Anomalies of the Axial-Vector Current in Two Dimensions*

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In a world with one space and one time dimension, it is shown that the axial-vector current in a vector-gluon model exhibits anomalies in perturbation theory analogous to those found by Adler in fermion electrodynamics. The analysis is extended to include a four-fermion (Thirring) interaction; this model has been solved exactly by Sommerfield, permitting an explicit verification of the perturbation-theory calculations. In analogy to results in four dimensions, a model is presented in which the anomalous properties of the axial-vector current, both in its divergence and in its commutation relations, follow immediately from the canonical structure of the theory.

I. INTRODUCTION

It has been shown by Adler[1] that in massless spinor electrodynamics the axial-vector current is not conserved, in contradiction with the prediction of Noether's theorem. Perturbation-theory arguments[1–4] indicate that the correct divergence condition reads

$$\partial^\mu j^5{}_\mu = \partial^\mu(\bar\psi\gamma_\mu\gamma_5\psi) = \frac{\alpha}{2\pi}\bar F^{\mu\nu}F_{\mu\nu}\,, \qquad (1.1)$$

where $F_{\mu\nu}$ and $\bar F^{\mu\nu}$ denote the electromagnetic field strength tensor and its dual, respectively; i.e.,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad \bar F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}\,.$$

One encounters this anomaly in a perturbation-theory derivation of the vector–vector–axial-vector Ward identity; there, linearly divergent expressions arising from triangle subgraphs invalidate the naive result. The triangle diagram (itself linearly divergent) must be evaluated by a gauge-invariant regularization procedure; inserting the result of this calculation into the Ward identity gives an answer consistent with (1.1). The singular nature of the axial-vector current in perturbation theory gives rise to other anomalies as well; e.g., some commutators involving $j^5{}_\mu$ differ from those calculated by use of the canonical commutation relations.[5]

We will show that, in a theory of a massless fermion interacting with a vector gluon in a world possessing one space and one time dimension, the axial-vector current obeys a divergence condition analogous to (1.1). Just as in the four-dimensional case, linearly divergent expressions arise in the derivation of the vector–vector–axial-vector Ward identity; however, in this case, the offending diagram (the "bubble" diagram) is only logarithmically divergent and hence can be evaluated unambiguously. An important distinction between the two-dimensional and four-dimensional cases is that the former can be solved exactly (this fact motivated our restriction to massless fermions). In fact, the model with an additional current-current (Thirring[6]) interaction has been solved exactly by Sommerfield.[7] We extend our perturbation-theory analysis to include this case and explicitly verify our conclusions via Ref. 7.

In the four-dimensional case, one can construct a model of a pseudoscalar meson interacting nonminimally with the electromagnetic field in which the divergence condition (1.1) and the anomalous commutation relations[5] of the axial-vector current follow from naive (canonical) manipulations.[8] We demonstrate the analog of this model in two dimensions, and extend this to include the Thirring interaction. The resulting theory involves two pseudoscalar mesons interacting only through mass mixing.

II. VECTOR-GLUON MODEL

We deal first with a massless fermion interacting with a vector gluon, as described by the Lagrangian

$$\mathcal{L}_0 = i\bar\psi\gamma^\mu\partial_\mu\psi - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} - \tfrac{1}{2}\mu^2A^\mu A_\mu + gj^\mu A_\mu\,,$$
$$B_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad j_\mu = \bar\psi\gamma_\mu\psi\,. \qquad (2.1)$$

We use natural units and describe the geometry by the diagonal metric tensor $\eta^{\mu\nu}$ with $-\eta^{00}=\eta^{11}=1$. The Dirac matrices satisfy

$$\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}\,,$$

as usual. Introducing the pseudoscalar matrix

$$\gamma_5 = \gamma^0\gamma^1$$

and the invariant pseudotensor density

$$\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}\,, \quad \epsilon^{01} = 1\,,$$

we find that

$$\gamma^\mu\gamma^\nu = -\eta^{\mu\nu} + \epsilon^{\mu\nu}\gamma^5\,, \qquad (2.2)$$

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[8] S. Deser and J. Rawls, Phys. Rev. **187**, 1935 (1969).

a fact that will be of use later. For definiteness, we will deal with the representation

$$\gamma^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The electrodynamics of a massless fermion in two dimensions, often referred to as the Schwinger[9] model, has been studied by a number of authors.[10–12] This case, which corresponds to Eq. (2.1) with $\mu=0$, is a very peculiar one, for there are no transverse degrees of freedom and the bad asymptotic behavior of the Coulomb potential leads to a total screening of the electric charge and to a nonzero renormalized "photon" mass. In general, the limit $\mu\to 0$ is singular[11,12]; however, all our arguments apply to this case as well.

Owing to the invariance of (2.1) under the transformation

$$\psi\to\exp[i(\alpha+\beta\gamma_5)]\psi\,,\quad \psi^\dagger\to\psi^\dagger\exp[-i(\alpha+\beta\gamma_5)]$$

for constant α and β, there are two formally conserved currents,

$$j_\mu=\bar\psi\gamma_\mu\psi\,,\quad j^5{}_\mu=\bar\psi\gamma_\mu\gamma_5\psi\,,$$

which are related by

$$j^5{}_\mu=\epsilon_{\mu\nu}j^\nu\,. \tag{2.3}$$

We will now demonstrate that, in analogy to spinor electrodynamics in four dimensions, $j^5{}_\mu$ is not conserved in perturbation theory.

We examine in perturbation theory the validity of the naive Ward identity

$$(p-p')^\mu\Gamma^5{}_\mu(p,p')=S_F'(p)^{-1}\gamma_5+\gamma_5S_F'(p')^{-1}\,, \tag{2.4}$$

where the vertex part $\Gamma^5{}_\mu$ is defined by

$$S_F'(p)\Gamma^5{}_\mu(p,p')S_F'(p')$$
$$=\int d^2x d^2y\, e^{-ipx}e^{ip'y}\langle 0|T(j^5{}_\mu(0)\psi(x)\bar\psi(y))|0\rangle. \tag{2.5}$$

Following Adler's[1] treatment, we divide the diagrams which contribute to the axial-vector vertex into two classes: (a) those in which $\gamma_\mu\gamma_5$ attaches to a fermion line in the main trunk of the diagram, i.e., that part which remains connected to the external fermions when the gluon lines are removed, and (b) those in which $\gamma_\mu\gamma_5$ attaches to a fermion line which is not in the main trunk of the diagram. Diagrams of type (a) give the entire right-hand side of (2.4). The diagrams of type (b) sum to an expression which vanishes, provided that one can shift the origin of the integral over the loop momentum. This operation is valid for all diagrams except the bubble diagram, illustrated in Fig. 1. It is amusing to note that this graph vanishes in four dimensions because of charge-conjugation invariance; however, as is seen immediately from Eq. (2.3), j_μ and $j^5{}_\mu$ have identical charge-conjugation properties in two dimensions. Although this diagram is only logarithmically divergent, it gives rise to a linearly divergent expression in the derivation of the Ward identity; hence we must calculate separately its contribution to $(p-p')^\mu\Gamma^5{}_\mu(p,p')$.

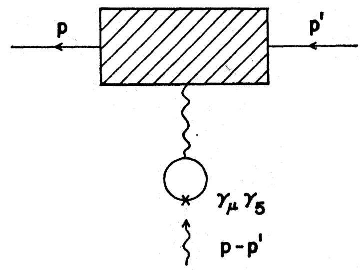

FIG. 1. Diagrams which give rise to an anomalous Ward identity in the vector-gluon model.

Since $j^5{}_\mu=\epsilon_{\mu\nu}j^\nu$, we can evaluate the contribution of Fig. 1 by calculating, to lowest order in g,

$$\epsilon^\mu{}_\alpha\int d^2x\, e^{-ikx}\langle 0|T^*(j^\alpha(x)A^\nu(0))|0\rangle.$$

The result is

$$g\,\epsilon^\mu{}_\alpha T^\alpha{}_\beta(k)\Delta^{\beta\nu}(k)=M^{\mu\nu}(k)\,,$$

where

$$\Delta^{\mu\nu}(k)=\frac{\eta^{\mu\nu}+k^\mu k^\nu/\mu^2}{k^2+\mu^2-i\epsilon}$$

is the free vector-boson propagator and

$$T^{\mu\nu}(k)=\int\frac{d^2q}{(2\pi)^2}\mathrm{Tr}\left(\gamma^\mu\frac{1}{\boldsymbol{k}+\boldsymbol{q}-i\epsilon}\gamma^\nu\frac{1}{\boldsymbol{q}-i\epsilon}\right) \tag{2.6}$$

is the contribution of the bubble. This is evaluated by means of the standard regularization procedure, in which an auxiliary field Ψ with large mass Λ is introduced, and the interaction Lagrangian is taken to be $gA^\mu j_\mu+igA^\mu J_\mu$, where $J_\mu=\bar\Psi\gamma_\mu\Psi$.

Evaluating the expression

$$\int d^2x\, e^{-ikx}\langle 0|T^*([j^\mu(x)+iJ^\mu(x)][j^\nu(0)+iJ^\nu(0)])|0\rangle$$

to lowest order, we find in the limit $\Lambda\to\infty$,

$$T^{\mu\nu}(k)=\frac{i}{\pi}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2-i\epsilon}\right).$$

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[9] J. Schwinger, Phys. Rev. **128**, 2425 (1962); *Seminar on Theoretical Physics, Trieste, 1962* (IAEA, Vienna, 1963).

[10] L. S. Brown, Nuovo Cimento **29**, 617 (1963); C. R. Hagen, *ibid.* **51B**, 169 (1967).

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[12] J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N. Y.) (to be published).

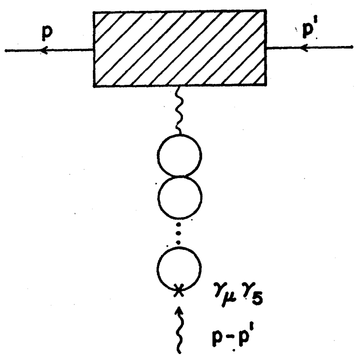

FIG. 2. Type-(b) diagrams which give rise to an anomalous term in the Ward identity in the Thirring model with a vector-gluon interaction.

Then,

$$k_\mu M^{\mu\nu}(k) = i(g/\pi)\epsilon_{\mu\alpha}k^\mu \Delta^{\alpha\nu}$$

$$= i(g/\pi)\int d^2x\, e^{-ikx}\langle 0|T^*(B(x)A^\nu(0))|0\rangle$$

to lowest order in g, where

$$B = \epsilon^{\mu\nu}\partial_\mu A_\nu = B_{01}\,.$$

Thus the contribution to (2.4) from a diagram of the type illustrated in Fig. 1 can be represented by a similar diagram with the bubble replaced by $-i(g/\pi)B$. The net effect is to modify the naive Ward identity to read

$$(p-p')^\mu \Gamma^5{}_\mu(p,p') = S_F'(p)^{-1}\gamma_5 + \gamma_5 S_F'(p')^{-1} - i(g/\pi)B(p,p')\,, \quad (2.7)$$

where

$$S_F'(p)B(p,p')S_F'(p') = \int d^2x d^2y\, e^{-ipx}e^{ip'y}\langle 0|T(B(0)\psi(x)\bar\psi(y))|0\rangle. \quad (2.8)$$

Equation (2.7) is consistent with

$$\partial^\mu j^5{}_\mu = -(g/\pi)B\,. \quad (2.9)$$

Here, as in four dimensions, the anomalous divergence is proportional to the simplest pseudoscalar object which can be formed from the vector field.

By virtue of the definition (2.3) and the method of regularization, we have imposed gauge invariance upon $j^5{}_\mu$. The anomalies just discussed then follow as a consequence. In analogy to the situation in four dimensions,[1,5] however, it is also possible to define an axial-vector current which *is* conserved but is not gauge invariant:

$$\tilde{j}^5{}_\mu = j^5{}_\mu + (g/\pi)\epsilon_{\mu\nu}A^\nu\,.$$

It is interesting to note that the corresponding time-independent axial charge

$$\tilde{Q}^5 = \int d^3x\, \tilde{j}^5{}_0$$

is gauge invariant. A further discussion of this point is is found in Sec. IV.

In order to calculate the contribution of the bubble diagram, we introduced an auxiliary fermion field with mass Λ and computed matrix elements of $j^5{}_\mu + iJ^5{}_\mu$. The formal identity

$$\partial^\mu(j^5{}_\mu + iJ^5{}_\mu) = -2\Lambda J^5\,,$$

where

$$J^5 = \bar\Psi\gamma_5\Psi\,,$$

suggests that the observed anomalous divergence arises because some matrix elements of $-2\Lambda J^5$ do not vanish as $\Lambda \to \infty$. This point of view was advanced by Hagen[13] in the case of quantum electrodynamics (QED) in four dimensions. To verify this in the model under discussion, we note that for the $-2\Lambda J^5$ vertex, all diagrams are of order Λ^{-1} except those of the form shown in Fig. 1 (with a massive fermion bubble and the vertex $-2\Lambda\gamma_5$ instead of $\gamma_\mu\gamma_5$). The contribution of the latter is found by computing

$$-2\Lambda\int d^2x\, e^{-ikx}\langle 0|T(J^5(x)A^\nu(0))|0\rangle$$

to first order in g. The result is

$$-\frac{g}{\pi}\epsilon_{\mu\alpha}k^\mu\Delta^{\alpha\nu}(k)\int_0^1 \frac{\Lambda^2 dz}{[z(1-z)k^2+\Lambda^2-i\epsilon]},$$

which reproduces the expected result $ik_\mu M^{\mu\nu}(k)$ as $\Lambda\to\infty$.

III. THIRRING MODEL

With the addition of a current-current interaction, the Lagrangian of Eq. (2.1) becomes

$$\mathcal{L} = i\bar\psi\gamma^\mu\partial_\mu\psi - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} - \tfrac{1}{2}\mu^2 A^\mu A_\mu + gj^\mu A_\mu + \tfrac{1}{2}\sigma j^\mu j_\mu. \quad (3.1)$$

The analysis of the Ward identity in this case is considerably simplified by the observation (proved in the Appendix) that fermion loops with more than two vertices do not contribute in perturbation theory.[14] The most general diagram of type (b) is illustrated in Fig. 2. It consists of a $\gamma_\mu\gamma_5$ vertex attached to a bubble which is in turn attached to a string of bubbles, the latter being connected to the blob by a single gluon line. We can sum the contributions to the Ward identity by evaluating, to first order in g and to all

[13] C. R. Hagen, Phys. Rev. 188, 2416 (1969).

[14] This is also true in the gluon model, but it was not necessary to use it.

orders in σ,

$$\epsilon^{\mu}{}_{\alpha}\int d^2x\, e^{-ikx}\langle 0|T(j^{\alpha}(x)A^{\nu}(0))|0\rangle .$$

The expansion in powers of σ is simply a geometric series because the "product" of two bubbles is given by

$$\left(-\frac{\sigma}{\pi}\right)\left(\eta^{\mu\alpha}-\frac{k^\mu k^\alpha}{k^2}\right)\left(-\frac{\sigma}{\pi}\right)\left(\eta_{\alpha}{}^{\nu}-\frac{k_\alpha k^\nu}{k^2}\right)=\left(-\frac{\sigma}{\pi}\right)^2\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right).$$

We find the result

$$(g/\lambda)\epsilon^{\mu}{}_{\alpha}T^{\alpha}{}_{\beta}(k)\Delta^{\beta\nu}(k)=(1/\lambda)M^{\mu\nu}(k)\,,$$

where $\lambda=1+\sigma/\pi$. Thus the current-current interaction changes the anomalous term in the Ward identity (2.7) by a factor λ^{-1}. In a similar analysis of type-(a) diagrams, summing strings of bubbles again produces a factor of λ^{-1}; hence the Ward identity becomes

$$(p-p')^\mu\Gamma^5{}_\mu(p,p')=\lambda^{-1}S_F'(p)^{-1}\gamma^5+\lambda^{-1}\gamma^5S_F'(p')^{-1}-i(g/\pi\lambda)B(p,p')\,. \quad (3.2)$$

Equation (3.2) is consistent with the relations

$$\partial^\mu j^5{}_\mu=-(g/\pi\lambda)B \quad (3.3)$$

and

$$[j^5{}_0(x^0,x^1),\psi(x^0,y^1)]=\lambda^{-1}\gamma^5\psi(x^0,x^1)\delta(x^1-y^1)\,. \quad (3.4)$$

Since the only fermion loop which contributes in perturbation theory is the bubble diagram, the boson fields A^μ and j^μ have very simple properties. The vacuum expectation value of any product of the boson fields is a sum of products of two-point functions.[15] Furthermore, we can compute the two-point functions by summing the perturbation series, which is essentially just a geometric series. In terms of the renormalized mass $\mu'^2=\mu^2+g^2/\pi\lambda$, the results can be written

$$\int d^2x\, e^{-ikx}\langle 0|T^*(j^\mu(x)j^\nu(0))|0\rangle=\frac{i}{\pi\lambda}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2-i\epsilon}\right)\left(1-\frac{g^2/\pi\lambda}{k^2+\mu'^2-i\epsilon}\right),$$

$$\int d^2x\, e^{-ikx}\langle 0|T^*(j^\mu(x)A^\nu(0))|0\rangle=\frac{ig}{\pi\lambda}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2-i\epsilon}\right)\frac{1}{k^2+\mu'^2-i\epsilon},$$

and

$$\int d^2x\, e^{-ikx}\langle 0|T^*(A^\mu(x)A^\nu(0))|0\rangle=-i\frac{\eta^{\mu\nu}+k^\mu k^\nu/\mu^2}{k^2+\mu^2-i\epsilon}+i\frac{g^2}{\pi\lambda}\left(\eta^{\mu\nu}-\frac{k^\mu k^\nu}{k^2-i\epsilon}\right)\times\frac{1}{(k^2+\mu^2-i\epsilon)(k^2+\mu'^2-i\epsilon)}$$
$$=-i\frac{\eta^{\mu\nu}+k^\mu k^\nu/\mu'^2}{k^2+\mu'^2-i\epsilon}-i\frac{g^2}{\pi\lambda\mu^2\mu'^2}\frac{k^\mu k^\nu}{k^2-i\epsilon}.$$

We now calculate matrix elements of the equal-time commutators of the theory by use of the Bjorken-Johnson-Low[16] technique, which identifies the commutators by the relation

$$i\int dx^1\, e^{-ik^1x^1}\langle\beta|[A(0,x^1),B(0)]|\alpha\rangle=\lim_{k^0\to\infty}k^0\int d^2x\, e^{-ikx}\langle\beta|T(A(x)B(0))|\alpha\rangle.$$

We find, in addition to the canonical commutation relation between A^1 and B, results consistent with the following nonvanishing commutators:

$$[j_0(x^0,x^1),j_1(x^0,y^1)]=(i/\pi\lambda)\partial_1\delta(x^1-y^1)\,, \quad (3.5a)$$

$$[A_0(x^0,x^1),A_1(x^0,y^1)]=(i/\mu^2)\partial_1\delta(x^1-y^1)\,, \quad (3.5b)$$

$$[j_1(x^0,x^1),B(x^0,y^1)]=i(g/\pi\lambda)\delta(x^1-y^1)\,. \quad (3.5c)$$

Sommerfield[7] has solved the model specified by (3.1) exactly in two dimensions.[17] In this reference, j_0 is defined as the limit of a spatially spread $\psi^\dagger\psi$, j_1 is determined by Lorentz invariance, and $j^5{}_\mu$ is defined by $j^5{}_\mu=\epsilon_{\mu\nu}j^\nu$. He finds that our perturbation-theory results, namely, the anomalous divergence condition (3.3) and the anomalous commutation relations (3.4) and (3.5), are valid as operator statements. In view of the differing opinions[18] concerning the validity of (1.1) as an operator statement in four dimensions, we regard it as interesting that, in the case studied here, perturbation-theory results *are* valid as operator statements.

IV. MASS-MIXING MODEL

In four dimensions, a model of a pseudoscalar meson interacting nonminimally with the electromagnetic field has been proposed[8] in which the anomalies of the

[15] This property suggests that we may be dealing with generalized free fields.

[16] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 74 (1966); J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

[17] A survey of the literature on the Thirring model is given by A. S. Wightman, in *High Energy Electromagnetic Interactions and Field Theory*, edited by M. Lévy (Gordon and Breach, New York, 1967), Vol. II.

[18] For opposing positions, see Adler and Bardeen (Ref. 3) and Jackiw (Ref. 4).

axial-vector current follow directly from canonical reasoning. We will exhibit a similar model in the two-dimensional vector-gluon case. The Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} - \tfrac{1}{2}\mu^2 A^\mu A_\mu + (g/\sqrt{\pi})\phi B \qquad (4.1)$$

gives rise to the field equations

$$\Box\phi = -(g/\sqrt{\pi})B\,, \qquad (4.2)$$

$$\partial^\nu B_{\mu\nu} + \mu^2 A_\mu = (g/\sqrt{\pi})\epsilon_{\mu\nu}\partial^\nu\phi\,. \qquad (4.3)$$

The vector current, defined to be the source of A,

$$j_\mu = (1/\sqrt{\pi})\epsilon_{\mu\nu}\partial^\nu\phi\,, \qquad (4.4)$$

is identically conserved by virtue of the antisymmetry of $\epsilon_{\mu\nu}$. Then, introducing the axial-vector current via

$$j^5{}_\mu = \epsilon_{\mu\nu}j^\nu\,, \qquad (4.5)$$

it is seen that

$$j^5{}_\mu = (1/\sqrt{\pi})\partial_\mu\phi\,, \qquad (4.6)$$

so that (4.2) may be reexpressed as

$$\partial^\mu j^5{}_\mu = -(g/\pi)B\,, \qquad (4.7)$$

as desired. It is easy to extend this model to include a Thirring interaction. Because

$$j^\mu j_\mu = -(1/\pi)\partial^\mu\phi\partial_\mu\phi\,,$$

the introduction of an interaction term $\frac{1}{2}\sigma j^\mu j_\mu$ alters (4.1) to read

$$\mathcal{L} = -\tfrac{1}{2}\lambda\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} - \tfrac{1}{2}\mu^2 A^\mu A_\mu + (g/\sqrt{\pi})\phi B\,, \qquad (4.8)$$

and gives for the divergence condition

$$\partial^\mu j^5{}_\mu = -(g/\pi\lambda)B\,, \qquad (4.9)$$

in agreement with (3.3). Equation (4.9) can be viewed in another way. By dropping a total derivative term, ϕB can be replaced by $\epsilon^{\nu\mu}\partial_\mu\phi A_\nu$ in (4.8) [note that this has transcribed the interaction into the form $-gj^\mu A_\mu$, just as in (3.1)]. The invariance of this modified Lagrangian under the transformation $\phi \to \phi+\eta$ for constant η implies that

$$\tilde{j}^5{}_\mu \equiv -\frac{\partial\mathcal{L}}{\partial\partial^\mu\eta} = \lambda\partial_\mu\phi + \frac{g}{\sqrt{\pi}}\epsilon_{\mu\nu}A^\nu \qquad (4.10)$$

is a conserved current, a statement which is equivalent to (4.9). This also clarifies the origin of the time-independent axial charge $\tilde{Q}^5$, defined by

$$\tilde{Q}^5 = \int dx^1\left(j^5{}_0 - \frac{g}{\pi\lambda}A_1\right). \qquad (4.11)$$

A similar result holds in four dimensions.[1,8] Further, it is easily verified that use of the canonical commutation relations of the theory specified by (4.8) results in complete agreement with Eq. (3.5), which are the "anomalous" commutators that characterize the Thirring model with an added vector-gluon interaction.

This model has a particularly simple energy-momentum tensor. Since $T_{\mu\nu}$ is defined as the response of the action to a change in the metric, i.e.,

$$T_{\mu\nu} \equiv 2\delta\int\frac{\mathcal{L}d^2x}{\delta g^{\mu\nu}}\,,$$

the interaction term ϕB does not contribute to $T_{\mu\nu}$ (ϕ is a scalar and B is a scalar density). It follows that

$$T_{\mu\nu} = T_{\mu\nu}(\phi) + T_{\mu\nu}(A)\,, \qquad (4.12)$$

where the terms on the right-hand side denote the standard (Belinfante) expressions for free pseudoscalar and vector-meson fields. But $T_{\mu\nu}(\phi)$, which is given by[19]

$$T_{\mu\nu}(\phi) = -\lambda(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}\eta_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi)\,, \qquad (4.13a)$$

can be written in the so-called Sugawara[20,21] form

$$T_{\mu\nu}(\phi) = -\pi\lambda(j_\mu j_\nu - \tfrac{1}{2}\eta_{\mu\nu}j^\alpha j_\alpha) \qquad (4.13b)$$

or, equivalently,

$$T_{\mu\nu}(\phi) = -\pi\lambda(j^5{}_\mu j^5{}_\nu - \tfrac{1}{2}\eta_{\mu\nu}j^{5\alpha}j^5{}_\alpha)\,. \qquad (4.13c)$$

In passing, we point out that the Thirring model also has a $T_{\mu\nu}$ of this form (see Refs. 21 and 22 for details). In the language of the dynamical theory of currents, in which a theory is specified by its $T_{\mu\nu}$ and the commutation relations of the operators entering in $T_{\mu\nu}$, we have introduced the interaction ϕB through the commutation relations. As mentioned above, the resulting "anomalous" commutators are precisely those of Eq. (3.5).

The fact that the interaction ϕB is quadratic suggests that the model we are discussing is really a free theory. This can be made more precise by first noting that the vector-gluon field, which possesses only one degree of freedom, is equivalent to a pseudoscalar field. This is seen most clearly in the first-order formalism, where $B^{\mu\nu}$ and A_μ are taken as independent variables, and

$$\mathcal{L}_A = \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} - \tfrac{1}{2}B^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \tfrac{1}{2}\mu^2 A^\mu A_\mu\,. \qquad (4.14)$$

Defining

$$\pi^\alpha = \mu A_\beta\epsilon^{\alpha\beta}\,, \quad \pi = \frac{1}{2\mu}\epsilon^{\alpha\beta}B_{\alpha\beta} = \frac{1}{\mu}B\,,$$

Eq. (4.14) may be reexpressed as

$$\mathcal{L}_\pi = \pi\partial_\mu\pi^\mu + \tfrac{1}{2}\pi_\mu\pi^\mu - \tfrac{1}{2}\mu^2\pi^2 \qquad (4.15a)$$

or

$$\mathcal{L}_\pi = -\pi^\mu\partial_\mu\pi + \tfrac{1}{2}\pi^\mu\pi_\mu - \tfrac{1}{2}\mu^2\pi^2\,, \qquad (4.15b)$$

[19] Note that the $T_{\mu\nu}$ for a free massless scalar field is traceless in two dimensions; hence the new improved $T_{\mu\nu}$ of Callan, Coleman, and Jackiw [Ann. Phys. (N.Y.) **59**, 42 (1970)] is identical to the canonical $T_{\mu\nu}$.

[20] H. Sugawara, Phys. Rev. **170**, 1659 (1968).

[21] C. Sommerfield, Phys. Rev. **176**, 2019 (1968).

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the standard first-order expression for a free (pseudo-) scalar field. Then (4.8) becomes (returning to the second-order form)

$$\mathcal{L}=-\tfrac{1}{2}\lambda\partial^\mu\phi\partial_\mu\phi-\tfrac{1}{2}\partial^\mu\pi\partial_\mu\pi-\tfrac{1}{2}\mu^2\pi^2+(g\mu/\sqrt{\pi})\phi\pi\,,$$

indicating clearly that the ϕB interaction is simply of the mass-mixing type. Thus, we are indeed dealing with a free theory. This suggests that the Thirring model with a vector-gluon interaction is a free theory. Sommerfield[7] reached a similar conclusion by studying the exact solution to the model, although infrared problems prevent the examination of the fermion spectrum. It is interesting to note that, in the case of QED in two dimensions ($\mu=\sigma=0$), the algebra of observables has in fact been shown[12] to be that of free (pseudo-) scalar fields; however, our analysis is not valid in this case because the electromagnetic field, which has no quantized degrees of freedom, is not equivalent to a pseudoscalar field.

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APPENDIX

We will show that for the Thirring model with a vector-gluon interaction, fermion loops with more than two vertices do not contribute in perturbation theory. This is equivalent to showing that in a theory of free massless fermions in two dimensions with $j^\mu=\bar{\psi}\gamma^\mu\psi$, the connected part of

$$\langle 0|T(j^{\mu_1}(z_1)\cdots j^{\mu_n}(z_n))|0\rangle\,, \tag{A1}$$

which we will denote by $T\{j_n\}$, vanishes for $n>2$. This is intuitively reasonable because $\partial_\mu j^\mu=\epsilon^{\mu\nu}\partial_\mu j_\nu=0$ implies $\Box j^\mu=0$, so we might expect j_μ to behave like a free field.

We will work in position space and define

$$G_0(x)=\int\frac{d^2p}{(2\pi)^2}e^{ipx}\frac{1}{p-i\epsilon}=\frac{1}{2\pi}\frac{\gamma^\mu x_\mu}{x^2+i\epsilon}. \tag{A2}$$

Using the symbol $P\{i(n)\}$ to denote the permutation $1, 2, \ldots, n\rightarrow i(1), i(2), \ldots, i(n)$, we have

$$\begin{aligned}T\{j_n\}\propto&\sum_{P\{i(n-1)\}}\mathrm{Tr}[\gamma^{\mu_n}G_0(z_n-z_{i(1)})\\&\times\gamma^{\mu_{i(1)}}G_0(z_{i(1)}-z_{i(2)})\cdots\gamma^{\mu_{i(n-1)}}G_0(z_{i(n-1)}-z_n)]\\&=\mathrm{Tr}[\gamma^{\mu_n}G^{\mu_1\cdots\mu_{n-1}}(z_1\cdots z_{n-1};x,y)]|_{x=y=z_n}\,,\end{aligned} \tag{A3}$$

where

$$\begin{aligned}G^{\mu_1\cdots\mu_{n-1}}&(z_1\cdots z_{n-1};x,y)\\&\equiv\sum_{P\{i(n-1)\}}G_0(x-z_{i(1)})\gamma^{\mu_{i(1)}}G_0(z_{i(1)}-z_{i(2)})\cdots\\&\qquad\times\gamma^{\mu_{i(n-1)}}G_0(z_{i(n-1)}-y)\,.\end{aligned} \tag{A4}$$

We will now show that

$$G^{\mu_1\cdots\mu_{n-1}}(z_1\cdots z_{n-1};x,y)=\Big[\prod_{i=1}^{n-1}G_0^{\mu_i}(z_i;x,y)\Big]G_0(x-y)\,, \tag{A5}$$

where

$$G_0^{\mu_i}(z_i;x,y)\equiv G_0(x-z_i)\gamma^{\mu_i}-G_0(y-z_i)\gamma^{\mu_i}\,. \tag{A6}$$

This is sufficient to establish $T\{j_n\}=0$ for $n>2$ because $G_0^{\mu_i}(z_i;x,y)\sim(x-y)$ and $G_0(x-y)\sim(x-y)^{-1}$ as $x\rightarrow y$. The proof of Eq. (A5) proceeds by induction. For $n=2$, it reduces to the identity

$$\begin{aligned}G_0(x-z)\gamma^\mu G_0(z-y)=&G_0(x-z)\gamma^\mu G_0(x-y)\\&-G_0(y-z)\gamma^\mu G_0(x-y)\,.\end{aligned} \tag{A7}$$

We have, for $n=k+1$,

$$\begin{aligned}G^{\mu_1\cdots\mu_k}&(z_1\cdots z_k;x,y)\\=&\sum_{P\{i(k-1)\}}\{G_0(x-z_k)\gamma^{\mu_k}G_0(z_k-z_{i(1)})\gamma^{\mu_{i(1)}}\cdots\\&\times G_0(z_{i(k-1)}-y)+\sum_{j=1}^{k-2}G_0(x-z_{i(1)})\cdots G_0(z_{i(j)}-z_k)\\&\times\gamma^{\mu_k}G_0(z_k-z_{i(j+1)})\cdots G_0(z_{i(k-1)}-y)\\&+G_0(x-z_{i(1)})\cdots G_0(z_{i(k-1)}-z_k)\gamma^{\mu_k}G_0(z_k-y)\}\,,\end{aligned} \tag{A8}$$

the three terms corresponding, respectively, to $i(k)=1$, $1<i(k)<k$, and $i(k)=k$ in (A4). Now applying (A7) to the vertices involving γ^{μ_k}, and using the fact that $G_0(a)\gamma^\mu$ commutes with $G_0(b)\gamma^\nu$ [this follows from Eq. (2.2), which shows that $\gamma^\mu\gamma^\nu$ is a diagonal matrix], we find that all but the first and last terms cancel, and the result can be written

$$\begin{aligned}G^{\mu_1\cdots\mu_k}&(z_1\cdots z_k;x,y)\\&=G_0^{\mu_k}(z_k;x,y)\sum_{P\{i(k-1)\}}G_0(x-z_{i(1)})\gamma^{\mu_{i(1)}}\cdots\\&\qquad\times G_0(z_{i(k-1)}-y)\,.\end{aligned} \tag{A9}$$

Use of the induction hypothesis completes the proof.