D}_1{}^b V_1{}^b D_2{}^b(1-P)^b V_2{}^b D_3{}^b V_3{}^b \hat{D}_4{}^b]W_{aba} \\
\times D_4{}^{a'} V_5{}^a D_5{}^a V_6{}^a|0\rangle_{ab}. \qquad (3.11)
\end{aligned}
$$

The "a" operators represent the base line and the "b" operators the loop. Using (3.4), we find

$$
\begin{aligned}
A=\sum_{n=0}^{\infty}\langle 0|{}_{ab}V_1{}^a D_1{}^a V_2{}^a D_2{}^{a'} W_{aba}' D_3{}^a \hat{D}_1{}^b \\
\times\binom{A_b{}^\dagger+\frac{1}{2}m^2+n-1}{n}x_2{}^n V_1{}^b D_2{}^b V_2{}^b D_3{}^b V_3{}^b \\
\times\binom{A_b+\frac{1}{2}m^2+n-1}{n}x_3{}^n \hat{D}_4{}^b W_{aba} D_4{}^{a'} V_5{}^a D_5{}^a V_6{}^a|0\rangle_{ab} \\
\times\binom{c+n-1}{n}^{-1}. \qquad (3.12)
\end{aligned}
$$

Using (3.8), we find

$$
\begin{aligned}
A=\sum_{n=0}^{\infty}\langle 0|{}_{ab}V_1{}^a D_1{}^a V_2{}^a D_2{}^{a'} W_{aba}' D_3{}^a \\
\times\binom{R_a-\alpha+n-1}{n}\hat{D}_1{}^b x_1{}^n x_2{}^n V_1{}^b D_2{}^b V_2{}^b D_3{}^b V_3{}^b \\
\times\binom{R_a-\alpha+n-1}{n}x_3{}^n x_4{}^n \hat{D}_4{}^b W_{aba} D_4{}^{a'} V_5{}^a D_5{}^a V_6{}^a|0\rangle_{ab} \\
\times\binom{c+n-1}{n}^{-1}. \qquad (3.13)
\end{aligned}
$$

Finally, using (3.10), we get

$$
\begin{aligned}
A=\langle 0|{}_{ab}V_1{}^a D_1{}^a V_2{}^a D_2{}^{a'} W_{aba}' D_3{}^{a''}(\hat{D}_1{}^b V_1{}^b D_2{}^b V_2{}^b \\
\times D_3{}^b V_3{}^b \hat{D}_4{}^b) W_{aba} D_4{}^{a'} V_5{}^a D_5{}^a V_6{}^a|0\rangle_{ab},
\end{aligned}
$$

where

$$
D_3{}^{a''}\equiv\int_0^1 x^{R_a-\alpha-1}(1-x)^{-c}\times\left[\frac{1-x}{1-x(1-x_1x_2x_3x_4)}\right]^{R_a-\alpha}dx. \qquad (3.14)
$$

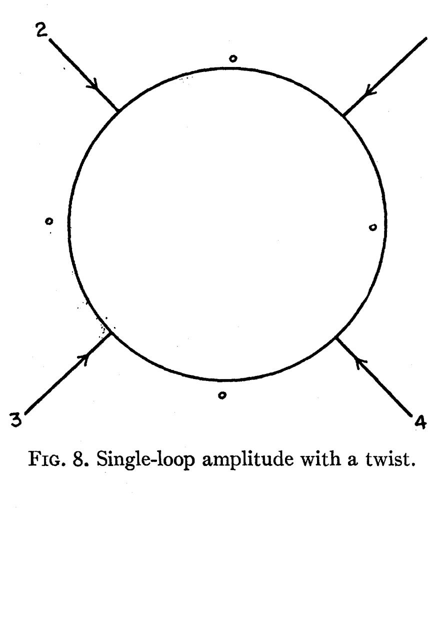

FIG. 8. Single-loop amplitude with a twist.

Similarly, linear dependences can be eliminated from amplitudes like that of Fig. 8.

The generalization to nonplanar and overlapping loops proceeds as before, except that the correction factor is even simpler than the planar case.

Consider the nonplanar and overlapping loop amplitudes of Fig. 9 or any combination. Projection operators along the base line disappear because of (3.1) and (3.2), while projection operators remaining in each loop decompose as before via (3.5) and (3.8). In contrast to the planar case, however, the two halves of the operator expressions in the nonplanar and overlapping case fail to combine. Nevertheless, we can re-express the operators appearing in each half as c-number expressions by extracting polynomials in the propagator variables. These c-number expressions *do* recombine, yielding a simple rational c-number expression as the correction term due to linear dependence.

As an illustration of this technique, linear dependences will be eliminated from the amplitude pictured in Fig. 10:

$$A=\langle 0|_{ab}V_1{}^a(1-P^\dagger)^a D_1{}^a V_2{}^a D_2{}^{a\prime} W_{aba}{}' \times D_3{}^a(1-P)^a V_4{}^a D_4{}^a V_5{}^a D_5{}^a[\hat{D}_1{}^b V_1{}^b (I-P^\dagger)^b \times D_2{}^b V_2{}^b D_3{}^b V_3{}^b \hat{D}_4{}^b] W_{aba} D_6{}^{a\prime} V_7{}^a D_7{}^a V_8{}^a|0\rangle_{ab}, \quad (3.15)$$

$$A=\sum_{n=0}^{\infty}\binom{c+n-1}{n}\left[\frac{z_3z_5x_1x_2x_3x_4}{(1-z_3)(1-z_5)}\right]^n \times\langle 0|_{ab}V_1{}^a D_1{}^a V_2{}^a D_2{}^{a\prime} W_{aba}{}' D_3{}^a V_4{}^a D_4{}^a \times V_5{}^a D_5{}^a(\hat{D}_1{}^b V_1{}^b D_2{}^b V_2{}^b D_3{}^b V_3{}^b \hat{D}_4{}^b) \times W_{aba} D_6{}^{a\prime} V_7{}^a D_7{}^a V_8{}^a|0\rangle_{ab}. \quad (3.16)$$

But

$$\sum_{n=0}^{\infty}\binom{c+n-1}{n}\left[\frac{z_3z_5x_1x_2x_3x_4}{(1-z_3)(1-z_5)}\right]^n =\left[1-\frac{z_3z_5x_1x_2x_3x_4}{(1-z_3)(1-z_5)}\right]^{-c}. \quad (3.17)$$

Quite simply, the linear dependence factor for nonplanar loop amplitudes is a c number. Since the order in which the Sciuto vertices occur along the base line is never specified, this procedure works equally well for arbitrary mixtures of nonplanar, planar, and overlapping amplitudes. We can generalize all the results in this section into two simple rules, which apply for configurations like that of Fig. 9.

Rule I: For every planar loop appearing along the base line, replace the standard beta function situated between the two Sciuto vertices with

$$D\to\int_0^1 t^{R-\alpha-1}(1-t)^{-c}\left[\frac{1-t}{1-t(1-x)}\right]^{R-\alpha}dt, \quad (3.18)$$

where x is the product of all propagator variables in the upper portion of the loop.

Rule II: For every nonplanar or overlapping loop configuration, merely add the c-number expression

$$[1-z_iz_jx/(1-z_i)(1-z_j)]^{-c}, \quad (3.19)$$

where x is again the product of all propagator variables along the upper portion of the loop, and i and j refer to the propagators adjacent to the position of the two Sciuto vertices (see Fig. 10).

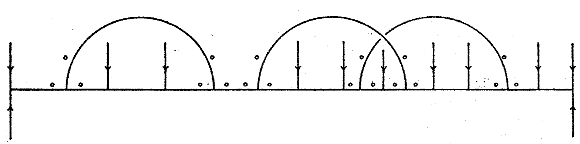

FIG. 9. Arbitrary planar, nonplanar, and overlapping n-loop diagram.

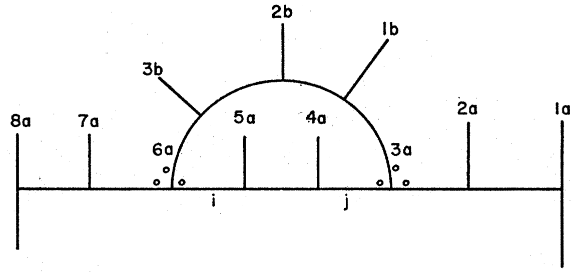

FIG. 10. Nonplanar diagram discussed in test.

IV. LINEAR DEPENDENCES AND PROJECTED PROPAGATOR OF GROSS AND SCHWARZ AND IDA

Since the projected propagator of Gross and Schwarz and Ida expresses the twisted propagator with all spurious states subtracted out, it is independent of the configuration and hence more general than the rules presented here. To establish an equivalence between the two, we shall remove linear dependences from planar, nonplanar, and overlapping loop amplitudes (constructed as before) via the projected propagator and rederive Rules I and II.

The rederivation of Rules I and II shall be performed as follows: Two symmetric vertex functions are joined together (as in Fig. 11) with projected propagators sandwiched in between; the twist operators contained in the projected propagators are used to reverse the position of the dots until they agree with Fig. 14; then a transformation of variables is made to achieve agreement with Rules I and II.

We shall use the form of the projected propagator expressed in terms of one twist operator:

$$\bar{D}\equiv(1-P^\dagger)D\Omega(1-P)=DS^{-1}\Omega. \quad (4.1)$$

(The variable contained in S is, as usual, understood to lie beneath the intergral contained in D.)

Since the manipulations involve repeated use of Eqs. (2.8)–(2.10), the procedure shall first be explained before any equations are presented. At each step, the twist operator contained in the projected propagator successively reverses the positions of the dots until they

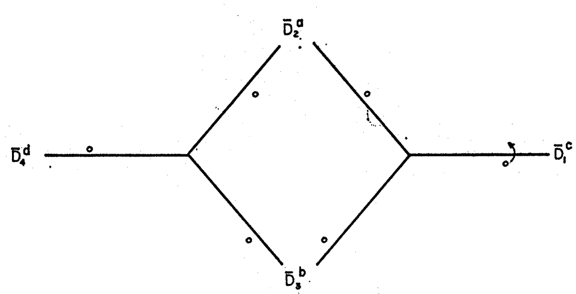

FIG. 11. Twist operator contained in $\bar{D}_1{}^c$ reverses the position of the dot.

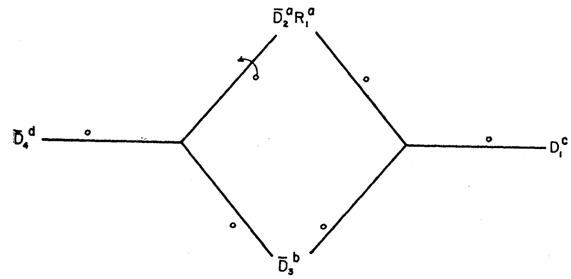

FIG. 12. Gauge term $(S_1{}^c)^{-1}$ contained in $\bar{D}_1{}^c$ converts to $R_1{}^a$ and $S_1{}^b$ (which annihilates on $\bar{D}_3{}^b$)—the twist operator in $\bar{D}_2{}^a$ reverses the dot.

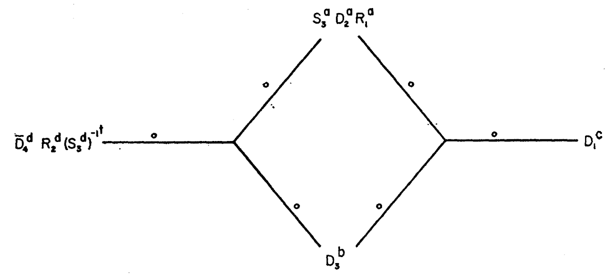

FIG. 14. Gauge term $(S_3{}^b)^{-1}$ contained in $\bar{D}_3{}^b$ converts to $S_3{}^a$ and $(S_3{}^d)^{-1\dagger}$—they commute past $R_2{}^d$ and $D_2{}^aR_1{}^a$, respectively.

agree with the configuration composed of two Sciuto vertices (Figs. 11–14). (Traditionally when the twist operator Ω_a acts on a three-Reggeon vertex, the "b" and "c" legs are reversed in the figure, while the "a" leg is kept constant. Since this convention leads to considerable geometric complications when performing several twists, we shall adopt the equivalent custom of reversing only the dot on the "a" leg in the figure whenever Ω_a acts on the vertex, contrary to tradition.)

First, the twist operator contained in $\bar{D}_1$ reverses the position of one dot (thereby creating a Sciuto vertex); the gauge term contained in $\bar{D}_1$ [via (2.8)] splits into a gauge term near $\bar{D}_3$ (which annihilates upon the projection operator) and a term $R_1=(1-x_1)^R$ situated near $\bar{D}_2$; see Fig. 12.

Second, the twist operator contained in $\bar{D}_2$ reverses the dot on the other vertex function; the gauge term in $\bar{D}_2$ [via (2.8)] again splits into a gauge term (which annihilates on the projection operator contained in $\bar{D}_3$) and $R_2(1-x_2)^R$ situated near $\bar{D}_4$ [see Fig. 13; $\bar{D}_4\equiv D_4(1-P)$].

Third, the twist operator in $\bar{D}_3$ makes the last dot change; two gauge terms (S_3^{-1} near D_4 and S_3 near D_2) are then created via (2.11). The gauge term near D_4 commutes past R_2 and annihilates on the projection operator contained in the adjacent loop function; the gauge term near D_2 commutes past R_1D_2, and finally converts into rational functions of R via (2.12). Notice now that the dot configuration is identical to the one studied previously; all that remains is to verify that all

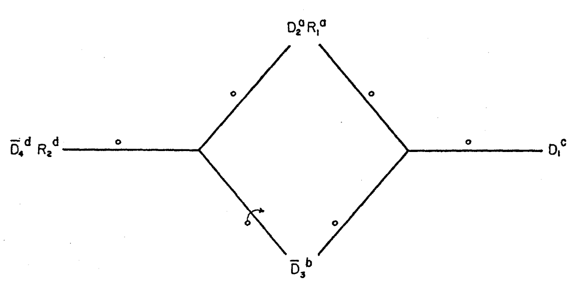

FIG. 13. Gauge term $(S_2{}^a)^{-1}$ contained in $\bar{D}_2{}^a$ converts to $R_2{}^d$ and $S_2{}^b$ (which annihilates on $\bar{D}_3{}^b$)—the twist operator in $\bar{D}_3{}^b$ reverses the next dot.

c-number expressions are identical once the transformation of variables is known (Fig. 14).

One last identity is needed before the proof is presented: We must know the commutation relation between $(1-z)^A$ and x^R. This is accomplished by using an operator identity presented in an earlier paper[2]:

$$e^{-z\bar{A}^\dagger(-\pi)}=(1-z)^{A^\dagger(-\pi)}(1-z)^{-R}(1-z)^{\frac{1}{2}\pi^2}. \quad (4.2)$$

But

$$y^R e^{-z\bar{A}^\dagger}=e^{-zy\bar{A}^\dagger}y^R.$$

Therefore,

$$y^R(1-z)^{A^\dagger}=(1-zy)^{A^\dagger}\left[\frac{y(1-z)}{1-zy}\right]^R\left[\frac{1-zy}{1-z}\right]^{\frac{1}{2}\pi^2}.$$

The proofs of Rules I and II are now straightforward:

$$A_{cd}=\sum_{\lambda_a,\lambda_b}\langle\lambda_a|\langle\lambda_b|\bar{D}_3{}^b\bar{D}_2{}^a\bar{D}_1{}^cV_{abc}'|0\rangle_{abc} \times\langle 0|_{abd}V_{abd}''\bar{D}_4{}^d|\lambda_a\rangle|\lambda_b\rangle, \quad (4.3)$$

where

$$\bar{D}_4{}^d\equiv D_4{}^d(1-P)^d,$$

$$V_{abc}'|0\rangle_{abc}\equiv\exp[\pi_1a^\dagger+\pi_2b^\dagger+\pi_3c^\dagger \times(c^\dagger,b^\dagger)_-+(b^\dagger,a^\dagger)_-+(a^\dagger,c^\dagger)_-]|0\rangle_{abc}$$

and

$$\langle 0|_{abd}V_{abd}''\equiv\langle 0|_{abd}\exp[-\pi_3a-\pi_2d+\pi_4b +(b,d)_-+(d,a)_-+(a,b)_-].$$

We now use (2.8) and (4.1):

$$A_{cd}=\sum_{\lambda_a,\lambda_b}\langle\lambda_a|\langle\lambda_b|\bar{D}_3{}^bD_2{}^aD_1{}^c(1-x_1)^{R_a}W_{abc}' \times|0\rangle_{abc}\langle 0|_{abd}V_{abd}''\Omega_a{}^\dagger(1-x_2)^{R_d-\frac{1}{2}\pi_4{}^2} \times|\lambda_a\rangle|\lambda_b\rangle(1-x_1)^{-\frac{1}{2}\pi_2{}^2-\frac{1}{2}m^2}(1-x_2)^{-\frac{1}{2}m^2}. \quad (4.4)$$

We shall now make use of (2.11) and (4.1):

$$A_{cd}=\sum_{\lambda_a,\lambda_b}\langle\lambda_a|\langle\lambda_b|D_3{}^bD_1{}^cS_3{}^aD_2{}^a(1-x_1)^{R_a} \times W_{abc}'|0\rangle_{abc}\langle 0|_{abd}W_{abd}(S_3{}^{-1})^d(1-x_2)^{R_d} \times\bar{D}_4{}^d|\lambda_a\rangle|\lambda_b\rangle(1-x_1)^{-\frac{1}{2}\pi_2{}^2-\frac{1}{2}m^2}(1-x_2)^{-\frac{1}{2}\pi_4{}^2-\frac{1}{2}m^2}. \quad (4.5)$$

At this point in the calculation, we shall make extensive use of (4.2):

$$S_3{}^aD_2{}^a(1-x_1)^{R_a}\rightarrow(1-x_3)^{A_a}x_2{}^{R_a}(1-x_1)^{R_a}$$
$$=\left[\frac{(1-x_1)(1-x_3)x_2}{1-x_3x_2(1-x_1)}\right]^{R_a}[1-x_2x_3(1-x_1)]^{A_a}$$
$$\times\left[\frac{1-x_2x_3(1-x_1)}{1-x_3}\right]^{\frac{1}{2}\pi^2} \tag{4.6}$$

and

$$(S_3{}^{-1})^d(1-x_2)^{R_d}=(1-x_3)^{-A_d}(1-x_2)^{R_d}$$
$$=\left(\frac{1-x_2}{1-x_2x_3}\right)^{R_d}\left(\frac{1-x_2x_3}{1-x_3}\right)^{A_d}(1-x_3x_3)^{\frac{1}{2}\pi^2}.$$

We now put everything together:

$$A_{cd}=\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\int_0^1 dx_4\sum_{\lambda_a,\lambda_b}\langle\lambda_a|\langle\lambda_b|\left[\frac{x_1}{1-x_2x_3(1-x_1)}\right]^{Rc}\left[\frac{x_2(1-x_1)(1-x_3)}{1-x_2x_3(1-x_1)}\right]^{R_a}$$
$$\times\{x_3[1-x_2x_3(1-x_1)]\}^{R_b}W_{abc}{}'|0\rangle_{abc}\langle 0|_{abd}W_{abd}\left[\frac{x_4(1-x_2)}{1-x_2x_3}\right]^{R_d}(1-P)^d|\lambda_a\rangle|\lambda_b\rangle$$
$$\times(1-x_1)^{-c}(1-x_2)^{-c}(1-x_3)^{-c}(1-x_4)^{-c}x_1{}^{-\alpha(\pi_1{}^2)-1}x_2{}^{-\alpha(\pi_2{}^2)-1}x_3{}^{-\alpha(\pi_3{}^2)-1}x_4{}^{-\alpha(\pi_4{}^2)-1}$$
$$\times(1-x_2x_3)^{\frac{1}{2}\pi_4{}^2}\left(\frac{1-x_2x_3}{1-x_3}\right)^{-\frac{1}{2}m^2}\left[\frac{1-x_2x_3(1-x_1)}{1-x_3}\right]^{\frac{1}{2}\pi_2{}^2}(1-x_1)^{-\frac{1}{2}\pi_2{}^2}(1-x_1)^{-\frac{1}{2}m^2}(1-x_2)^{-\frac{1}{2}m^2-\frac{1}{2}\pi_4{}^2}. \tag{4.7}$$

Now compare this to the expression that we desire [note that we make use of (2.4) so that the joining of two "b" legs is well-defined]:

$$A_{cd}=\int_0^1 dz_1\int_0^1 dz_2\int_0^1 dz_3\int_0^1 dz_4\sum_{\lambda_a,\lambda_b}\langle\lambda_a|\langle\lambda_b|z_1{}^{R_c}z_2{}^{R_a}(1-z_1)^{R_a}\left\{\frac{z_3(1-z_3)}{1-z_3[1-z_2(1-z_1)]}\right\}^{R_b}$$
$$\times W_{abc}{}'|0\rangle_{abc}\langle 0|_{abd}W_{abd}z_4{}^{R_d}(1-z_2)^{R_d}(1-P)^d|\lambda_a\rangle|\lambda_b\rangle(1-z_1)^{-c}(1-z_2)^{-c}(1-z_3)^{-c}(1-z_4)^{-c}$$
$$\times(1-z_1)^{-\alpha(\pi_2{}^2)}z_3{}^{-\alpha(\pi_3{}^2)-1}\left\{\frac{(1-z_3)}{1-z_3[1-z_2(1-z_1)]}\right\}^{-\alpha(\pi_3{}^2)}(1-z_2)^{-\alpha(\pi_4{}^2)}z_4{}^{-\alpha(\pi_4{}^2)-1}z_1{}^{-\alpha(\pi_1{}^2)-1}z_2{}^{-\alpha(\pi_2{}^2)-1}. \tag{4.8}$$

We can now simply read off the transformation of variables:

$$\begin{aligned}z_1&=x_1[1-x_2x_3(1-x_1)]^{-1},\\ z_2(1-z_1)&=x_2(1-x_1)(1-x_3)[1-x_2x_3(1-x_1)]^{-1},\\ z_3(1-z_3)\{1-z_3[1-z_2(1-z_1)]\}^{-1}&=x_3[1-x_2x_3(1-x_1)],\\ z_4(1-z_2)&=x_4(1-x_2)(1-x_2x_3)^{-1}.\end{aligned} \tag{4.9}$$

Verifying the equivalence of the c-number expressions is straightforward but tedious.

The case with nonplanar and overlapping loops proceeds just like before, except that an arbitrary number of ordinary or Sciuto vertices are allowed to sit within the loop. The one critical step which differs from the previous derivation is the last. $S_3{}^a$ splits into rational functions of R_c and R_b via (2.12) except that the function of R_b does not combine with $D_3{}^a$:

$$A_{cd}=\sum_{\lambda_a,\lambda_b}\langle\lambda_a|\langle\lambda_b|(\prod_{i=5}^{n}D_i{}^bV_i{}^b)\bar{D}_3{}^b\bar{D}_2{}^b\bar{D}_1{}^bV_{abc}{}'|0\rangle_{abc}\langle 0|_{abd}V_{abd}{}''\bar{D}_4{}^d|\lambda_a\rangle|\lambda_b\rangle$$
$$=\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\int_0^1 dx_4\int_0^1 dx_5\sum_{\lambda_a,\lambda_b}\langle\lambda_a|\langle\lambda_b|V_5{}^b(\prod_{i=6}^{n}D_i{}^bV_i{}^b)\left[\frac{x_1}{1-x_2x_5(1-x_1)}\right]^{R_c}$$
$$\times\left[\frac{x_2(1-x_1)(1-x_5)}{1-x_5x_2(1-x_1)}\right]^{R_a}W_{abc}{}'|0\rangle_{abc}\langle 0|_{abd}W_{abd}\left[\frac{x_4(1-x_2)}{1-x_2x_5}\right]^{R_d}\{x_3[1-x_5x_2(1-x_1)]\}^{R_b}$$
$$\times(1-P)^dx_5{}^{R_a}|\lambda_a\rangle|\lambda_b\rangle x_1{}^{-\alpha(\pi_1{}^2)-1}x_2{}^{-\alpha(\pi_2{}^2)-1}x_3{}^{-\alpha(\pi_3{}^2)-1}x_4{}^{-\alpha(\pi_4{}^2)-1}$$
$$\times x_5{}^{-\alpha(\pi_5{}^2)-1}(1-x_1)^{-c}(1-x_2)^{-c}(1-x_3)^{-c}(1-x_4)^{-c}(1-x_5)^{-c}\left[\frac{(1-x_1)(1-x_5)}{1-x_5x_2(1-x_1)}\right]^{-\frac{1}{2}\pi_2{}^2}\left[\frac{(1-x_2)}{1-x_2x_3}\right]^{-\frac{1}{2}\pi_4}$$
$$\times[1-x_5x_2(1-x_1)]^{-\frac{1}{2}\pi_1{}^2}[1-x_5x_2(1-x_1)]^{-\frac{1}{2}\pi_3{}^2}(1-x_2x_5)^{-\frac{1}{2}m^2}(1-x_5)^{\frac{1}{2}m^2}(1-x_1)^{-\frac{1}{2}m^2}(1-x_2)^{-\frac{1}{2}m^2}.$$

If our rules are correct, then this should equal

$$A_{cd}=\int_0^1 dz_1\int_0^1 dz_2\int_0^1 dz_3\int_0^1 dz_4\int_0^1 dz_5 \sum_{\lambda_a,\lambda_b} \langle\lambda_a|\langle\lambda_b|V_5{}^b(\prod_{i=6}^{n} D_i{}^bV_i{}^b)z_1{}^{R_c}z_2{}^{R_a}(1-z_1)^{R_a}$$

$$\times z_3{}^{R_b}W_{abc}{}'|0\rangle_{abc}\langle 0|_{abd}W_{abd}z_4{}^{R_d}(1-z_2)^{R_d}z_5{}^{R_a}|\lambda_a\rangle|\lambda_b\rangle$$

$$\times z_1{}^{-\alpha(\pi_1{}^2)-1}z_2{}^{-\alpha(\pi_2{}^2)-1}z_3{}^{-\alpha(\pi_3{}^2)-1}z_5{}^{-\alpha(\pi_5{}^2)-1}z_4{}^{-\alpha(\pi_4{}^2)-1}(1-z_1)^{-c}(1-z_2)^{-c}(1-z_3)^{-c}$$

$$\times(1-z_4)^{-c}(1-z_5)^{-c}(1-z_1)^{-\alpha(\pi_2{}^2)}(1-z_2)^{-\alpha(\pi_4{}^2)}\left[1-\frac{z_3z_5z_2(1-z_1)}{[(1-z_3)(1-z_5)]}\right]^{-c}.$$

The last term is the correction factor. We can now simply read off the transformation of variables:

$$z_1=x_1[1-x_5x_2(1-x_1)]^{-1},$$
$$z_2(1-z_1)=x_2(1-x_1)(1-x_5)[1-x_5x_2(1-x_1)]^{-1},$$
$$z_3=x_3[1-x_2x_5(1-x_1)],$$
$$(1-z_2)z_4=x_4(1-x_2)(1-x_2x_5)^{-1},$$
$$z_5=x_5.$$

Proving the equivalence of all c-number expressions is again a straightforward but tedious process.

The objection may be raised that linear dependences have not been removed for an arbitrary configuration. The dual properties of the three-Reggeon vertex function, however, allow us to pass from the configuration studied here to an arbitrary one. Several previous authors have elegantly proved the dual properties of the vertex function by expressing it in a symmetric four-Reggeon vertex function. Moreover, we have checked the dual properties of the Sciuto three-Reggeon vertex function directly by explicitly showing the equivalence between two four-Reggeon vertex functions composed out of unsymmetric vertex functions. The demonstration that the unsymmetric four-Reggeon vertex function is dual is straightforward but rather long and tedious, and hence will not be presented here. In addition, the fact that a scalar vertex may "pass" through a symmetric three-Reggeon vertex has also been shown directly. The major consequence of these two calculations is to prove the periodicity properties of planar and nonplanar multiloop amplitudes, even before actual traces are performed. In a previous paper on the nonplanar loop amplitude,[4] for example, the periodicity of the imaginary part was proven algebraically. But because the operators themselves have dual properties built in, the periodicity may be seen before the calculation is actually performed. In passing, we note that both calculations depend critically on the presence of $1-P$ on at least one leg.

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