For the exceptional algebras,

$$G_3:\ \langle 0,1\rangle\otimes\langle 0,1\rangle=[\langle 0,0\rangle\oplus\langle 2,0\rangle\oplus\langle 0,2\rangle]_{\text{symm}}\oplus[\langle 0,1\rangle\oplus\langle 3,0\rangle]_{\text{antisym}},$$
$$F_4:\ \langle 0,1,0,0\rangle\otimes\langle 0,1,0,0\rangle=[\langle 0,0,0,0\rangle\oplus\langle 2,0,0,0\rangle\oplus\langle 0,2,0,0\rangle]_{\text{symm}}\oplus[\langle 0,1,0,0\rangle\oplus\langle 1,0,1,0\rangle]_{\text{antisym}},$$
$$E_6:\ \langle 0,0,1,0,0,0\rangle\otimes\langle 0,0,1,0,0,0\rangle$$
$$=[\langle 0,0,0,0,0,0\rangle\oplus\langle 0,0,2,0,0,0\rangle\oplus\langle 1,1,0,0,0,0\rangle]_{\text{symm}}\oplus[\langle 0,0,1,0,0,0\rangle\oplus\langle 0,0,0,0,0,1\rangle]_{\text{antisym}},$$
$$E_7:\ \langle 0,1,0,0,0,0,0\rangle\otimes\langle 0,1,0,0,0,0,0\rangle \qquad \text{(B10)}$$
$$=[\langle 0,0,0,0,0,0,0\rangle\oplus\langle 0,0,0,1,0,0,0\rangle\oplus\langle 0,2,0,0,0,0,0\rangle]_{\text{symm}}\oplus[\langle 0,1,0,0,0,0,0\rangle\oplus\langle 0,0,0,0,1,0,0\rangle]_{\text{antisym}},$$
$$E_8:\ \langle 1,0,0,0,0,0,0,0\rangle\otimes\langle 1,0,0,0,0,0,0,0\rangle$$
$$=[\langle 0,0,0,0,0,0,0,0\rangle\oplus\langle 0,1,0,0,0,0,0,0\rangle\oplus\langle 2,0,0,0,0,0,0,0\rangle]_{\text{symm}}$$
$$\oplus[\langle 1,0,0,0,0,0,0,0\rangle\oplus\langle 0,0,1,0,0,0,0,0\rangle]_{\text{antisym}}.$$

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Relativistic Hydrodynamics in One Dimension*

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Hydrodynamic equations for one-dimensional motion, of interest in supernova explosions, are integrated in the relativistic limit. A simple solution is found for free expansion into a vacuum. The propagation of a shock into a medium of decreasing density is determined, and the solution for the subsequent flow behind the shock is also obtained.

I. INTRODUCTION

EXTREME relativistic motions of a fluid can occur in supernova explosions as the result of a strong shock propagating through the outermost mantle of the star. It has been proposed that cosmic radiation is matter ejected from the surface of the star in this manner.[1]

In this paper the hydrodynamic equations for one-dimensional motion are integrated in the relativistic limit. A simple solution is found for free expansion into a vacuum. The propagation of a shock into a medium of decreasing density is also determined, and the solution for the subsequent flow behind the shock front is obtained.

II. HYDRODYNAMIC EQUATIONS

The equations for the motion of a fluid in the absence of external forces are obtained by setting the divergence of the energy-momentum tensor equal to zero.[2] Let p be the pressure, E the proper energy density, c the speed of light, and βc the fluid speed. For one-dimensional motion in the x direction, the vanishing of the divergence gives

$$\frac{\partial}{\partial x}\left(\frac{p+\beta^2E}{1-\beta^2}\right)+\frac{\partial}{\partial ct}\left(\frac{\beta(p+E)}{1-\beta^2}\right)=0, \qquad (1)$$

$$\frac{\partial}{\partial x}\left(\frac{\beta(p+E)}{1-\beta^2}\right)+\frac{\partial}{\partial ct}\left(\frac{E+\beta^2p}{1-\beta^2}\right)=0. \qquad (2)$$

There is also a conservation law for the nucleon number density,

$$\frac{\partial}{\partial x}\frac{n\beta}{(1-\beta^2)^{1/2}}+\frac{\partial}{\partial ct}\frac{n}{(1-\beta^2)^{1/2}}=0, \qquad (3)$$

where n is the nucleon number density in the proper frame of reference. In the nonrelativistic limit, (1)–(3) reduce to the classical forms of momentum, energy, and mass conservation.

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[1] S. A. Colgate and M. H. Johnson, Phys. Rev. Letters **5**, 235 (1960).
[2] L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Addison-Wesley, Reading, Mass., 1959).

An equation of state is needed to complete the description of the fluid. In the extreme relativistic limit, when most of the energy is in radiation or lepton pairs,

$$p=\tfrac{1}{3}E. \qquad (4)$$

When the rest energy is not negligible, a better approximation is

$$p=\tfrac{1}{3}(E-nmc^2), \qquad (5)$$

where m is the nucleon mass.[3]

Sound propagation may be examined by eliminating the term $\beta(p+E)/(1-\beta^2)$ between (1) and (2), with the result

$$\frac{\partial^2}{\partial x^2}\frac{p+\beta^2E}{1-\beta^2}-\frac{\partial^2}{\partial(ct)^2}\frac{E+\beta^2p}{1-\beta^2}=0. \qquad (6)$$

If the amplitude of the sound wave is not too large, it is always possible to choose a reference frame where $\beta^2 \ll 1$, so that

$$\frac{\partial^2 p}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}=0. \qquad (7)$$

For adiabatic variations, ΔE and Δp are related by

$$\Delta E=(\partial E/\partial p)_a\Delta p=\beta_a^{-2}\Delta p, \qquad (8)$$

where $(\partial E/\partial p)_a$ is the adiabatic derivative of the energy density with respect to pressure. Then (7) becomes a wave equation with a propagation speed $c\beta_a$. For the equation of state (4), Eq. (8) holds for finite displacements and, through second-order terms in the amplitude, sound waves propagate with a constant speed $c/\sqrt{3}$. The constancy of β_a significantly simplifies the hydrodynamics.

The conditions across a strong shock front are conveniently treated in the frame of reference in which the shocked fluid is at rest.[4] The internal energy of the unshocked fluid is ignored. Let β be the velocity of the unshocked fluid, $c\beta_s$ the speed of the shock front relative to the shocked fluid, and define $\gamma=(1-\beta^2)^{-1/2}$. Then (1)–(3) are replaced by corresponding continuity conditions across the shock:

$$p=\gamma^2 n_0 mc^2(\beta+\beta_s)\beta, \qquad (9)$$

$$E\beta_s c=\gamma^2 n_0 mc^2(\beta+\beta_s)c, \qquad (10)$$

$$n\beta_s c=\gamma n_0(\beta+\beta_s)c. \qquad (11)$$

Dividing (10) by (11) gives

$$E/nmc^2=\gamma. \qquad (12)$$

[3] Both forms of the equation of state assume that the lepton pairs are either absent or are extremely relativistic. For a gas of photons and $e^{\pm}$ pairs this approximation leads to an error in the pressure of at most about 20% [this is readily derived from Table II of C. F. McKee, Astrophys. J. **151**, 647 (1968)]. Equation (4) further assumes $nmc^2 \ll aT^4$, or $n \ll 5\times10^{-12}\,T^4$ (T in °K), while (5) neglects the nucleon pressure, which is valid for $n \ll 20T^3$.

[4] E. Teller (unpublished).

If μ is the ratio of proper energy to the nucleon rest energy, (12) becomes

$$\gamma=\mu. \qquad (13)$$

Equation (13) expresses the condition that the energy per nucleon ahead of the shock γmc^2 is equal to the energy per nucleon behind the shock μmc^2. In the nonrelativistic limit (13) reduces to the usual condition that the internal energy and kinetic energy behind a strong shock are equal. The shock velocity can be obtained by dividing (10) by (9):

$$\beta_s=p/E\beta=\tfrac{1}{3}[(\mu-1)/(\mu+1)]^{1/2}. \qquad (14)$$

In the last step (5) and (13) have been used. In the extreme relativistic limit $\mu\to\infty$, so that $\beta_s=\frac{1}{3}$. Since the speed of sound is $c/\sqrt{3}$, the shock is subsonic. Furthermore, in this limit $\gamma_s'=\sqrt{2}\gamma$, where $c\beta_s'$ is the shock velocity in the reference frame of the unshocked fluid.

The compression, defined as the ratio of proper densities, is given from (11), (13), and (14):

$$n/n_0=\gamma(1+\beta/\beta_s)=4\mu+3. \qquad (15)$$

Because of the Lorentz contraction, the "apparent compression" $n/\gamma n_0$ is $4+3/\mu$, or just 4 in the relativistic limit. However, the apparent compression in the reference frame of the unshocked fluid is $\gamma n/n_0=\mu(4\mu+3)$, or $4\mu^2$ in the relativistic limit: the fluid appears highly compressed by the shock. The energy density can also be expressed in terms of μ through Eq. (15):

$$E\equiv\mu nmc^2=n_0mc^2\mu(4\mu+3). \qquad (16)$$

III. RIEMANN INVARIANTS

As originally shown by Taub,[5,6] the relativistic hydrodynamic equations can be integrated by the method of characteristics. Multiplying (2) by β and subtracting from (1), and vice versa, yields the two equations

$$\beta\frac{\partial\beta}{\partial x}+\frac{\partial\beta}{\partial ct}=-\left(\frac{1-\beta^2}{p+E}\right)\left(\frac{\partial p}{\partial x}+\beta\frac{\partial p}{\partial ct}\right), \qquad (17)$$

$$\beta\frac{\partial E}{\partial x}+\frac{\partial E}{\partial ct}=-\left(\frac{p+E}{1-\beta^2}\right)\left(\frac{\partial\beta}{\partial x}+\beta\frac{\partial\beta}{\partial ct}\right). \qquad (18)$$

The combination of derivatives on the left-hand sides of (17) and (18) are convective time derivatives following the motion of a fluid element. Because heat conduction and viscosity terms have been omitted from the energy-momentum tensor, the volume changes of a particular fluid element are adiabatic. Consequently, the adiabatic derivative (8) can be introduced to rewrite (18) as

$$\beta\frac{\partial p}{\partial x}+\frac{\partial p}{\partial ct}=-\left(\frac{\beta_a^2(p+E)}{1-\beta^2}\right)\left(\frac{\partial\beta}{\partial x}+\beta\frac{\partial\beta}{\partial ct}\right). \qquad (19)$$

[5] A. H. Taub, Phys. Rev. **74**, 328 (1948).

[6] A. H. Taub, Phys. Rev. **107**, 884 (1957).

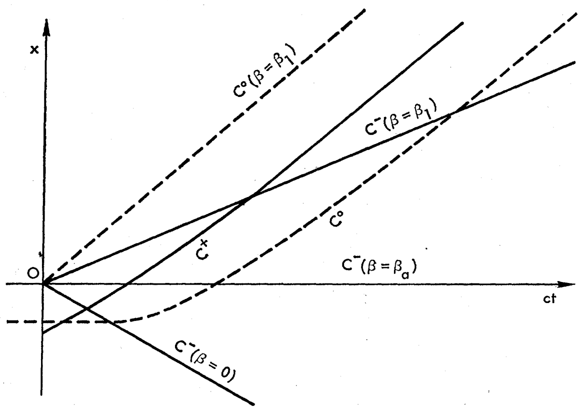

FIG. 1. Expansion into a vacuum. At $t=0$ the fluid is at rest and extends from $x=0$ to $x=-\infty$. Three backward characteristics C^- of the rarefaction fan at O are shown. Above the characteristic $C^-(\beta=\beta_1)$ the fluid moves with constant velocity β_1, the velocity of the leading edge of the expanding fluid. Along $C^-(\beta=\beta_a)$ the fluid speed and speed of sound are equal. A sound wave propagates backward into the fluid at rest along $C^-(\beta=0)$. A typical particle world line C^0 and a typical forward characteristic C^+ are also shown. Above $C^-(\beta=\beta_1)$ the speed of sound is zero and C^+ is parallel to C^0. Since all points in the fluid can be reached by forward characteristics originating from the fluid at rest at $t=0$, the forward invariant has the same value $E_i^{(\sqrt{3})/2}$ throughout the fluid.

If p is a unique function of E, it is convenient to introduce a new dependent variable σ:

$$\sigma=\int\frac{dp}{\beta_a(p+E)}. \tag{20}$$

Two cases will be considered: the radiation-dominated equation of state (4), and isentropic flow, in which p and E are connected by the adiabatic expansion law. Then (17) and (19) become

$$\left(\beta\frac{\partial}{\partial x}+\frac{\partial}{\partial ct}\right)\tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)=-\beta_a\left(\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial ct}\right)\sigma, \tag{21}$$

$$\left(\beta\frac{\partial}{\partial x}+\frac{\partial}{\partial ct}\right)\sigma=-\beta_a\left(\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial ct}\right)\tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right). \tag{22}$$

By addition and subtraction, (21) and (22) become

$$\left[(\beta+\beta_a)\frac{\partial}{\partial x}+(1+\beta\beta_a)\frac{\partial}{\partial ct}\right]\left[\sigma+\tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]=0, \tag{23}$$

$$\left[(\beta-\beta_a)\frac{\partial}{\partial x}+(1-\beta\beta_a)\frac{\partial}{\partial ct}\right]\left[\sigma-\tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]=0. \tag{24}$$

Since $(\beta+\beta_a)(1+\beta\beta_a)^{-1}$ is the relativistic sum of the fluid velocity and sound velocity, (23) states that the quantity $\sigma+\frac{1}{2}\ln[(1+\beta)(/1-\beta)]$ is constant on the world line of a sound wave traveling in the same direction as the fluid. This world line is the forward characteristic, designated C^+. Similarly, (24) states that $\sigma-\frac{1}{2}\ln[(1+\beta)/(1-\beta)]$ is constant on the world line of a sound wave traveling in the opposite direction to the fluid, i.e., on the backward characteristic C^-. In the nonrelativistic limit, $\sigma\pm\frac{1}{2}\ln[(1+\beta)/(1-\beta)]$ reduce to the Riemann invariants of classical hydrodynamics.

The evaluation of σ is immediate for the equation of state (4). Then $\beta_a=\frac{1}{3}\sqrt{3}$, and (20) becomes

$$\sigma=\tfrac{1}{4}\sqrt{3}\int\frac{dE}{E}=\tfrac{1}{4}\sqrt{3}\ln E. \tag{25}$$

The invariants on the forward and backward characteristics can then be taken as

$$\begin{aligned}I^{\pm}&=[(1+\beta)/(1-\beta)]E^{\pm(\sqrt{3})/2}\\&\cong 4\gamma^2E^{\pm(\sqrt{3})/2},\end{aligned} \tag{26}$$

where the second line is the relativistic limit. The adiabatic expansion laws are those for enclosed radiation:

$$p=p_0(V_0/V)^{4/3}, \tag{27}$$

$$E=E_0(V_0/V)^{4/3}, \tag{28}$$

$$\mu=E/nmc^2=\mu_0(V_0/V)^{1/3}, \tag{29}$$

where the subscript 0 designates the initial value of a quantity.

To evaluate σ using the equation of state (5), the adiabatic expansion laws are needed. Since the pressure and internal energy density $E-nmc^2$ are still connected by the laws for radiation, the pressure law (27) remains valid. Equation (29) is replaced by

$$\mu-1=3p/nmc^2=(\mu_0-1)(V_0/V)^{1/3}, \tag{30}$$

while (28) becomes

$$E=nmc^2\mu=E_0(V_0/V)[\mu_0^{-1}+(1-\mu_0^{-1})(V_0/V)^{1/3}]. \tag{31}$$

The adiabatic derivative $(\partial E/\partial p)_a$ is formed by differentiating (27) and (31) with respect to V and forming the ratio. The result is

$$\beta_a=(\partial E/\partial p)_a^{-1/2}=\tfrac{1}{3}\sqrt{3}(\mu-1)^{1/2}(\mu-\tfrac{1}{4})^{-1/2}. \tag{32}$$

Since $(\mu-1)(\mu-\frac{1}{4})^{-1}<1$ for $\mu>1$, the speed of sound is always less than the extreme relativistic value $c/\sqrt{3}$. Evaluation of (20) gives

$$\sigma=2\sqrt{3}\ln[2(\mu-1)^{1/2}/\sqrt{3}+2(\mu-\tfrac{1}{4})^{1/2}/\sqrt{3}]. \tag{33}$$

The invariant on the forward and backward characteristic can now be taken as

$$\begin{aligned}I^{\pm}&=(1+\beta)(1-\beta)^{-1}[2(\mu-1)^{1/2}/\sqrt{3}+2(\mu-\tfrac{1}{4})^{1/4}/\sqrt{3}]\\&\cong 4^{1\pm4\sqrt{3}}\gamma^2(\tfrac{1}{3}\mu)^{\pm2\sqrt{3}}.\end{aligned} \tag{34}$$

The second line is the relativistic limit, which differs only by a constant from (26) since $E\propto\mu^4$ in an adiabatic expansion.

IV. EXPANSION INTO A VACUUM

Let the fluid be at rest at $t=0$, extending from $x=0$ to $x=-\infty$. Also let the initial energy density E_i be the same throughout the fluid so that the quantity σ in (25) also has the same value σ_i throughout the fluid at $t=0$. Since every point in the expanding fluid can be reached by a sound wave originating from a point in the fluid at rest, the invariant on the forward characteristic $(1+\beta)(1-\beta)^{-1}e^{2\sigma}$ is equal to $e^{2\sigma_i}$ throughout the entire fluid. Now on a backward characteristic the quantity $(1+\beta)(1-\beta)^{-1}e^{-2\sigma}$ must be constant, so that its product with the forward invariant, given by $(1+\beta)^2(1-\beta)^{-2}$, must be constant on a backward characteristic. This implies that β is a constant and the backward characteristics are straight lines. A rarefaction fan of backward characteristics emanates from the surface of the fluid at $t=0$ (the point O in Fig. 1). At one extreme is the characteristic with $\beta=0$, which represents a sound wave propagating backward in the resting fluid. The other extreme of the fan is the characteristic for the maximum velocity β_1 attained by the leading edge of the expanding fluid.

The velocity of the leading edge is obtained by solving the equation

$$(1+\beta_1)(1-\beta_1)^{-1}=e^{2(\sigma_i-\sigma_1)}. \qquad (35)$$

For the equation of state (4), Eq. (35) becomes

$$(1+\beta_1)(1-\beta_1)^{-1}\cong 4\gamma_1^2=(E_i/E_1)^{(\sqrt{3})/2}. \qquad (36)$$

The result diverges as $E_1\rightarrow 0$. The difficulty can be evaded by assuming that (4) only holds for $\mu>1$ and that the pressure drops to zero when μ reaches the value 1; thereafter the fluid expands without acceleration. Combining (28) and (29) gives

$$E_i/E=(\mu_i/\mu)^4, \qquad (37)$$

where $\mu_i=E_i(n_imc^2)^{-1}$ and E attains its least value when $\mu=1$. Setting E equal to this least value in (36) gives

$$\gamma_1=\tfrac{1}{2}\mu_i^{\sqrt{3}}. \qquad (38)$$

Thus the leading edge acquires a directed energy which is $\frac{1}{2}\mu_i^{\sqrt{3}-1}$ times as large as its initial energy.

If the equation of state (5) is used, (35) becomes

$$(1+\beta_1)(1-\beta_1)^{-1}=[2(\mu_i-1)^{1/2}/\sqrt{3}+2(\mu_i-\tfrac{1}{4})^{1/2}/\sqrt{3}]^{4\sqrt{3}}. \qquad (39)$$

In the relativistic limit, (39) reduces to

$$\gamma_1=\tfrac{1}{2}(16/3)^{\sqrt{3}}\mu_i^{\sqrt{3}}. \qquad (40)$$

Comparison with (44) shows that the cutoff procedure used above underestimates the final energy by a factor $(16/3)^{\sqrt{3}}$. In the nonrelativistic limit, (39) becomes

$$\beta_1=6\beta_a. \qquad (41)$$

This is the result to be expected from classical hydrodynamics for a fluid obeying the adiabatic expansion laws (30) and (31).[2]

V. SHOCK PROPAGATION

In the relativistic limit a solution can be obtained for the propagation of a strong shock into a medium of de-

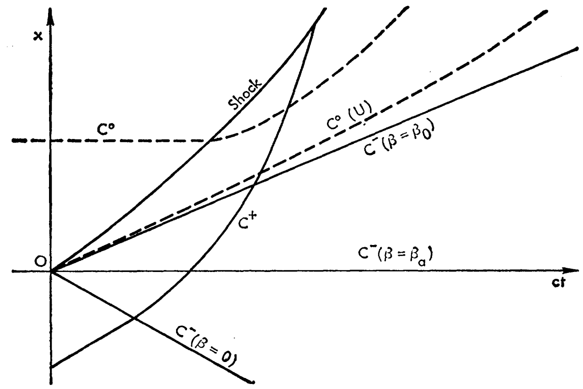

FIG. 2. Shock driven by expanding fluid. The initial conditions are the same as for the expansion into a vacuum problem shown in Fig. 1, except that the region $x>0$ contains a fluid of decreasing nucleon density n and energy density nmc^2. As before, three backward characteristics C^- of the rarefaction fan emanating from O are shown. The characteristic C^- $(\beta=\beta_0)$ corresponds to the velocity β_0 of the shocked fluid at O. A contact discontinuity [labeled $C^0(U)$] separates the shocked fluid from the unshocked expanding fluid.

creasing density. At $t=0$, let a hot fluid of constant energy density E_i occupy the region $x<0$ and a cold fluid with $\mu=1$ and decreasing density n occupy the region $x>0$; both are at rest. Forward characteristics originating in the hot fluid carry the constant value of the forward invariant $E_i^{(\sqrt{3})/2}$ throughout the region behind the shock. Then, just as in the problem of expansion into a vacuum, the backward characteristics are straight lines which β and γ are constant. A rarefaction fan of backward characteristics emanates from O (Fig. 2). A contact discontinuity separates the shocked from the unshocked fluid. Since the pressure is continuous across the discontinuity, the medium appears continuous insofar as hydrodynamic motion is concerned (the discontinuity is in the nucleon density, which does not enter the hydrodynamic equations in the extreme relativistic limit). The characteristic C^- $(\beta=0)$ is a sound wave propagating backward into the fluid at rest.

Any point on the shock front can be reached by forward characteristics carrying the invariant $E_i^{(\sqrt{3})/2}$, so that for $\gamma\gg 1$

$$4\gamma^2E^{(\sqrt{3})/2}=E_i^{(\sqrt{3})/2}, \qquad (42)$$

where γ and E are at the shock front. By using (13) and (16), Eq. (42) becomes

$$\gamma=\tfrac{1}{2}(E_i/nmc^2)^s, \quad s=\tfrac{1}{2}\sqrt{3}(2+\sqrt{3})^{-1}. \qquad (43)$$

Thus the shock strength increases approximately as $n^{-1/4}$, where n is the density just ahead of the shock.

The shocked fluid expands until $\mu=1$ and the available internal energy is exhausted. Since the forward invariant remains at the same value at the shock front, it follows that

$$\gamma_f^2E_f^{(\sqrt{3})/2}=\gamma^2E^{(\sqrt{3})/2}, \qquad (44)$$

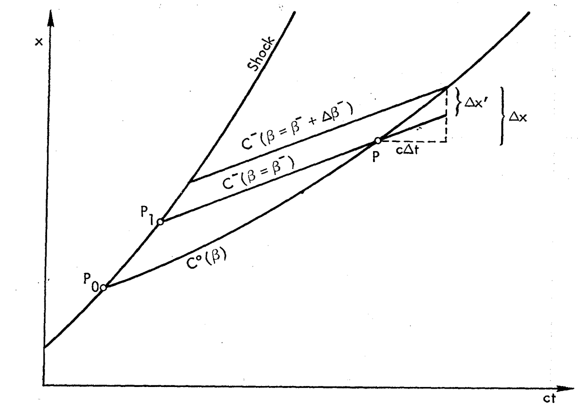

FIG. 3. Post-shock acceleration. A fluid element is shocked at $P_0(x_0,ct_0)$ and travels along C^0 to $P(x,ct)$; there it is connected to a corresponding point $P_1(x_1,ct)$ on the shock trajectory by a backward characteristic C^-. From the figure, $\Delta x=\Delta x'+(\beta^-/\beta)\Delta x$, $\Delta x'=\Delta x_1'+\Delta\beta^-(x-x_1)$, and $\Delta x_1=\Delta x_1'+(\beta^-/\beta_s)\Delta x_1$, where Δx_1 and $\Delta x_1'$ are defined analogously to Δx and $\Delta x'$. The second equation contains the approximation $c(t-t_1)=(x-x_1)/\beta^-\simeq x-x_1$.

where the subscript f denotes values at the end of the expansion. Estimating E_f from (37) just as in the problem of expansion into a vacuum, (44) becomes

$$\gamma_f=\gamma\mu^{\sqrt{3}}=\gamma^{1+\sqrt{3}}. \tag{45}$$

Consequently, γ_f is proportional to n^{-r}, with $r=\frac{1}{2}\sqrt{3}(\sqrt{3}-1)$. With a reasonable model of the stellar atmosphere in a supernova, (45) predicts a cosmic-ray energy distribution in fair agreement with experiment. The solutions of the problem of the expansion into a vacuum indicates that the method of estimating E_f underestimates the final energy by a factor like 18.

Unfortunately, a similar solution is not possible with the equation of state (5). The flow behind a shock of variable strength is not isentropic, so that E depends on both p and the entropy. Then the Riemann invariants no longer satisfy (23) and (24). Only in the extreme relativistic case is p a unique function of E.

VI. FLUID TRAJECTORIES BEHIND SHOCK

The path of a particle after passing through the shock front can be determined by relating the displacement Δx on the trajectory to a corresponding displacement Δx_1 on the shock trajectory. The corresponding points are connected by backward characteristics originating at the shock front so that the fluid velocities (and γ) are the same at corresponding points. The geometry in Fig. 3 yields

$$\Delta x(1-\beta^-/\beta)=\Delta x_1(1-\beta^-/\beta_s)+\Delta\beta^-(x-x_1), \tag{46}$$

where all quantities are defined in the reference frame of the unshocked fluid. Then, in the relativistic limit,

$$\Delta\beta^-=\Delta\left(\frac{\beta-\frac{1}{3}\sqrt{3}}{1-\beta/\sqrt{3}}\right)=(2+\sqrt{3})\Delta\beta=\frac{(2+\sqrt{3})\Delta\gamma}{\gamma^3}, \tag{47}$$

$$1-\frac{\beta^-}{\beta_s}=1-\frac{1}{2\gamma_s^2}-\left(1-\frac{1}{2\gamma_-^2}\right)=\frac{1}{4\gamma^2}\sqrt{3}(2+\sqrt{3}), \tag{48}$$

$$1-\frac{\beta^-}{\beta}=\frac{1}{2\gamma^2}(1+\sqrt{3}). \tag{49}$$

Hence (46) becomes

$$\frac{dx}{dx_1}=\frac{1}{4}(3+\sqrt{3})\left[1+\frac{4}{\sqrt{3}}(x-x_1)\frac{d}{dx_1}\ln\gamma\right]. \tag{50}$$

The integral of (50) which satisfies the boundary condition $x=x_0$ at $x_1=x_0$ is

$$x-x_1=\frac{1}{4}(\sqrt{3}-1)\gamma^{1+\sqrt{3}}\int_{x_0}^{x_1}(\gamma')^{-(1+\sqrt{3})}dx_1', \tag{51}$$

where the integration is on the shock trajectory. An appropriate form for the preshock density is that of a polytropic stellar atmosphere, in which $p\propto n^{(q+1)/q}$; then the density n_1' at x_1' is given by

$$n_1'=n_0\left(\frac{R-x_1'}{R-x_0}\right)^q \quad \text{for} \quad R-x_1'\ll R, \tag{52}$$

where R is the radius of the star and n_0 is the density at the radius x_0. Equations (43) and (52) determine γ at the shock front as a function of the shock position x_1':

$$(R-x_1')/(R-x_0)=(\gamma_0/\gamma')^{1/sq}. \tag{53}$$

The integration of (51) leads to

$$\frac{x-x_1}{R-x_1}=c_1\left[\left(\frac{\gamma}{\gamma_0}\right)^{1+\sqrt{3}+1/sq}-1\right], \tag{54}$$

or, from (53),

$$\frac{x-x_0}{R-x_0}=\left[1+c_1\left(\frac{\gamma}{\gamma_0}\right)^{1+\sqrt{3}}-(1+c_1)\left(\frac{\gamma_0}{\gamma}\right)^{1/sq}\right], \tag{54'}$$

where

$$c_1=[2(\sqrt{3}+1+q\sqrt{3})]^{-1}.$$

According to (54), when the fluid trajectory reaches the edge of the star (i.e., $x=R$), the fluid γ is given by $\gamma=\gamma_0(1+1/c_1)^l$, with $l=(1+\sqrt{3}+1/sq)^{-1}$. For a polytropic index $q=3$, this yields $\gamma\simeq 2\gamma_0$: The fluid γ has about doubled in traversing the distance $R-x_0$.

The result for an exponential atmosphere is obtained by taking the limit $q\to\infty$. Define the scale height

$$h=|d\ln n/dx|^{-1}, \tag{55}$$

so that for the density distribution (52),

$$qh=R-x. \tag{56}$$

It is readily verified that the limiting form of (52) as

$q\to\infty$ is an exponential, while (54) becomes

$$x-x_0\cong h_0(2\sqrt{3})^{-1}[(\gamma/\gamma_0)^{1+\sqrt{3}}-1]+(h_0/s)\ln(\gamma/\gamma_0). \quad (57)$$

For large γ only the first term on the right-hand side of (57) contributes, and the use of (45) gives

$$x-x_0\cong h_0(2\sqrt{3})^{-1}\gamma_f^{\sqrt{3}}. \quad (58)$$

For large γ_f, the acceleration takes place over many scale heights and indeed may take place over many stellar radii. In the latter case the approximation of one-dimensional flow breaks down.

VII. FORWARD CHARACTERISTICS

The forward characteristics C^+ behind the shock can be determined in the same way as the fluid trajectories. The differential equation replacing (50) is

$$\frac{dx}{dx_1}=\tfrac{1}{4}(2+\sqrt{3})\left[1+\frac{4}{\sqrt{3}}(x-x_1)\frac{d}{dx_1}\ln\gamma\right], \quad (59)$$

where x is now a point on C^+. Integration of (59) leads to

$$\frac{x-x_1}{R-x_1}=c_2\left[1-K\left(\frac{\gamma}{\gamma_0}\right)^{(q+2)/2qs}\right], \quad (60)$$

$$c_2=(2-\sqrt{3})/(4+2q),$$

where K is an integration constant and γ_0 is the value of the fluid γ at x_0, just behind the shock.

To evaluate K, consider a fluid element shocked at x_0 which intercepts the forward characteristic at $\gamma=\gamma_c$; then (54) and (60) yield

$$K=\left(1+\frac{c_1}{c_2}\right)\left(\frac{\gamma_c}{\gamma_0}\right)^{-(q+2)/2qs}-\frac{c_1}{c_2}\left(\frac{\gamma_c}{\gamma_0}\right)^{1/\sqrt{3}}. \quad (61)$$

One class of characteristics intersects the shock front, so that $\gamma_c=\gamma_0$ and $K=1$. The remaining characteristics end at fluid elements which are just reaching their terminal velocity; thereafter the characteristic follows a fluid path since the local speed of sound is zero. For the latter characteristics $\gamma_c=\gamma_f=\gamma_0^{1+\sqrt{3}}$ from (45), and $K=-c_1\gamma_0/c_2$ since $\gamma_0\gg 1$.

The limiting forward characteristic C_l^+, which intersects the shock at $x=R$, separates the two classes of characteristics. The equation for C_l^+ is obtained by letting $\gamma_0\to\infty$ in (60):

$$x-x_1=c_2(R-x_1); \quad (62)$$

it is the same for both classes of characteristics. Since x and x_1 are corresponding points on a backward characteristic (see Fig. 3), they are not simultaneous. The simultaneous spatial separation between the point x and the shock front is of order h/γ^2, because the apparent compression behind the shock is of order γ^2.

The limiting characteristic can also be approached by allowing γ/γ_0 to become small for fixed K and γ_0. Thus, when traced backward in time, all forward characteristics converge on C_l^+ in spite of the decreasing compression and the increasing scale height; the convergence is more rapid for the characteristics intersecting the shock line. This suggests that the results in Secs. V and VI are asymptotically correct even if all the forward characteristics do not originate in a spatially uniform medium: The spatial variation in the forward invariant can be made arbitrarily small for sufficiently large γ_c.

The results for an exponential atmosphere are qualitatively similar to those for a polytropic atmosphere. The limit as $q\to\infty$ in (60) and (61) gives

$$x-x_1=(1-\tfrac{1}{2}\sqrt{3})h[1-K(\gamma/\gamma_0)^{1/2s}],$$
$$K=\left(1+\frac{2}{\sqrt{3}}\right)\left[2(\sqrt{3}-1)\left(\frac{\gamma_c}{\gamma_0}\right)^{-1/2s}-\left(\frac{\gamma_c}{\gamma_0}\right)^{1/\sqrt{3}}\right], \quad (63)$$

as the equation for the forward characteristics. The existence of a limiting characteristic, given by

$$x-x_1=(1-\tfrac{1}{2}\sqrt{3})h, \quad (64)$$

implies that the shock outruns disturbances sufficiently far behind the shock, just as in the nonrelativistic case.[7]

[7] R. Grover and J. W. Hardy, Astrophys. J. **143**, 48 (1966).