Classical Electrodynamics in Two Dimensions: Exact Solution*

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The classical electrodynamics of N charged pointlike particles, moving in the space-time of two dimensions, is studied. The exact solution of this model is found, and its properties connected with the Poincaré group in two dimensions are described. In the case of $N=2$, our solution reduces to a model of relativistic classical mechanics discovered by Currie and Jordan and by Beard and Fong. Nonlinear generalizations of the model are also discussed.

I. INTRODUCTION

EVEN in classical relativistic theories, exact solutions are not easily found. In this paper we lower the number of space dimensions from three to one—the trick that has recently become very popular in quantum theories—in order to obtain an exactly soluble model of a relativistic theory.

The first model, which we discuss in Secs. II and III, describes N charged pointlike particles coupled to the electromagnetic field.[1] Except for the number of space dimensions, this model does not differ from Maxwell-Lorentz electrodynamics.

When there are only two particles present we reproduce the results found by Currie and Jordan[2] and by Beard and Fong[3] in their studies of relativistic theories of particles.

In Sec. IV we briefly describe nonlinear generalizations of our model.

II. FIELD AND PARTICLE EQUATIONS

Following the standard procedure, we shall derive the equations of motion for the field and the particles from the variational principle, starting from the following action integral W:

$$W=-\frac{1}{4}\int d^2x\, f_{\mu\nu}(x)f^{\mu\nu}(x)-\sum_A m_A\int ds_A -\sum_A e_A\int d\xi_A{}^\mu\, A_\mu(\xi_A)\,, \quad (1)$$

where m_A, e_A, and $\xi_A{}^\mu$ denote, respectively, the mass, charge, and space-time coordinates of the Ath particle. The differential of the proper time ds_A is defined, as usual, as

$$ds_A=dt(1-\mathbf{v}_A{}^2)^{1/2}. \quad (2)$$

The variations of W with respect to the vector potential A_μ and the particle coordinates $\xi_A{}^\mu$ give the following equations of motion:

$$\partial_\mu f^{\mu\nu}(x)=\sum_A e_A\int d\xi_A{}^\nu\,\delta_{(2)}(x-\xi_A)\,, \quad (3a)$$

$$m_A\frac{d^2\xi_A{}^\mu}{ds^2}=e_Af^\mu{}_\nu(\xi_A)\frac{d\xi_A{}^\nu}{ds}. \quad (3b)$$

The homogeneous Maxwell equations are satisfied identically in the space-time of two dimensions. In nonrelativistic notation Eqs. (3) become

$$\partial_t f(\mathbf{x},t)=-\sum e_A\mathbf{v}_A\delta(\mathbf{x}-\boldsymbol{\xi}_A(t))\,, \quad (4a)$$

$$\partial_x f(\mathbf{x},t)=\sum e_A\delta(\mathbf{x}-\boldsymbol{\xi}_A(t))\,, \quad (4b)$$

$$m_A\frac{d}{dt}[\mathbf{v}_A(t)(1-\mathbf{v}_A{}^2)^{-1/2}]=e_Af(\boldsymbol{\xi}_A(t),t)\,, \quad (4c)$$

where $\mathbf{x}$ and $\boldsymbol{\xi}$ are the space components of x and ξ and $f(x)=f_{01}(x)$. The field equations (4a) and (4b) can be easily integrated to give

$$f(\mathbf{x},t)=\sum e_A\theta(\mathbf{x}-\boldsymbol{\xi}_A(t))+c_1 =-\sum e_A\theta(\boldsymbol{\xi}_A(t)-\mathbf{x})+c_2. \quad (5)$$

The field strength, therefore, has a discontinuity equal to e_A at the position of the Ath particle. These discontinuities make it impossible to find the motion of the particles directly from Eq. (4c), because the right-hand side of this equation is ambiguous. In the next section we shall show that an alternative method based on the canonical formalism leads to an unambiguous expression for the force.

III. CANONICAL FORMALISM

For the canonical formulation of our model, we introduce the energy-momentum tensor $T_{\mu\nu}$,

$$T_{\mu\nu}(x)=f_{\mu\lambda}(x)f^\lambda{}_\nu(x)+\tfrac{1}{4}g_{\mu\nu}f_{\lambda\rho}(x)f^{\lambda\rho}(x) +\sum_A m_A\int d\xi_{A\mu}\frac{d\xi_{A\nu}}{ds}\delta_{(2)}(x-\xi_A)\,, \quad (6)$$

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[1] The term "electromagnetic" is probably not quite appropriate here, since the field-strength tensor $f_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ has only one independent component. We shall use it here, however, to stress the formal similarity between our model and standard electrodynamics.

[2] D. G. Currie and T. F. Jordan, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA, p. 91.

[3] A. N. Beard and R. Fong, Phys. Rev. **182**, 1397 (1969).

and calculate the total energy E,

$$E=\int d\mathbf{x}\, T_{00}(\mathbf{x},t)=\sum_A m_A(1-\mathbf{v}_A{}^2)^{-1/2}+\frac{1}{2}\int d\mathbf{x}\, f^2(\mathbf{x},t). \quad (7)$$

After expressing the total energy in terms of the canonically conjugate particle variables ξ_A and $\mathbf{p}_A$,

$$\mathbf{p}_A=m_A\mathbf{v}_A(1-\mathbf{v}_A{}^2)^{-1/2}, \quad (8)$$

we obtain the Hamiltonian H of the system,

$$H(\xi_A,\mathbf{p}_A)=\sum_A (m_A{}^2+\mathbf{p}_A{}^2)^{1/2}+\frac{1}{2}\int d\mathbf{x}\, f^2(\mathbf{x},t), \quad (9)$$

where $f(\mathbf{x},t)$ stands for a known function of the ξ_A's given by Eq. (5). In order to make the integral of f^2 finite, we must restrict ourselves to systems of particles which are neutral as a whole. In that case the constants c_1 and c_2 in Eq. (5) can be chosen to be zero so that the field f vanishes both to the right and to the left of the system of particles. The Hamiltonian can now be written in the following form:

$$\begin{aligned}H(\xi_A,\mathbf{p}_A)&=\sum_A (m_A{}^2+\mathbf{p}_A{}^2)^{1/2}-\tfrac{1}{2}\sum_{AB} e_Ae_B\int d\mathbf{x}\,\theta(\xi_A-\mathbf{x})\theta(\mathbf{x}-\xi_B)\\&=\sum_A (m_A{}^2+\mathbf{p}_A{}^2)^{1/2}-\tfrac{1}{2}\sum_{AB} e_Ae_B|\xi_A-\xi_B|. \quad (10)\end{aligned}$$

The canonical equations of motion derived with the use of this Hamiltonian are

$$\frac{d\xi_A}{dt}=\frac{\partial H}{\partial \mathbf{p}_A}=\mathbf{p}_A(m_A{}^2+\mathbf{p}_A{}^2)^{-1/2}, \quad (11a)$$

$$\frac{d\mathbf{p}_A}{dt}=-\frac{\partial H}{\partial \xi_A}=\tfrac{1}{2}e_A\sum_B e_B\epsilon(\xi_A-\xi_B). \quad (11b)$$

To test the relativistic covariance of this model, we construct from the energy-momentum tensor the two remaining generators of the Poincaré group. In terms of the canonical variables, they read

$$P=\sum_A \mathbf{p}_A, \quad (12a)$$

$$K=\sum_A \xi_A(m_A{}^2+\mathbf{p}_A{}^2)^{1/2}-\tfrac{1}{4}\sum_{AB} e_Ae_B|\xi_A{}^2-\xi_B{}^2|. \quad (12b)$$

The generators H, P, and K obey the Poisson-bracket relations of the Poincaré group:

$$\{H,P\}=0, \quad \{K,H\}=P, \quad \{K,P\}=H. \quad (13)$$

Also, the coordinates of the particles transform properly[4] under the action of these generators,

$$\{\xi_A,P\}=1, \quad \{\xi_A,H\}=\mathbf{v}_A, \quad \{\xi_A,K\}=\xi_A\mathbf{v}_A. \quad (14)$$

Equations (13) and (14) express the relativistic covariance of our model. The Dirac-Schwinger condition[5]

$$\{T_{00}(\mathbf{x},t),T_{00}(\mathbf{y},t)\}=[T_{01}(\mathbf{x},t)+T_{01}(\mathbf{y},t)]\partial_x\delta(\mathbf{x}-\mathbf{y}), \quad (15)$$

which expresses the local relativistic covariance, can also be verified in this model. [To evaluate the Poisson bracket in Eq. (15), we eliminate the field with Eq. (5).]

IV. NONLINEAR GENERALIZATIONS OF MODEL

Any nonlinear version of our model, in which the field Lagrangian is chosen to be an arbitrary function of the invariant $-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}=\frac{1}{2}f^2$, is also exactly soluble. Such modifications do not change the main feature of the model, that forces acting between particles are constant. What is changed, however, is the simple two-body character of the forces. As we shall see, the force acting on one of the particles cannot be decomposed in the nonlinear case into a sum of contributions, each attributable to one particle. To show this character of the forces, we will describe briefly a general nonlinear model of electrodynamics in two dimensions.

The action integral for the field will now be

$$W_f=\int d^2x\, \mathcal{L}(x), \quad (16)$$

where $\mathcal{L}(x)$ is any[6] local function of f^2,

$$\mathcal{L}(x)=F(f^2(x)). \quad (17)$$

The field equations in this case are

$$\partial_\mu h^{\mu\nu}(x)=\sum_A e_A\int d\xi_A{}^\nu\, \delta_{(2)}(x-\xi_A), \quad (18)$$

where

$$h^{\mu\nu}(x)=-\frac{\partial\mathcal{L}(x)}{\partial f_{\mu\nu}}. \quad (19)$$

The independent component $h^{10}(x)$ of this field will be called $h(x)$,

$$h=\frac{\partial F}{\partial f}. \quad (20)$$

The total energy of the system is

$$E=\sum_A m_A(1-\mathbf{v}_A{}^2)^{-1/2}+\int d\mathbf{x}\left(f\frac{\partial F}{\partial f}-F\right), \quad (21)$$

[4] Reference 2, p. 93.

[5] P. A. M. Dirac, Rev. Mod. Phys. **34**, 592 (1962); J. Schwinger, Phys. Rev. **130**, 800 (1963).

and the Hamiltonian is

$$H(\xi_A,\mathbf{p}_A)=\sum_A (m_A{}^2+\mathbf{p}_A{}^2)^{1/2}+\int d\mathbf{x}\, G(h^2)\,, \quad (22)$$

where G is the energy density of the field expressed as a function of h,

$$G(h^2)=\left(f\frac{\partial F}{\partial f}-F\right)_{f=f(h)}. \quad (23)$$

The field h in formulas (22) and (23) is a function of the ξ_A's obtained by integrating Eq. (18),

$$h(\mathbf{x},t)=\sum_A e_A\theta(\mathbf{x}-\xi_A(t))=-\sum_A e_A\theta(\xi_A(t)-\mathbf{x})\,. \quad (24)$$

The integrated expression for the interaction energy can be simplified if we number the particles according to their instantaneous configuration—say, from the right to the left. The Hamiltonian (22) then becomes

$$\begin{aligned}H(\xi_A,\mathbf{p}_A)=&\sum_A (m_A{}^2+\mathbf{p}_A{}^2)^{1/2}+(\xi_2-\xi_1)G(e_1{}^2)\\&+(\xi_3-\xi_2)G((e_1+e_2)^2)+\cdots\\&+(\xi_{B+1}-\xi_B)G((\sum_{A=1}^{B} e_A)^2)+\cdots\\&+(\xi_N-\xi_{N-1})G(e_N{}^2)\,, \quad (25)\end{aligned}$$

and the force acting on the Bth particle is

$$G((e_1+\cdots+e_B)^2)-G((e_1+\cdots+e_{B-1})^2)\,. \quad (26)$$

It is only in the linear case that this expression breaks up into a sum of terms e_Be_A.

V. DISCUSSION

We believe that the analysis of the two-dimensional version of electrodynamics which is presented in this paper sheds some light on the interesting and intensely studied problem of the existence of interactions in relativistic classical particle mechanics.[7] The model of a relativistic two-particle system with the constant interparticle force has been introduced in Refs. 2 and 3 to provide an example of a system of interacting particles. Our results show that this interaction can be attributed to a field, even though the only dynamical variables which need be introduced in this model are particle coordinates and momenta. The field generating the interaction contains only the one-dimensional analog of the Coulomb field and has no degrees of freedom which could be excited. However, we can discover the field behind these particle interactions if we search for the local theory which underlies this relativistic model of particle mechanics. The local aspects are most easily described in terms of the energy-momentum tensor, and the formula for the energy density shows that the interaction energy is distributed in the space *between* the particles. This is a clear indication that the space is filled with the energy-carrying field, which suggests to me that the system of interacting particles with constant two-particle forces is not complete as a particle system.

Note added in proof. After this paper was submitted for publication, Dr. A. Staruszkiewicz has pointed out to me that he derived the motion of two particles in two-dimensional electrodynamics from the Fokker action principle.[8]

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[6] There are some obvious requirements that this function must obey; for example, the energy density should be positive.

[7] In Ref. 2 one can find an extensive bibliography of papers on relativistic classical particle mechanics. Most of the papers written on this subject were concerned with particle interactions in four-dimensional space-time. We fully realize that "no-interaction" theorems [D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963)] tell us that in four dimensions we cannot eliminate the field and obtain a particle model with canonical transformations for the Poincaré group.

[8] A. Staruszkiewicz, Am. J. Phys. **35**, 437 (1967).