in the combinations $q_n{}^a\delta^{ab}q_m{}^b$ and $q_n{}^a\epsilon^{ab}q_m{}^b$ which are invariant under rotations in the two-dimensional charge space, so the Feynman diagrams individually manifest chiral invariance discussed previously.[4] If all charge vectors are parallel, then only the δ^{ab} term contributes to the photon propagator and ordinary electrodynamics in a particular gauge is recovered.

The infinite series of Feynman diagrams representing the S operator (7.1) is not useful for practical calculations because of the large magnetic coupling constant $g^2/4\pi\approx(137)n$ implied by the charge-quantization condition, but its formal properties may be of interest. For example, one might hope to deduce from it the behavior of the scattering amplitude under Lorentz transformation. However, even this would not be simple, for the group property is not satisfied order by order [as is obvious from the transformation law (6.17)], and when the charge quantization condition holds, the same power of the charge appears in an infinite number of diagrams of different order. (By "order" one means here the number of vertices in a Feynman diagram.) We hope to return to the transformation law of Green's functions on another occasion.

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Kinematic Superstructures for Abnormal Couplings in the 8-π Dual Amplitude*

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We present a general method for including abnormal couplings in multiparticle dual amplitudes. We construct amplitudes from a sum of terms having kinematic superstructure and dual substructure; and we show how a tensorial analysis of the kinematic superstructure in various channels determines the normality of the trajectories involved. As an illustration, we give a specific solution for abnormal couplings in the 8-π amplitude in which the above analysis is carried out in detail, and is compared with previous 6-π amplitudes and four-point amplitudes for spin-1–spin-0 scattering.

I. INTRODUCTION

ANALYSES of the structure of the N-scalar-particle dual amplitude by Chan *et al.*[1] and by Koba and Nielsen[2] have shown that it predicts normal coupling at all three-point vertices. [A coupling is normal or abnormal if the product of the normalities of the three particles is $+1$ or -1. A particle is normal, $n=+1$, if it has parity $(-1)^J$ or abnormal, $n=-1$, if it has parity $-(-1)^J$.] The existence of abnormal vertices is essentially a complication due to spin; any vertex with two spinless particles conserves normality. The problem of choosing the normality of an internal trajectory thus first arises in the four-point functions in reactions like $\rho\pi\to\rho\pi$; the problem already occurs in the 3-π trajectories of the 6-π amplitude. We reexamine the previous analyses of four-point[3–7] and six-point[8,9] amplitudes which are concerned with prescribing the normalities of internal trajectories. From this analysis we are able to propose a procedure for writing amplitudes for N-π's with defined leading normality on internal trajectories. In particular we concern ourselves with having the ω-A_2 trajectory in certain 3-π channels of the 8-π amplitude.

For four-point amplitudes with external spinning particles, one usually writes the invariant Lorentz tensors contracted against helicities and makes use of the analysis of Gell-Mann *et al.*[10] to determine the normalities of the invariant amplitudes associated with them. This method is clearly impractical when we come to analyze processes involving high spins. This is especially true since invariant amplitudes tend to give normal couplings. The method we adopt involves the use of "noninvariant" amplitudes, whose use has been de-

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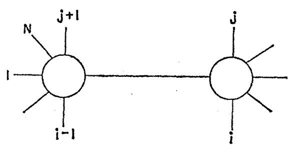

FIG. 1. Trajectory $\alpha(ij)$ in the N-point amplitude.

scribed by Canning.[11] By this we mean that we write an amplitude in the following form:

$$A=\sum_a K_a(P_i)B_N{}^a(\alpha_{jk}), \tag{1}$$

where

$$\alpha_{jk}=\alpha^0{}_{jk}+\alpha'(P_{jk})^2, \quad P_{jk}=P_j+P_{j+1}+\cdots+P_{k-1}+P_k.$$

The function $B_N{}^a$, which we call the dual substructure, is a function only of the internal trajectories and is of the form

$$B_N{}^a(\alpha_{jk})=\sum_b f_b{}^a(\alpha_{jk})B_N(\alpha_{jk}-m_{jk}{}^{a,b}), \tag{2}$$

B_N being the usual N-point Veneziano function, $m_{jk}{}^{a,b}$ small non-negative integers, and $f_b{}^a$ a simple polynomial in the α's used essentially for ghost-eliminating purposes. $K_a(P_i)$, which we call the kinematic superstructure, is a Lorentz scalar for all spinless external particles or a Lorentz tensor of appropriate rank to be contracted against the "wave function" of the external spinning particles. It is a function of the external momenta P_i and is designed to require certain trajectories to be of fixed leading normality. The sums over a and b are due to the different ghost-eliminating choices or trajectory-starting values and the fact that some channels may have more than one trajectory.

In practice, both K_a and $B_N{}^a$ are dependent on a particular ordering of the external particles $1, \ldots, N$ and, for all external pions, the sums over a and b should be taken over cyclic and anticyclic permutations with a given ordering. Then, the complete amplitude is given by

$$A=\sum_p \mathrm{Tr}(\tau_{1(p)}\tau_{2(p)}\cdots\tau_{N(p)})\sum_{a(p)} K_{a(p)}B_N{}^{a(p)}. \tag{3}$$

The sum over p is over all permutations of $1, \ldots, N$ and the trace is the Paton-Chan[12] isospin factor. This automatically guarantees that pions obey Bose statistics. Accordingly, it also gives alternating isospin along normal-parity trajectories, and allows no exotic isospins.

We now return to our discussion of K_a and $B_N{}^a$. As has been shown,[1] at a pole in $\alpha_{ij}=J$, the residue of B_N looks like

$$\sim\frac{1}{(J-m)!}T_{\mathrm{initial}}{}^{\mu_1\cdots\mu_{J-m}}T_{\mathrm{final}}{}^{\mu_1\cdots\mu_{J-m}}, \tag{4}$$

where $m=m_{ij}{}^{a,b}$ and the usual summation conventions on repeated Lorentz indices hold with the metric having $g_{00}=+1$ [see comment after Eq. (6) for further summation conventions]. T_{initial} is composed of momenta from one side of the graph in Fig. 1 and T_{final} from the other side only, and both T_{initial} and T_{final} contain leading-rank $(J-m)$ symmetric tensors (not pseudotensors). In general they also contain lower-rank tensors and pseudotensors.

We write the kinematic superstructure as

$$K_a=K_{\mathrm{initial}}{}^{\nu_1\cdots\nu_r}K_{\mathrm{final}}{}^{\nu_1\cdots\nu_r}, \quad r\geqslant m \tag{5}$$

with $K_{\mathrm{initial\ (final)}}$ constructed out of momenta on the one (other) side of Fig. 1 only. Then, the combination of superstructure and substructure has a leading trajectory α_{ij} coupling normally, abnormally, or parity doubled—depending on whether the highest-rank tensor in $K_{\mathrm{initial\ (final)}}$ (which rank is constructed to be m) is a tensor, pseudotensor, or linear combination of both. Thus, with appropriate choices of superstructure and substructure, internal trajectories can be made to have certain desired leading properties and the general N-point amplitudes can be written. The normalities of the leading particles on trajectories are determined solely by the superstructure, and it is this which we desire to analyze in the following sections.

In Sec. II we discuss certain superstructures in the four- and six-point amplitudes. From this analysis we are able to propose a constructive technique for the superstructures of the N(even)-π amplitudes. In Sec. III we analyze in detail those superstructures which are appropriate to the 8-π problem with an ω-A_2 trajectory in certain 3-π channels. Section IV is devoted to finding the substructures which occur with these superstructures, and Sec. V presents the conclusions.

II. ANALYSIS OF ABNORMAL COUPLING

In this section we discuss the properties of superstructures in four-point amplitudes, with specific reference to obtaining pure abnormally coupled trajectories. We then show how these superstructures generalize to the six-point amplitude and suggest a set of rules for their implementation in the N-point amplitude.

The first occurrence of both normality trajectories in one channel—2-to-2 scattering—is in spin $1_1{}^-\otimes 0_2{}^- \to 1_3{}^-\otimes 0_4{}^-$ reactions. The Veneziano model has usually been applied to the reactions by writing sums of modified beta functions for the invariant amplitudes associated with the Lorentz operators

$$g^{\mu\nu},\ P_2{}^\mu P_4{}^\nu,\ P_4{}^\mu P_2{}^\nu,\ (P_2{}^\mu P_2{}^\nu+P_4{}^\mu P_4{}^\nu),\ (P_2{}^\mu P_2{}^\nu-P_4{}^\mu P_4{}^\nu)$$

(only the first four if elastic), or other equivalent operators. If the 1^- particles are ρ, ω, or A_1, the leading normally coupled trajectory is the π-A_1, B, ρ-f, respectively. In order to have these trajectories non-parity-doubled, it has been shown[3–7] that linear combinations

[11] G. P. Canning, Phys. Rev. D 2, 1146 (1970).
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of these amplitudes with numerical coefficients must have prescribed asymptotic behavior. To obtain the abnormally coupled ω-A_2, ρ-f, or B trajectories, respectively, not parity doubled, one must relate certain asymptotic powers of the amplitudes by polynomials in Mandelstam variables and external masses.[3-7] Indeed, even to get a parity-doubled trajectory, the asymptotics are still related by Mandelstam variables. This result also holds true for the reactions $J_1\otimes 0_2^- \to J_3'\otimes 0_4^-$ (normality of J equals the normality of J'). There it has been shown by Jacobs[7] that combinations of invariant amplitudes with numerical coefficients and prescribed asymptotics give normally coupled trajectories, while parity-doubled or pure abnormally coupled trajectories require asymptotics related by Mandelstam variables and external masses.

These requirements of polynomials in external momenta are distinct from the polynomials in internal trajectory functions obtainable by identities among beta functions. These polynomials can be replaced by "noninvariant" amplitude superstructures. The structures

$$K^{\mu\nu}=\epsilon^{\mu\alpha\cdot\cdot}P_i{}^{\cdot}P_j{}^{\cdot}\epsilon^{\nu\alpha\cdot\cdot}P_k{}^{\cdot}P_1{}^{\cdot}A_{ijkl} \tag{6}$$

have been studied by Kosterlitz[4] and by Canning.[11] (In the interest of keeping extraneous superscripts to a minimum we adopt the following convention: a missing superscript in an $\epsilon^{\mu\nu\rho\sigma}$ shown by a dot indicates a contraction of that index with the first available Lorentz index of a four-momentum in the expression, which is also shown by a dot.) The Latin subscripts i, j, k, and l in Eq. (6) are intended to be labels only and are not summed over. In the direct (s) channel, A_{1234} contributes only to leading, abnormally coupled particles and lower parity-doubled particles. Thus, the form

$$K_{\text{initial}}{}^{\alpha}=\epsilon^{\cdot\alpha\cdot\cdot}(1^-)_1{}^{\cdot}P_1{}^{\cdot}P_2{}^{\cdot} \tag{7}$$

can be thought of as a superstructure coupling $1^-\otimes 0^-$ to $(1^-)_{12}{}^{\alpha}$ abnormally. In the crossed (u) channel, however (or equivalently A_{1432} in the s channel), it gives leading parity doublets starting at spin 2 which must be depressed.

The reaction $1_1^-\otimes 0_2^- \to 1^{3+}\otimes 0_4^-$ allows both normalities of trajectories, each coupling normally at one end and abnormally at the other. Analyzing this reaction, Canning[11] found that the operator

$$K^{\mu\nu}=\epsilon^{\mu\nu\cdot\cdot}P_1{}^{\cdot}P_2{}^{\cdot} \tag{8}$$

had the normal vertex at the (34) end, i.e., in its coupling to particles 3 and 4,

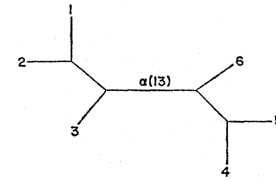
$$K^{\mu\nu}=\epsilon^{\mu\nu\cdot\cdot}P_3{}^{\cdot}P_4{}^{\cdot} \tag{9}$$

had the normal vertex at the (12) end, and

$$K^{\mu\nu}=\epsilon^{\mu\nu\cdot\cdot}P_1{}^{\cdot}P_3{}^{\cdot} \tag{10}$$

gave parity doubling. The remaining amplitudes involved more factors of P. All three of these operators,

FIG. 2. Trajectory $\alpha(13)$ in the 6-π reaction.



separately, have mixed parity in the u channel. The fact that Eq. (8) is normal at the (34) end is immediate from observing that $K_{\text{final}}{}^{\nu}$ is a vector (index ν) not involving momenta P_3 or P_4. Conversely, we have the form

$$K^{\nu}=\epsilon^{\cdot\nu\cdot\cdot}(1^-)_1{}^{\cdot}P_1{}^{\cdot}P_2{}^{\cdot}, \tag{11}$$

which couples abnormally ($\rho_1\pi_2 \to \omega_{12}$), since $K_{\text{initial}}{}^{\nu}$ is a pseudovector. If we had only the superstructures (8) and (9), which have definite normality in the s channel, we would obviously predict restricted ratios among couplings since we have eliminated some allowed operators.

In the 6-π amplitude it is possible to have the π-A_1 or ω-A_2 in 3-π channels. In the reduced graph of Fig. 2, for instance, we can be sure that the $\alpha(13)$ trajectory is the ω-A_2 by comparison with (8) when the $\alpha(12)$ is at the ρ pole by setting up

$$K_{\text{initial}}{}^{\mu}=\epsilon^{\mu\cdot\cdot\cdot}(1^-)_{12}{}^{\cdot}P_{12}{}^{\cdot}P_3{}^{\cdot}. \tag{12}$$

The "polarization vector" of the ρ, $(1^-)_{12}{}^{\nu}$, is made up of $P_1{}^{\nu}$ and $P_2{}^{\nu}$ and is perpendicular to $P_{12}{}^{\nu}$, i.e., for $P_1{}^2=P_2{}^2=m_\pi{}^2$, it is $P_1{}^{\nu}-P_2{}^{\nu}$. Corresponding results hold for $K_{\text{final}}{}^{\nu}$ at the (45) trajectory ρ pole. Our method of tensor analysis gives us directly that $K_{\text{initial}}{}^{\nu}$ is precisely a rank-one pseudotensor $(1^+)^{\nu}$. Hence the trajectory must start at 1^- since there are an odd number of pions, and it always has normal parity. More symmetrically, we have the identical form

$$K=\epsilon^{\mu\cdot\cdot\cdot}P_1{}^{\cdot}P_2{}^{\cdot}P_3{}^{\cdot}\epsilon^{\mu\cdot\cdot\cdot}P_4{}^{\cdot}P_5{}^{\cdot}P_6{}^{\cdot}, \tag{13}$$

showing that the superstructure is good also for the $\alpha(23)$ and $\alpha(56)$ trajectories. Dorren *et al.*[8] and Gabarro and Mestres[9] have, in fact, proposed this superstructure with appropriate substructure for the 6-π amplitude. We here derive its complete properties using our method of tensorial analysis.

For the trajectory $\alpha(12)$, we obtain from Eq. (13) $K_{\text{initial}}{}^{\mu}=(1^-)_{12}{}^{\mu}$, which, from our comments about substructure in the Introduction, shows that $\alpha(12)$ is of positive normality starting at $J=1$. Equivalent results hold for $\alpha(23)$, $\alpha(45)$, and $\alpha(56)$.

For the trajectory $\alpha(34)$, we have

$$\begin{aligned}K_{\text{initial}}{}^{\mu\nu}=\tfrac{1}{4}[&-(1^-)_{34}{}^{\mu}(1^-)_{34}{}^{\nu}-(1^-)_{34}{}^{\mu}P_{52}{}^{\nu}\\&+P_{52}{}^{\mu}(1^-)_{34}{}^{\nu}+P_{52}{}^{\mu}P_{52}{}^{\nu}],\end{aligned} \tag{14}$$

$$K_{\text{final}}{}^{\mu\nu}=\epsilon^{\sigma\mu\cdot\cdot}P_1{}^{\cdot}P_2{}^{\cdot}\epsilon^{\sigma\nu\cdot\cdot}P_5{}^{\cdot}P_6{}^{\cdot}.$$

In K_{initial}, the first of the terms contains a $(2^+)_{34}{}^{\mu\nu}$ in its reduction because $\epsilon^{\sigma\mu\cdot\cdot}\epsilon^{\sigma\nu\cdot\cdot}$ in K_{final} contains a sym-

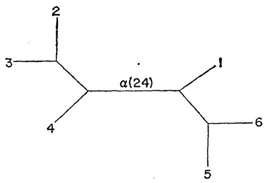


FIG. 3. Trajectory $\alpha(24)$ in the 6-π reaction.

metric part in $\mu\leftrightarrow\nu$. Hence $\alpha(34)$ is of positive normality starting at $J^P=2^+$, and similarly for $\alpha(61)$.

By construction $\alpha(13)$ is the ω-A_2 trajectory, abnormally coupled to three pions.

For the $\alpha(24)$ trajectory we consider the graph of Fig. 3. Equation (13) can be written as

$$K=\epsilon^{\mu\cdots}P_1\cdot(1^-)_{23}\cdot P_{23}\cdot\epsilon^{\mu\cdots}P_4\cdot(1^-)_{56}\cdot P_{56}\cdot. \qquad (15)$$

This corresponds to the operator with A_{1432} in $\rho\pi\to\rho\pi$ subscattering [see Eq. (6)]. Using helicity techniques, Canning[11] has shown this to be parity doubled. From our techniques, this can be seen if we write

$$\begin{aligned}K_{\text{initial}}{}^{\mu\nu\tau}&=(1^-)_{23}{}^{\mu}\{(1^-)_{24}{}^{\nu}\oplus P_{51}{}^{\nu}\}\{(1^-)_{24}{}^{\tau}\oplus P_{51}{}^{\tau}\},\\ K_{\text{final}}{}^{\mu\nu\tau}&=\epsilon^{\sigma\mu\nu}\cdot P_1\cdot\epsilon^{\sigma\tau\cdot}\cdot P_5\cdot P_6\cdot.\end{aligned} \qquad (16)$$

Here we have written

$$\begin{aligned}P_{23}{}^{\mu}&=P_{24}{}^{\mu}\oplus(1^-)_{24}{}^{\mu},\\ P_4{}^{\mu}&=P_{24}{}^{\mu}\oplus(1^-)_{24}{}^{\mu},\end{aligned} \qquad (17)$$

meaning that, at the (23)-4-(234) vertex, P_{23} (or P_4) is effectively a linear combination of P_{24} and $(1^-)_{24}$. It transforms as a scalar and a (1^-) which is an object perpendicular to P_{24} and, in the P_{24} rest frame, transforms as a vector under $O(3)$. The general expression for the linear combination at a vertex a-b-c (see Fig. 4) is

$$P^{\mu}=(1/2S_{ab})[(1^-)_{ab}{}^{\mu}+(S_{ab}-P_a{}^2+P_b{}^2)P_{ab}{}^{\mu}]. \quad (18)$$

Now, in K_{initial} of Eq. (16), from the part like $P_{51}{}^{\nu}(1^-)_{23}{}^{\mu}(1^-)_{24}{}^{\tau}$, we find a symmetric part in K_{final} in $\mu\leftrightarrow\tau$, which must correspond to a 2^+ tensor as above. From $(1^-)_{23}{}^{\mu}(1^-)_{24}{}^{\nu}(1^-)_{24}{}^{\tau}$, however, we obtain from the antisymmetry in $\mu\leftrightarrow\nu$ in K_{final}, a $(1^+)^{\mu\nu}$ pseudotensor, which, when combined with a symmetric part in K_{final} in $\mu\leftrightarrow\tau$, $\nu\leftrightarrow\tau$, gives a 2^- pseudotensor. This shows that $\alpha(24)$ is parity doubled and begins at $J=2^+$, 2^-. We have checked in the $\pi^+\pi^-\pi^+\pi^-\pi^+\pi^-$ amplitude, symmetrizing it suitably, and have found that there are no terms canceling this doubling. Equivalent results hold for the $\alpha(35)$ trajectory, and both $\alpha(24)$ and $\alpha(35)$ must be depressed at least three units not to have leading parity doubling. Although this same form (13) of superstructure has been proposed by Dorren *et al.*,[8] they are in disagreement with this analysis and the helicity analysis of Canning,[11] and they claim that these trajectories are solely ω-A_2. Nevertheless, they too recommend lowering them by three units.

With these brief descriptions of procedures for analyzing superstructures, we can now propose requirements that the structures for the N-π amplitudes should have. An optimal superstructure would have the property that, when evaluated for all tree graphs included in its dual substructure, it corresponds to particles of spin equal to the starting point of each internal trajectory in that graph and three-point couplings of prescribed normality. As we have seen in the $\pi\rho\to\pi\rho$, $\pi\omega\to\pi A_1$, and 6-π analyses, it usually turns out that these properties are obtainable only on a limited subset of graphs. Hence we must make do with superstructures with more restricted properties. We might try requiring that all graphs having a common trajectory (or set of nondual trajectories) have all leading non-parity-doubled behavior. Crossed (dual) channels which are doubled can then be depressed. We would then require that, in our sum of such terms, for each graph there is at least one term with all correct leading trajectories.

We note that, in $\pi\omega\to\pi A_1$ for instance,[11] use of the simplest superstructures satisfying these criteria does not allow all couplings to be independent, and we expect, therefore, that we will have restrictions in the N-point case too. We defer this point to a later paper.

We propose that the simplest superstructures with prescribed normality should include the following.

(a) A vertex of $2n$-π to the ρ-f trajectory, $n>1$. This is a normal vertex, and we use $K=1$ or a linear combination of the P_i (for $n=1$, such as P_1-P_2). The $K=1$ superstructure allows a scalar σ to couple. Since this would be a ghost, it must be eliminated by the substructure.

(b) A vertex of $2n$-π to the B trajectory, $n>2$. This is an abnormal vertex. Since the B is usually considered to lie on a trajectory in a daughterlike relation to the ρ-f trajectory and is, in any event, low lying, we do not couple this trajectory as a leading one.

(c) A vertex of $(2n+1)$-π to the π-A_1, $n>1$. This is a normal vertex and again we use $K=1$. Since this trajectory does start at $J=0$, the ghost-eliminating and B_N substructure is simpler for this than in (a).

(d) A vertex of $(2n+1)$-π to the ω-A_2, $n>1$. This is abnormal. We propose the forms

$$K^{\mu}=\epsilon^{\mu\cdots}P_{1,i}\cdot P_{i+1,j}\cdot P_{j+1,2n+1}\cdot. \qquad (19)$$

This is patently a pseudovector, orthogonal to $P_{1,2n+1}{}^{\mu}$, the momentum of the ω, and hence corresponds to a trajectory beginning at $J^P=1^-$. The change from apparent starting value of the trajectory of $J^P=1^+$ to $J^P=1^-$ is due to the odd number of pions forming the

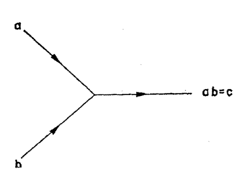


FIG. 4. A vector from two scalars in a tree subgraph.

trajectory. In general, the vertex

$$\epsilon^{\mu\cdots}P_1{\cdot}P_{2,2n}{\cdot}P_{2n+1}{\cdot} \tag{20}$$

can be shown to have defined parity trajectories on all trajectories involving only a subset of the labels 1, ..., $2n+1$; however, since it predicts trajectories starting at $J=0$ for the two-body channels (23), (34), ..., $(2n-1, 2n)$ and precludes pions on some three-body trajectories which should have them, we have the option of including forms other than (20) which have defined parity on fewer graphs, but start various of the two-body trajectories at $J=1$, and the pion trajectories at $J=0$.

(e) A $\pi\omega\rho$ coupling. This is abnormal, the ω and ρ coming from a type (d) and type (a) (P_1-P_2 variety) structure.

$$\epsilon^{\cdots}\omega{\cdot}\rho{\cdot}P_\omega{\cdot}P_\rho{\cdot}. \tag{21}$$

In any real $2N$-π amplitude there are always an even number of abnormal vertices. Thus, there may arise questions as to which pairs of vector indices to contract. We propose to take up this question in a later paper.

III. KINEMATIC SUPERSTRUCTURE FOR 8-π AMPLITUDE

The simplest type of superstructure is of course where there are no abnormally coupling trajectories. Here the only problem is that of constructing α factors to systematically eliminate all spin-0 ghosts on the ρ trajectory without eliminating too many π poles. For this the kinematic superstructure is the identity.

The next simplest type of process is one which reduces to the four-point $\omega\pi$-$\omega\pi$ in the eight-point amplitude with four abnormal couplings. For the tree graph of Fig. 5, the kinematic superstructure is of the type

$$\epsilon^{\mu\cdots}P_1{\cdot}P_2{\cdot}P_3{\cdot}\epsilon^{\nu\mu\cdots}P_4{\cdot}P_{13}{\cdot}\epsilon^{\nu\rho\cdots}P_5{\cdot}P_{68}{\cdot}\epsilon^{\rho\cdots}P_6{\cdot}P_7{\cdot}P_8{\cdot}. \tag{22}$$

This can easily be seen to be correct for the $\alpha(13)$ trajectory by observing that the form of $K_{\text{initial}}{}^\mu$ has exactly the form already discussed of Eq. (12), so that the $\alpha(13)$ trajectory starts at 1^-. For the $\alpha(14)$ trajectory we see that $K_{\text{initial}}{}^\nu$ is precisely a rank-1 tensor, which determines that the trajectory $\alpha(14)$ will start at $J^P=1^-$ and will always have normal parity.

Another type of abnormal coupling would be that illustrated in Fig. 6, for which we would select from the general form given in Eq. (20):

$$\epsilon^{\mu\cdots}P_1{\cdot}P_4{\cdot}P_{14}{\cdot}\epsilon^{\mu\cdots}P_5{\cdot}P_8{\cdot}P_{58}{\cdot}\,, \tag{23}$$

or possibly

$$\epsilon^{\cdots}P_1{\cdot}P_2{\cdot}P_3{\cdot}P_4{\cdot}\epsilon^{\cdots}P_5{\cdot}P_6{\cdot}P_7{\cdot}P_8{\cdot}. \tag{24}$$

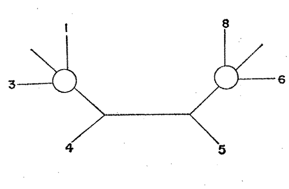


FIG. 5. $\omega\pi\rightarrow\omega\pi$ in the 8-π reaction.

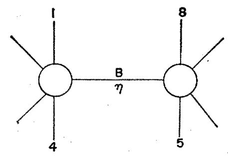


FIG. 6. The $4-\pi\rightarrow B$ trajectory in the 8-π reaction.

In Eq. (23), $K_{\text{initial}}{}^\mu$ is precisely a rank-1 pseudotensor, and in Eq. (24), K_{initial} is precisely a pseudoscalar, so that on the $\alpha(14)$ trajectory only abnormal parity particles will be found. However, we do not investigate these forms any further, since the $\alpha(14)$ trajectory of Fig. 6 would correspond to the B trajectory, which is in a daughterlike relation to the ρ trajectory, and which therefore we do not need to put in separately.

The only nontrivial problem is the single, abnormally coupling trajectory of the type illustrated in Fig. 7(a). This trajectory is meant to couple abnormally to pions 1, 2, and 3 at one "end" and to couple abnormally to pions 4, 5, 6, 7, and 8 at the other "end." Of the general solution to this problem given in Eq. (20), we choose the following forms for investigation:

$$K(1,2,3;4,57,8)=\epsilon^{\mu\cdots}P_1{\cdot}P_2{\cdot}P_3{\cdot}\epsilon^{\mu\cdots}P_4{\cdot}P_{57}{\cdot}P_8{\cdot}\,, \tag{25}$$

$$K(1,2,3;46,7,8)=\epsilon^{\mu\cdots}P_1{\cdot}P_2{\cdot}P_3{\cdot}\epsilon^{\mu\cdots}P_{46}{\cdot}P_7{\cdot}P_8{\cdot}\,, \tag{26}$$

$$K(1,2,3;4,5,68)=\epsilon^{\mu\cdots}P_1{\cdot}P_2{\cdot}P_3{\cdot}\epsilon^{\mu\cdots}P_4{\cdot}P_5{\cdot}P_{68}{\cdot}. \tag{27}$$

In fact, only the form of Eq. (25) determines that the two abnormal couplings should be on adjacent vertices (123)(45678) in all tree graphs including the $\alpha(13)$ trajectory. The other two forms achieve this for only certain of the tree graphs including the $\alpha(13)$ trajectory. They are, however, necessary to permit pions on the trajectories $\alpha(46)$ and $\alpha(68)$ in the superstructures of Eqs. (26) and (27), respectively. This is why the coupling of five pions to make an ω is not a trivial problem like the coupling of three pions to make an ω.

We now examine the kinematic superstructure of Eq. (25) by referring to various general types of tree graphs chosen to elucidate the normalities of the individual trajectories present. Our task is to perform the analysis implicit in Eq. (5), i.e., to ascertain the maximum rank of tensor in the product $K_{\text{initial}}{\cdot}K_{\text{final}}$ and to find whether it is a tensor or a pseudotensor or both. For this purpose we shall use the tensorial methods already developed in Sec. II.

$\alpha(56)$—see Fig. 7(b). Here we may rearrange and factorize K to give simply $K_{\text{initial}}=1$. Clearly K_{initial} corresponds to a scalar 0^+. This determines that the trajectory $\alpha(56)$ has normal parity starting at $J^P=0^+$.

$\alpha(57)$—see Fig. 7(c). Here again we may arrange K to give simply $K_{\text{initial}}=1$, corresponding to a scalar 0^+. Since there are an odd number of pions, this determines that the trajectory $\alpha(57)$ has abnormal parity starting at $J^P=0^-$.

$\alpha(12)$—see Fig. 7(d). Here we may rearrange K, using Eq. (18), to give

$$K_{\text{initial}}{}^\mu=(1^-)_{12}{}^\mu. \tag{28}$$

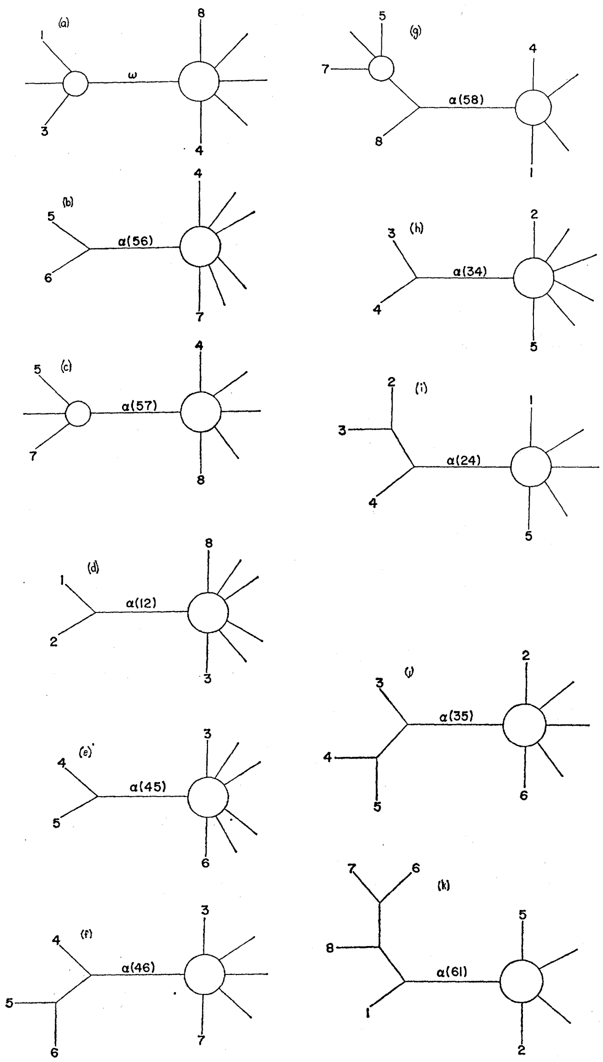


FIG. 7. Various trajectories used in the analysis of 8-π reaction.

K_{initial} is thus precisely a rank-1 tensor, and this determines that the trajectory $\alpha(12)$ has normal parity starting at $J^P=1^-$.

$\alpha(45)$—see Fig. 7(e). Here we may rearrange K, using Eq. (18), to give

$$K_{\text{initial}}{}^{\mu}=(1^-)_{45}{}^{\mu}+P_{63}{}^{\mu}. \tag{29}$$

K_{initial} clearly represents a rank-1 tensor and a scalar. This determines that trajectory $\alpha(45)$ has normal parity starting at $J^P=1^-$.

$\alpha(46)$—see Fig. 7(f). Using Eqs. (17) and (18), we may rearrange K to give

$$K_{\text{initial}}{}^{\mu}=(1^-)_{46}{}^{\mu}\oplus P_{73}{}^{\mu}. \tag{30}$$

As above, K_{initial} represents a rank-one tensor and a scalar. Since there are an odd number of pions, this determines that the trajectory $\alpha(46)$ has abnormal parity and starts at $J^P=1^+$.

$\alpha(13)$—see Fig. 7(a). We see directly from the form of K that

$$K_{\text{initial}}{}^{\mu}=\epsilon^{\mu\cdots}P_1{\cdot}P_2{\cdot}P_3{\cdot}. \tag{31}$$

Clearly this corresponds to a rank-1 pseudotensor, since it has exactly the form of Eq. (12). Since there are an odd number of pions, this determines that the trajectory $\alpha(13)$ has only normal parity starting at $J^P=1^-$.

$\alpha(58)$—see Fig. 7(g). Using Eqs. (17) and (18), we may rearrange K to give

$$K_{\text{initial}}{}^{\mu}=(1^-)_{58}{}^{\mu}\oplus P_{14}{}^{\mu}. \tag{32}$$

This corresponds to a rank-1 tensor and scalar, determining that the trajectory $\alpha(58)$ has only normal parity starting at $J^P=1^-$.

$\alpha(34)$—see Fig. 7(h). Here we may rearrange K, using Eq. (18), to give

$$\begin{aligned} K_{\text{initial}}{}^{\mu\nu}&=[(1^-)_{34}{}^{\mu}+P_{52}{}^{\mu}][(1^-)_{34}{}^{\nu}-P_{52}{}^{\nu}],\\ K_{\text{final}}{}^{\mu\nu}&=\tfrac{1}{4}\epsilon^{\sigma\mu\cdot\cdot}P_1{\cdot}P_2{\cdot}\epsilon^{\sigma\nu\cdot\cdot}P_{58}{\cdot}P_8{\cdot}. \end{aligned} \tag{33}$$

By comparison with the form of Eq. (14), we see that there will be some contribution to K_{initial} like a rank-2 tensor. This determines that the trajectory $\alpha(34)$ has only normal parity starting at $J^P=2^+$.

$\alpha(24)$—see Fig. 7(i). Using Eqs. (17) and (18), we may rearrange K to give

$$\begin{aligned} K_{\text{initial}}{}^{\mu\nu\tau}&=[(1^-)_{24}{}^{\mu}\oplus P_{51}{}^{\mu}](1^-)_{23}{}^{\nu}[(1^-)_{24}{}^{\tau}\oplus P_{51}{}^{\tau}],\\ K_{\text{final}}{}^{\mu\nu\tau}&=\epsilon^{\sigma\mu\nu\cdot}P_1{\cdot}\epsilon^{\sigma\tau\cdot\cdot}P_8{\cdot}P_{57}{\cdot}. \end{aligned} \tag{34}$$

By comparison with the form of Eq. (16), we see that there will be some contribution to K_{initial} like a rank-2 tensor and pseudotensor. This determines that the trajectory $\alpha(24)$ is parity doubled starting at $J^P=2^+$, 2^-.

$\alpha(35)$—see Fig. 7(j). Using Eqs. (17) and (18), we may rearrange K to give

$$\begin{aligned} K_{\text{initial}}{}^{\mu\nu\tau}&=[(1^-)_{45}{}^{\mu}\oplus(1^-)_{35}{}^{\mu}\oplus P_{62}{}^{\mu}]\\ &\quad\times[(1^-)_{34}{}^{\nu}\oplus P_{62}{}^{\nu}\oplus P_{68}{}^{\nu}][(1^-)_{35}{}^{\tau}\oplus P_{62}{}^{\tau}],\\ K_{\text{final}}{}^{\mu\nu\tau}&=\epsilon^{\sigma\mu\nu\cdot}P_8{\cdot}\epsilon^{\sigma\tau\cdot\cdot}P_1{\cdot}P_2{\cdot}. \end{aligned} \tag{35}$$

As above, comparison with Eq. (16) tells us that there will be some contribution to K_{initial} from rank-2 tensors and pseudotensors. Accordingly, the trajectory $\alpha(35)$ is parity doubled and starts at $J^P=2^+$, 2^-.

$\alpha(61)$—see Fig. 7(k). Using Eqs. (17) and (18), we may rearrange K to give

$$\begin{aligned} K_{\text{initial}}{}^{\mu\nu\tau}&=[(1^-)_{61}{}^{\mu}\oplus P_{25}{}^{\mu}\oplus P_{23}{}^{\mu}]\\ &\quad\times[(1^-)_{68}{}^{\nu}\oplus(1^-)_{61}{}^{\nu}\oplus P_{25}{}^{\nu}]\\ &\quad\times[(1^-)_{61}{}^{\tau}\oplus P_{25}{}^{\tau}],\\ K_{\text{final}}{}^{\mu\nu\tau}&=\epsilon^{\sigma\mu\nu\cdot}P_4{\cdot}\epsilon^{\sigma\tau\cdot\cdot}P_2{\cdot}P_3{\cdot}. \end{aligned} \tag{36}$$

As above, comparison with Eq. (16) determines that the trajectory $\alpha(61)$ is parity doubled starting at $J^P=2^+$, 2^-.

TABLE I. Properties of two superstructures for 8-π reaction. Integers to be used in the B_N of the substructure for various α factors used with these superstructures. NL signifies nonleading behavior in that channel.

Channel	Behavior of superstructure $K(1,2,3;4,57,8)$	Integers used in B_N for α factors of 1	$\alpha(56)$	$\alpha(67)$	Behavior of superstructure $K(1,2,3;46,7,8)$	Integers used in B_N for α factors of 1	$\alpha(45)$	$\alpha(56)$
$\alpha(12)$	1^-	1	1	1	1^-	1	1	1
$\alpha(23)$	1^-	1	1	1	1^-	1	1	1
$\alpha(34)$	2^+	2	2	2	2^+	2	3	2
$\alpha(45)$	1^-	1	2	1	0^+	1-NL	0	1
$\alpha(56)$	0^+	1-NL	0	1	0^+	1-NL	1	0
$\alpha(67)$	0^+	1-NL	1	0	1^-	1	1	2
$\alpha(78)$	1^-	1	1	2	1^-	1	1	1
$\alpha(81)$	2^+	2	2	2	2^+	2	2	2
$\alpha(13)$	1^-	1	1	1	1^-	1	1	1
$\alpha(24)$	$2^+, 2^-$	3-NL	3-NL	3-NL	$2^+, 2^-$	3-NL	4-NL	3-NL
$\alpha(35)$	$2^+, 2^-$	3-NL	4-NL	3-NL	2^-	2	2	3
$\alpha(46)$	1^+	1	1	2	0^-	0	0	0
$\alpha(57)$	0^-	0	0	0	1^+	1	2	1
$\alpha(68)$	1^+	1	2	1	$1^+, 1^-$	2-NL	2-NL	3-NL
$\alpha(71)$	$2^+, 2^-$	3-NL	3-NL	4-NL	$2^+, 2^-$	3-NL	3-NL	3-NL
$\alpha(82)$	$2^+, 2^-$	3-NL	3-NL	3-NL	$2^+, 2^-$	3-NL	3-NL	3-NL
$\alpha(14)$	1^-	1	1	1	$1^+, 1^-$	2-NL	3-NL	2-NL
$\alpha(25)$	$2^+, 2^-$	3-NL	4-NL	3-NL	$2^+, 2^-$	3-NL	3-NL	4-NL
$\alpha(36)$	$2^+, 2^-$	3-NL	3-NL	4-NL	$2^+, 2^-$	3-NL	3-NL	3-NL
$\alpha(47)$	1^-	1	1	1	1^-	1	1	1

The results of the above analysis are recorded in column 2 of Table I in the form of the J^P of the lowest-spin particle allowed on each trajectory. Here we have made the compilation for all trajectories, the others being strictly analogous with the ones we have analyzed above. In column 6 we have tabulated the similar results obtained by an analysis of the superstructure of Eq. (26), which proceeds along similar lines.

IV. ANALYSIS OF EIGHT-POINT AMPLITUDE α-FACTORS

In columns 2 and 6 of Table I we have tabulated the behavior of the kinematic superstructures $\epsilon^{\mu}\cdots P_1\cdot P_2\cdot P^3\cdot \times \epsilon^{\mu}\cdots P_4\cdot P_{57}\cdot P_8\cdot$ and $\epsilon^{\mu}\cdots P_1\cdot P_2\cdot P_3\cdot \epsilon^{\mu}\cdots P_{46}\cdot P_7\cdot P_8\cdot$, respectively. With the first of these we note that trajectories $\alpha(56)$ and $\alpha(67)$ which are ρ-f may start at $J=0$. We have the option of starting them at $J=1$ and making them nonleading trajectories or starting them at $J=0$ and putting in ghost-eliminating factors. A factor $\alpha(ij)$ has the property that it eliminates a pole at $\alpha(ij)=0$ from the B_N function and that it raises by one the highest angular momentum present in all channels dual to $\alpha(ij)$. However, it does not change the normality of the highest angular momentum present and hence does not parity-double leading single-normality trajectories.

In columns 3–5 of Table I we list the small integers m_{ij} for the various choices of α factors, sufficient to eliminate all ghosts. We indicate by NL those trajectories which are nonleading.

All of the trajectories nondual to $\alpha(13)$ are of single normality and start at their appropriate values of J except $\alpha(46)$ and $\alpha(68)$, both of which have the π precluded. In order to allow reduced graphs with $\alpha(13)$ and $\alpha(46)$ to have a π-(46), we consider superstructure 2. The m_{ij} for a set of various α factors, sufficient for the requirements listed in Sec. II, are presented in columns 7–9. An exactly analogous result holds to allow the π-(68). We note that, with these additional superstructures, not all trajectories compatible with $\alpha(13)$ are of single normality.

V. CONCLUSIONS

In the N-point amplitude it is necessary physically to be able to specify the normalities of particles on all internal leading trajectories. We have described a method of constructing amplitudes from a sum of terms each with a kinematic superstructure and dual substructure. By means of the superstructure, we prescribe the normalities of a complete set of compatible leading trajectories. By means of the substructure, we arrange to make appropriate trajectories leading, to suppress parity doublets, and to eliminate ghosts.

We have suggested a procedure for the construction of these kinematic superstructures appropriate to securing the normalities of the internal trajectories. The techniques for a more systematic analysis of the N-point problem will be presented in a later paper. Here we have discussed the superstructures appropriate for four-point and six-point amplitudes in which we use our method of tensorial analysis to rederive the results of previous analyses using helicity amplitudes.

Effectively, this method of tensorial analysis involves decomposing the momenta which appear in the super-superstructure, and which are Lorentz rank-1 tensors, as $O(3)$ vectors 1^- and scalars 0^+ in a chosen rest frame. Then we analyze products of these vectors involving ϵ pseudotensors in order to determine the highest representation of $O(3)$ present and to determine its parity. This method is effective, much simpler, and more general than the use of helicity amplitudes employed by Dorren *et al.*[8] We also avoid any need to work out Gram determinants with this method.

The problem of incorporating the ω-A_2 trajectory as an abnormally coupled trajectory to 3-π and the remaining 5-π in the 8-π amplitude was then analyzed in detail. There we showed, using our tensorial technique, how our proposed form for the N-point superstructures could be applied to achieve the desired result.

From these considerations, we are able to formulate a minimality hypothesis. In $\pi\pi \to \pi\pi$ [13] and $\pi\pi \to \pi\omega$,[14] it was possible to obtain the desired leading behavior in all channels by writing effectively only one term for each of the reactions. The minimality hypothesis then claimed that these single terms gave the complete amplitude. By contrast, for the four-point amplitudes with two spinning particles, no such minimal hypothesis can be formulated. However, for the N-point, all-spin-0 amplitude, we may take as an acceptable minimal hypothesis that we should only include the minimum number of terms necessary to include all possible subgraphs with minimum starting points on all trajectories, which have the necessary property of defined normalities on all leading trajectories.

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[14] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

Accordingly, by careful examination of Table I, we see that the unit α factors for both kinematic superstructures are superfluous. The minimality hypothesis leads us to reject these forms, especially as they have a higher number of nonleading trajectories. Then, in fact, we obtain the correct minimal forms for $\pi\pi \to \pi\pi$ when $\alpha(46)$, $\alpha(57)$, or $\alpha(68)=0$. In any case, the correct minimal form for $\pi\pi \to \pi\omega$ is obtained when either $\alpha(46)$ or $\alpha(68)=0$, and $\alpha(13)=1$. At least in this examples the hypothesis of minimality applied to $2n$-π amplitude, is consistent with minimality for a smaller $2n'<2n$-π amplitudes.

Since changing the set of B_N functions used in defining an amplitude changes drastically the degeneracy structure of the amplitude, we feel that such a hypothesis of minimality is critically necessary to a discussion of the factorization and level structure. We do not, however, discuss this here, since we have not included specifically all the forms of tree diagrams with two ω-A_2 trajectories and no ω-A_2 trajectories. Moreover, from the discussion given by Canning,[11] we may expect some conflict between minimality, duality, and factorization.

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Wave Equations with Compact and Noncompact Spectrum

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We present a unified approach to compact and noncompact wave equations based on the algebra of $O(6,C)$. The Bhabha and Nambu equations emerge as the simplest possibility for describing multiplets of relativistic particles. Dynamical quantities such as mass, spin, and magnetic moment are evaluated in terms of the spectra of relevant operators.

INTRODUCTION

Among the most interesting phenomena of the physics of strongly interacting particles is the fact that hadrons and their resonances seem to fall into more or less well-defined groups and families. Some of these are described, on a phenomenological level, by internal symmetry groups such as $SU(2)$ and $SU(3)$. Other approaches yield families or trajectories of particles with different spins. One would welcome, perhaps, a description of such supermultiplets in terms of fields and wave equations. Attempts in this direction are the finite-component wave equation of Bhabha[1] and the infinite-component equation of Nambu.[2] While neither of these equations is perhaps very physical, it seems, nontheless, to be rewarding to explore them as models of an eventually more complete theory. After all, the Dirac equation, which is physically quite relevant, is indeed the Bhabha equation of lowest order.

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