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# Bootstrap Models and a Special Property of the Unitary Lie Algebras

C. M. ANDERSEN
*Department of Physics, College of William and Mary, Williamsburg, Virginia 23185*
AND
JOEL YELLIN*
*Institute for Advanced Study, Princeton, New Jersey 08540*
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It is shown that within the dynamical framework of $N/D$ bootstraps, $Z=0$ field theories, finite-energy sum rules, or narrow-resonance models, a self-consistent world of mesons of both parities, in which all particles interact symmetrically and belong to representations transforming as *adjoint*$\oplus$*singlet* of an arbitrary, compact, simple Lie algebra, may be constructed only for the algebras $SU(n)$.

## INTRODUCTION

IN this paper, a previous conjecture by one of us and Schmid[1] is verified by explicit construction. The conjecture amounts to the statement that in the framework of the approximations commonly made in models of the bootstrap, a self-consistent world of even- and odd-parity mesons, in which all particles interact symmetrically and belong to degenerate multiplets classifiable according to representations transforming as *adjoint*$\oplus$*singlet* with respect to an arbitrary, compact, simple, real Lie algebra,[2] makes sense only if the algebra is of unitary type.[3]

This statement, in which the choice of representation fixes the algebraic type, may be compared to the usual procedure in which a phenomenological choice is made for both algebra and representation.[4]

Though the existence of a set of conserved charges leads very naturally to the supposition that a Lie algebra is present,[5] the effectively linear nature of the constraints on amplitudes considered here[6] precludes one from fixing *a priori* either the number of conserved quantities (i.e., the rank of the associated algebra) or the identity of the input representations. Our choice of *adjoint*$\oplus$*singlet* is therefore arbitrary and the final result is consequently only mildly interesting.[7]

The mathematical nub of our result is the fact, discussed in detail in Appendix A below, that in the Clebsch-Gordan series for the direct product with itself of the adjoint representation of an arbitrary, compact, simple Lie algebra, *adjoint* appears only once, antisymmetrically, for all such algebras, except for the algebras $SU(n)$, where it appears twice, once symmetrically and once antisymmetrically.

Another way of stating our result is as follows. In the usual bootstrap approaches, one derives, among other things, linear constraints on the pure particle pieces of amplitudes (i.e., on the pole terms), and these are equivalent to eigenvalue equations for the crossing

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* Present address: Belfer Graduate School, Yeshiva University, New York, N. Y.

[1] C. Schmid and J. Yellin, Phys. Rev. **182**, 1449 (1969), Sec. III; Phys. Rev. D **2**, 1354(E) (1970).

[2] A glossary for some of the mathematical terms of this paper is as follows. A *Lie algebra* $L$ is a vector space (for our purposes, over the real or complex numbers) in which a product, denoted $[\ ,\ ]$, has been defined having the properties (i) $[x,y]=-[y,x]$, (ii) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$, (iii) $[x+y,\ z]=[x,z]+[y,z]$, and (iv) $\alpha[x,y]=[\alpha x,y]=[x,\alpha y]$ for all $x$, $y$, $z\in L$ and all real or complex numbers $\alpha$. A subspace $B$ of $L$ is an *ideal* if and only if for all $b\in B$ and $l\in L$, $[b,l]\in B$. The *derived algebra* $L'=[L,L]$, which is the subspace spanned by all $[l_1,l_2]$ where $(l_1,l_2)\in L$, is an ideal of $L$. We say that $L$ is *simple* if it contains no ideals except $\{0\}$ and $L$ and if $L'=L$. We say that $L$ is *Abelian* if $L'=\{0\}$. Let $L'=[L,L]$, $L''=[L',L']$, $\ldots$, $L^{(k)}=[L^{(k-1)},L^{(k-1)}]$. We say that $L$ is *solvable* if $L^{(h)}=0$ for some positive integer $h$. $L$ is *semisimple* if $L$ has no nonzero solvable ideal. Let $L$ be semisimple. Then we may choose a basis $\{e_1,e_2,\ldots\}$ for $L$ such that $[e_i,e_j]=if_{ijk}e_k$, where $f_{ijk}$ is completely antisymmetric under interchange of its indices. The *order* (or *dimension*) of an algebra is the number of its linearly independent $e_i$'s. A real semisimple Lie algebra $L$ is *compact* if all eignevalues of $g_{ij}\equiv f_{ikl}f_{jkl}$ have the same sign. Compactness implies the reality of the associated coupling constants. All finite-dimensional Lie algebras can be realized in matrix form. Sets of matrices of various dimensions can realize the same Lie algebra and correspond to its different *representations*. The *dimensionality* of the representation is the dimension of the corresponding matrices. In this paper we deal only with finite-dimensional representations. Two representations are *inequivalent* if one set of matrices cannot be obtained from the other by a fixed similarity transformation. Representations of different dimensionalities are necessarily inequivalent, but inequivalent representations may sometimes have the same dimensionality. A representation of a semisimple Lie albegra is *irreducible* if the corresponding matrices cannot be brought to block diagonal form by a fixed similarity transformation. The *adjoint* (or *regular*) representation, which we repeatedly use here, is realized by matrices having dimensionality equal to the order of the algebra.

[3] It is to be emphasized that our statement is one of necessity only. The induction of the symmetry itself remains an open question.

[4] See the review article of J. Mandula, J. Weyers, and G. Zweig, Ann. Rev. Nucl. Sci. (to be published). The question of the proper representations to choose in the presence of baryons is subtle, difficult, and presently under dispute. [Compare, for example, Mandula *et al.* with R. H. Capps, Phys. Rev. **185**, 2008 (1969).]

[5] Compare S. Weinberg, University of California, Berkeley, report, 1964 (unpublished). So far as is known to the authors, general arguments with respect to the induction of symmetries in bootstrap systems originate with the work of R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[6] One must be careful to distinguish between linear constraints on pole pieces of bootstrapped amplitudes, and the accompanying nonlinear constraints on cut-plus-pole contributions which fix, for example, absolute sizes of couplings. This will be discussed further below.

[7] To be fair, it should be pointed out that *adjoint*$\oplus$*singlet* may possibly be unique, in the sense of being the self-consistent choice with the smallest dimensionality. If this is true, our present result becomes somewhat more interesting. This particular question is now under investigation.

operator.[8] What we are asserting here is that assuming the system possesses a particular internal symmetry and that its particles have trilinear interactions which are both symmetric and antisymmetric, an *adjoint*$\oplus$*singlet* eigenvector, with eigenvalue one, of the crossing operator exists only for the Lie algebras $SU(n)$ $(n\geq 3)$, because these are the only Lie algebras which allow a symmetric trilinear coupling of the adjoint representation to itself.

Below, in Sec. I, we briefly summarize and review a line of reasoning which boils the basic conjecture down to the bare mathematics, using as an example spinless-spinless scattering. In Sec. II we list the Clebsch-Gordan series for the Kronecker products for *adjoint* $\otimes$*adjoint* over the entire Cartan classification,[9] leaving mathematical methods and details to the appendices.

## I. BOOTSTRAP SYSTEMS

The equations of interest here are algebraic, linear relations between bilinear combinations of coupling constants. Such relations arise in bootstrap systems based on $Z=0$ field theories,[10] $N/D$ models,[11] models using finite-energy sum rules,[12] and narrow-resonance models.[13] One example of such a relation is provided by the Jacobi identity satisfied by the structure constants of a Lie algebra[2]:

$$F_{abs}F_{cds}+F_{bcu}F_{adu}+F_{cat}F_{bdt}=0. \qquad (1.1)$$

If one takes the subscripts to be particle labels, the $F$'s can be thought of as coupling constants and (1.1) gives the sum of $s$-, $t$-, and $u$-channel exchanges of a virtual multiplet of particles, in the scattering process $a+b\rightarrow c+d$.

The four bootstrap approaches listed above yield relations of one or more of the following generic forms:

$$G_{abs}G_{abs'}=\lambda_1\delta_{ss'}, \qquad (1.2a)$$

$$V_{abfg}G_{fgc}=\lambda_2 G_{abc}, \qquad (1.2b)$$

$$G_{abs}G_{cds}+G_{bcu}G_{adu}=\lambda_3 G_{cat}G_{bdt}, \qquad (1.2c)$$

where the $\lambda_i$ are functions of the masses and couplings of the system, the $G$'s are proportional to the trilinear vertices, the matrix $V$ is to be evaluated in terms of $G$'s, and (1.2a) and (1.2c) are symbolic in the sense that in general one will get sums over $(GG)$ bilinears, each term corresponding to a different particle or Regge multiplet being exchanged.[14]

In principle, having obtained (1.2), one would then proceed to break these equations into two sets: (a) relations among the $\lambda_i$ alone and (b) relations between $(GG)$'s, $\delta_{ij}$, and $V_{ijkl}$. The form of the type-(a) relations depends violently on which bootstrap approach one has taken.[15] Type-(b) relations, from whatever starting point, allow a symmetric solution[16] which, under the assumption that all particles fall into *adjoint*$\oplus$*singlet*, turns out in the general case, as will be shown below, to be $SU(n)$, with $n\geq 3$.

To make all this clear, we will briefly remind the reader of the procedure for obtaining (1.2) in the finite-energy sum-rule approach, using the example of $2\rightarrow 2$ scattering of spinless particles.

Provided an amplitude for the scattering of spinless particles satisfies analyticity and crossing and has Regge asymptotic behavior, its discontinuity $D_\nu(\nu,t)$ in $\nu=\frac{1}{2}(s-u)$ at fixed $t$ satisfies

$$\frac{1}{2}\int_{-N}^{N}D_\nu(\nu,t)\nu^n d\nu$$

$$=\int_{-N}^{N}\Big(\sum_{\text{Regge poles}}+\sum_{\text{Regge cuts}}+\text{background integral}\Big)\times\nu^n d\nu, \qquad (1.3)$$

where $n$ is a positive integer, $N$ is arbitrary, and we have suppressed internal quantum numbers. If one drastically truncates the right-hand side of (1.3) leaving the

[8] See, for example, S. Mandelstam, Phys. Rev. **166**, 1539 (1968). By crossing operator we mean the usual matrices which take internal and ordinary spin amplitudes from channel to channel, plus the correct interchanges of four-momenta.

[9] E. Cartan, *thèse, University of Paris, 1894*, 2nd ed. (Vuibert, Paris, 1933). Cartan's list of simple Lie algebras, to which we will constantly refer below, contains the unitary algebras $A_l$ $(SU_{l+1})$, the odd-dimensional orthogonal algebras $B_l$ $(l\geq 2)$, the symplectic algebras $C_l$ $(l\geq 3)$, the even-dimensional orthogonal algebras $D_l$ $(l\geq 4)$, and the "exceptional" algebras $G_2$, $F_4$, $E_6$, $E_7$, $E_8$. (The subscripts indicate the rank of the algebra and the inequality conditions arise because of low-rank isomorphism among the various families.) Note that dimensionality in general does not uniquely identify a representation. A complete specification will be given below in Appendix B.

[10] See P. Kaus and F. Zachariasen, Phys. Rev. **171**, 1597 (1968).

[11] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963); H.-M. Chan, P. C. DeCelles, and J. E. Paton, Phys. Rev. Letters **11**, 521 (1963).

[12] See Ref. 1 for a listing of early work on this subject.

[13] See the review article of D. Sivers and J. Yellin, Rev. Mod. Phys. (to be published April 1971).

[14] For example, in the pseudoscalar-vector-tensor ($PVT$) example considered in Ref. 1, there is a $G$ for the antisymmetric $PPV$ coupling, and another $G$ for the symmetric $PPT$ coupling.

[15] It is important to distinguish between bootstraps of the first kind which fix all observables up to a single arbitrary parameter, which can be taken to be the scale of mass, and those of the second kind which leave the scale of mass and the absolute size of coupling strengths free. The $N/D$ approach is of the first kind while the narrow-resonance approach belongs to the second. The relations of type (b) are crossing-symmetric, corresponding to the fact that all the approaches treat poles in a crossing-symmetric way. The differences between bootstraps of the first and second kinds arise from the treatment of cut contributions. For example, it is well known that in any of the known approximation schemes the $N/D$ cut structure is incorrect [see A. W. Martin, Phys. Rev. **161**, 1528 (1967; C. Cronström, Acta Phys. Acad. Sci. Hung. **26**, 101 (1969)] and if one eliminates all cut structure the four approaches become identical.

[16] Evidently there are also symmetric solutions for (1.2) at least for finite systems of self-consistent families of particles such as those discussed in Ref. 1. What the general solution of the type (b) equations is, and specifically whether a nonsymmetric solution is possible for infinite families of Regge trajectories, remains an open question. A closely related question is whether or not bootstraps of the second kind (see Ref. 15) allow an adjustable amount of symmetry breaking and also whether or not they are stable under a weak external perturbation. In the classic bootstrap of the first kind the allowed solutions should form a discrete set, and no such freedom would be allowed.

Regge-pole contribution only, one gets

$$\frac{1}{2}\int_{-N}^{N} D_\nu(\nu,t)\nu^n d\nu \cong \sum_i \beta_i(t)\frac{N^{\alpha_i(t)+n+1}}{\alpha_i(t)+n+1}, \quad (1.4)$$

where $\alpha_i$ and $\beta_i$ are the trajectory and residue of the $i$th Regge pole.

To obtain equations of the form (1.2), we do two more things. First we assume the integral on the left-hand side of (1.4) can be saturated by narrow resonances; second, we use (1.4) at $t=$(mass of any resonance)$^2$.[17]

By standard arguments it is now straightforward to reduce every term in (1.4) to a product of a coupling-constant bilinear times a kinematic factor, so that we are left with a relation of the generic form of (1.2c)[18]:

$$\sum_j L_j(G_{abs}G_{cds}+G_{bcu}G_{adu})_j = \sum_i L_i'(G_{cat}G_{bdt})_i, \quad (1.5)$$

where the primed and unprimed $L$'s are kinematic factors depending on masses and coupling ratios, and the sums run over the various resonances included on the left-hand side of (1.4) and the Regge trajectories included on the right-hand side[19] of (1.4), respectively.

At this point we interpose our violently strong assumptions that a Lie algebra is present and that all particles are classifiable as *adjoint*$\oplus$*singlet* under it.[7] The $G$'s must then be proportional to the appropriate Clebsch-Gordan coefficients, and provided the class of particles included is large enough to possess both symmetric and antisymmetric trilinear couplings,[14] the Lie algebra must be one of the family $SU(n)$ with $n\geq 3$ since, as we will show in Appendix A below, these are the only algebras which allow trilinear symmetric coupling between adjoints.[20] The assumption that the system possesses a symmetry therefore breaks (1.5) down to a set of type-(b) relations purely between $(GG)$'s, and a set of type-(a) relations purely between the $L_i'$. The type-(b) relations concern us here.

## II. MATHEMATICAL RESULTS

In this section we give the solution for the mathematical problem of deciding which simple Lie algebras allow symmetric trilinear couplings of their adjoint representations $\mathfrak{D}^{(\mathrm{adj})}$ (to the trivial one-dimensional representation). Our considerations are somewhat simplified by the fact that the only possibilities turn out to be the totally antisymmetric $F$-type coupling—possessed by every simple Lie algebra since the appropriate Clebsch-Gordan coefficients are just the structure constants—and the totally symmetric $D$-type couplings[21] in the case of $SU(n)$ $(n\geq 3)$. Trilinear couplings of mixed symmetry are absent. To show that this is the case, we take the straightforward route of computing the Clebsch-Gordan series of $\mathfrak{D}^{(\mathrm{adj})}\otimes\mathfrak{D}^{(\mathrm{adj})}$ for every compact, simple Lie algebra.[22] These series each contain a symmetric and an antisymmetric part, and we want to determine the number of times $\mathfrak{D}^{(\mathrm{adj})}$ appears in each part. The calculations are discussed in detail in the appendices below.

In the formulas of this section the representations are denoted by their dimensionalities; the symmetric terms are bracketed first, antisymmetric terms second; and for the higher members of the Cartan families[9] $A_l$ through $D_l$, the dimensionalities are expressed in terms of the dimension $n$ of the self-representation of the algebra,

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[17] This procedure for using finite-energy sum rules at $t\geq 0$ is by no means universally accepted. For example, in Ref. 4 algebraic relations of type (b) are obtained by using superconvergence relations—equations like (1.4) with zero on the right-hand side—near $t=0$. The appropriate way to proceed is under dispute. Near $t=0$ some of the dynamic approximations look more reasonable, but it is not clear whether one is relating coupling bilinears (partial widths) or total widths. One of us (J. Y.) would like to thank C. Schmid for a useful discussion of this point.

[18] The details are given in the Appendix of Ref. 1. Partial verifications of the statement in the text may be instantly acquired by recalling that, for elastic scattering, Breit-Wigner numerators, entering on the left-hand side of (1.4), and Regge residues evaluated on resonance, are both proportional to a Feynman diagram for single-particle $s$- or $u$-channel exchange, while each term on the right-hand side corresponds to the same in the $t$ channel, provided we sit on resonance in $t$.

[19] The form of (1.5) brings us very close to the narrow-resonance-amplitude approach. By using the type-(a) relations for the $L$'s and $L'$'s and inserting the infinite rising Regge-trajectory spectrum, it should be possible to obtain all coupling ratios and show, provided we insist there are no ghosts with negative couplings squared, that a simple narrow-resonance formula results.

[20] For remarks in a related context see R. H. Capps, Phys. Rev. **171**, 1591 (1968). For consideration of representations other than *adjoint*, especially for $G_2$, see Chan *et al.*, Ref. 11. The reader may ask why, even without adjoint contributions, singlet representations by themselves cannot saturate the symmetric parts of bootstrap equations. In adjoint-adjoint scattering this could happen in two possible ways: (1) with singlets in all three channels; (2) with singlets in two channels and an adjoint exchanged antisymmetrically in the third. Case (1) never works for any Lie algebra. This is because $SU(2)$ is a subalgebra of any $L$; any adjoint has an $SU(2)$ decomposition containing at least one isotriplet, and the $\pi$-$\pi$ crossing matrix does not allow a solution with singlets in all channels. (This is one way to see that the Pomeranchon is not trivially crossing-symmetric.) Case (2) in fact works for $SU(2)$ and for no other algebra. This is because the $SU(2)$ structure constants satisfy the particularly simple relation $f_{ijk}f_{pqk}=\epsilon_{ijk}\epsilon_{pqk}=\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp}$, which makes explicit the fact that an $SU(2)$ quartet, $\mathbf{3}\oplus\mathbf{1}$, is a self-consistent possibility. For other algebras $f_{ijk}f_{pqk}$ is more complicated and this kind of simple crossing does not hold. The only role of the singlet representation here is to provide enough freedom so that the symmetrically coupled particles satisfy the crossing constraints for $SU(n)$. Any necessity for more detailed consideration of crossing matrices for arbitrary algebras is obviated by the fantastically strong restrictions on trilinear couplings allowable for adjoints. When we pass to the full minimal bootstrap problem (cf. Ref. 7) more detail is unavoidable. For example the representation **7** of $G_2$ provides an (antisymmetric) eigenvector, but it is physically unadmissible because its eigenvalue is $-1$. [See Chan *et al.* (Ref. 11) and Y. Ne'eman, Nuovo Cimento **33**, 133 (1964)]. The reader can easily check that *adjoint*$\oplus$*singlet* is a proper eigenvector for $SU(n)$ by examining the relations for $d_{ijk}$ and $f_{ijk}$ given in Ref. 21.

[21] A convenient reference for information regarding the $f_{ijk}$ and $d_{ijk}$ of $SU(n)$ is the Appendix of the paper of L. M. Kaplan and M. Resnikoff, J. Math. Phys. **8**, 2194 (1967).

[22] The symbol $\otimes$ indicates the Kronecker or direct product and is just the generalization of addition of vector angular momenta in $SU(2)$ to arbitrary algebras. The Kronecker-product representation is the direct sum, $\oplus$, of irreducible representations. This direct sum is called the *Clebsch-Gordan series* of the product representation.

i.e., the dimension of the corresponding unitary, orthogonal, or symplectic matrices.

For the unitary algebras[23] $A_l$, with $n=l+1$,

$$n=2:\ \{3\}\otimes\{3\}=[\{1\}\oplus\{5\}\oplus\{3\}]_{\text{symm}}, \tag{2.1a}$$

$$n=3:\ \{8\}\otimes\{8\}=[\{1\}\oplus\{8\}\oplus\{27\}\oplus\{8\}\oplus\{10\}\oplus\{10\}^*]_{\text{antisym}}, \tag{2.1b}$$

$$\begin{aligned} n\geqq 4:\ &\{n^2-1\}\otimes\{n^2-1\}\\ &=[\{1\}\oplus\{n^2-1\}\oplus\{\tfrac{1}{4}n^2(n-1)(n+3)\}\\ &\quad\oplus\{\tfrac{1}{4}n^2(n+1)(n-3)\}]_{\text{symm}}\\ &\quad\oplus[\{n^2-1\}\oplus\{\tfrac{1}{4}(n^2-1)(n^2-4)\}\\ &\quad\oplus\{\tfrac{1}{4}(n^2-1)(n^2-4)\}^*]_{\text{antisym}}. \end{aligned} \tag{2.1c}$$

The asterisk after the last term in Eq. (2.1c) indicates that it is the complex conjugate representation to the term before it.

For the orthogonal algebras,[24] $B_l$ $(l\geq 2)$ with $n=2l+1$ and $D_l$ $(l\geq 4)$ with $n=2l$,

$$\begin{aligned} n=5,\ n\geq 7:\ &\{\tfrac{1}{2}n(n-1)\}\otimes\{\tfrac{1}{2}n(n-1)\}\\ &=[\{1\}\oplus\{\tfrac{1}{2}(n-1)(n+2)\}\\ &\quad\oplus\{\tfrac{1}{12}n(n+1)(n+2)(n-3)\}\\ &\quad\oplus\{(1/24)n(n-1)(n-2)(n-3)\}]_{\text{symm}}\\ &\quad\oplus[\{\tfrac{1}{2}n(n-1)\}\\ &\quad\oplus\{\tfrac{1}{8}n(n-1)(n+2)(n-3)\}]_{\text{antisym}}. \end{aligned} \tag{2.2a}$$

For the orthogonal algebra $D_3$ with $n=6$,

$$\begin{aligned} n=6:\ &\{15\}\otimes\{15\}\\ &=[\{1\}\oplus\{20\}\oplus\{84\}\oplus\{15\}]_{\text{symm}}\\ &\quad\oplus[\{15\}\oplus\{45\}\oplus\{45\}^*]_{\text{antisym}}. \end{aligned} \tag{2.2b}$$

This follows from (2.1c) since $D_3$ is isomorphic to $A_3$. $D_2$ is not listed here since it is not a simple Lie algebra; $B_1$ and $D_1$ are isomorphic to $A_1$.

For the symplectic algebras[25] $C_l$ $(l\geq 2)$ with $n=2l$,

$$\begin{aligned} n\geq 4 \text{ and even}:\ &\{\tfrac{1}{2}n(n+1)\}\otimes\{\tfrac{1}{2}n(n+1)\}\\ &=[\{1\}\oplus\{\tfrac{1}{2}(n+1)(n-2)\}\\ &\quad\oplus\{\tfrac{1}{12}n(n-1)(n-2)(n+3)\}\\ &\quad\oplus\{(1/24)n(n+1)(n+2)(n+3)\}]_{\text{symm}}\\ &\quad\oplus[\{\tfrac{1}{2}n(n+1)\}\\ &\quad\oplus\{\tfrac{1}{8}n(n+1)(n-2)(n+3)\}]_{\text{antisym}}. \end{aligned} \tag{2.3}$$

This equation is exactly like (2.2a) except that all signs in the dimensionality formulas are reversed.

[23] $A_l$ is the Lie algebra of traceless $(l+1)\times(l+1)$ Hermitian matrices.

[24] $B_l$ is the Lie algebra of real $(2l+1)\times(2l+1)$ matrices $A_{ij}$ such that $A_{ij}=-A_{ji}$. $D_l$ is the Lie algebra of real $(2l)\times(2l)$ matrices such that $A_{ij}=-A_{ji}$.

[25] $C_l$ is the Lie algebra of $(2l)\times(2l)$ Hermitian matrices

$$G=\begin{pmatrix} A & B\\ C & D\end{pmatrix}$$

such that the transpose

$$G^T=\begin{pmatrix} -D & C\\ B & -A\end{pmatrix}.$$

Finally, for the exceptional Lie algebras,

$$\begin{aligned} G_2:\ &\{14\}\otimes\{14\}=[\{1\}\oplus\{27\}\oplus\{77\}_2]_{\text{symm}}\\ &\qquad\oplus[\{14\}\oplus\{77\}_1]_{\text{antisym}},\\ F_4:\ &\{52\}\otimes\{52\}=[\{1\}\oplus\{324\}\oplus\{1053\}]_{\text{symm}}\\ &\qquad\oplus[\{52\}\oplus\{1274\}]_{\text{antisym}},\\ E_6:\ &\{78\}\otimes\{78\}=[\{1\}\oplus\{650\}\oplus\{2430\}]_{\text{symm}}\\ &\qquad\oplus[\{78\}\oplus\{2925\}]_{\text{antisym}},\\ E_7:\ &\{133\}\otimes\{133\}=[\{1\}\oplus\{1539\}\oplus\{7371\}]_{\text{symm}}\\ &\qquad\oplus[\{133\}\oplus\{8645\}]_{\text{antisym}},\\ E_8:\ &\{248\}\otimes\{248\}=[\{1\}\oplus\{3875\}\oplus\{27000\}]_{\text{symm}}\\ &\qquad\oplus[\{248\}\oplus\{30380\}]_{\text{antisym}}. \end{aligned} \tag{2.4}$$

Note that despite the very large dimensionality of the representations dealt with, there never appear more than seven inequivalent representations in any of these Clebsch-Gordan series.

It now becomes clear that adjoint representations appear only once in the antisymmetric products. These terms are directly associated with the structure constants and thus correspond to totally antisymmetric trilinear couplings of the adjoint representations to the trivial representations. It also becomes clear that $\mathfrak{D}^{(\text{adj})}$ only appears once in the symmetric product for the $SU(n)$ algebras with $n\geq 3$ and not at all in the symmetric products for the other simple, compact Lie algebras. For $SU(n)$ we know from studying the matrices $d_{ijk}$ that this corresponds to a totally symmetric trilinear coupling.[21]

*Note added in manuscript.* After the preparation of this manuscript it was independently pointed out to us by R. Roskies and M. Whippman that our mathematical result follows from a theorem of J. Ginibre, J. Math. Phys. **4**, 720 (1963), on Kronecker products of an adjoint representation with any arbitrary representation. For the case considered here, the theorem states that the number of times the adjoint representation appears in the Clebsch-Gordan series for its Kronecker square, is the number of nonzero Dynkin indices needed to specify the adjoint. This statement may be verified by inspection of Eqs. (B6)–(B10) below. We thank Dr. Roskies and Dr. Whippman for their very helpful communications.

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## APPENDIX A

In order to compute the Clebsch-Gordan series for the Kronecker squares of adjoint representations, we use a method which applies to all semisimple Lie algebras and which gives complete information as to which terms of the Clebsch-Gordan series belong to the symmetrized Kronecker square and which to the antisymmetrized square. The method is based on the Racah-Speiser lemma,[26] which leans heavily on the concept of weight diagrams.

In the present appendix we will outline the method employed and briefly summarize the necessary background information, hopefully making our remarks self-contained. In Appendix B we will implement the method and derive the results of Sec. II.

Our method of computation of the Clebsch-Gordan series is rather more general than is really required here, since it applies to the decomposition of an arbitrary Kronecker product. However, this generality will be useful in the discussion of the minimal bootstrap[7,27] and in preparation for that effort we give the full discussion.

### General Remarks

In order to explain our operational procedure, we need to use *root* and *weight* diagrams, and for completeness, we define these below.

It is well known[28] that a basis (with elements labeled as $H$'s and $E$'s) of a finite-dimensional, simple Lie algebra may be so chosen that

$$[H_i,H_j]=0\,, \qquad i,j=1,2,\ldots,l \qquad \text{(A1)}$$

$$[H_i,E_\alpha]=\alpha_i E_\alpha\,, \qquad i=1,2,\ldots,l \qquad \text{(A2)}$$

where $l$ is the rank[2] of the algebra. The subalgebra spanned by the $H$'s alone is known as the *Cartan subalgebra* and in many physical applications these mutually commuting $H$'s are associated with additively conserved quantities [e.g., in $SU(3)$, with hypercharge and the third component of isospin]. We may regard the number $\alpha_i$ as the $i$th component of the *root* $\boldsymbol{\alpha}$, a vector in an $l$-dimensional space called the *root space.* Since the set of simultaneous eigenvalues for each $E$ is distinct we may use the vector $\boldsymbol{\alpha}$ to label the $E$'s.

The set of roots form a *root diagram* which can be shown to have the followlng two properties[29]: (1) If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are roots, then $A\equiv 2(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})/(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})$ is an integer[30] and $\boldsymbol{\beta}-A\boldsymbol{\alpha}$ is a root. (2) If $\boldsymbol{\alpha}\neq 0$ is a root, then $k\boldsymbol{\alpha}\neq 0$ cannot be a root unless $k=\pm 1$.

The quantity $\boldsymbol{\beta}-A\boldsymbol{\alpha}$ is the vector obtained by reflecting $\boldsymbol{\beta}$ through a hyperplane, called a *Weyl plane,* which passes through the origin and is normal to $\boldsymbol{\alpha}$. In the particular case that $\boldsymbol{\alpha}=\boldsymbol{\beta}$, we see that if $\boldsymbol{\alpha}$ is a root, then $-\boldsymbol{\alpha}$ is also a root and the corresponding basis vector is denoted by $E_{-\alpha}$. The finite group of rotations and rotation-reflections in this $l$-dimensional space generated by reflections through Weyl planes is known as the *Weyl group*[31] $\mathcal{W}$. Property (1) states in part that the root diagram is invariant under the action of $\mathcal{W}$.

Property (2) follows from (1) and from the statement that if $E_\alpha$ and $E_\beta$ are basis vectors with roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively, and if $\boldsymbol{\alpha}+\boldsymbol{\beta}$ is also a root, then

$$[E_\alpha,E_\beta]=N_{\alpha\beta}E_{(\alpha+\beta)} \qquad \text{(A3)}$$

with $N_{\alpha\beta}\neq 0$. Similarly, $[E_\alpha,E_{-\alpha}]$ is a linear combination of the $H$'s, since the $H_i$ $(i=1,2,\ldots,l)$ may each be thought of as having its root equal to zero.

To each diagram with properties (1) and (2) there corresponds a unique, simple Lie algebra. The various root diagrams are described in detail in Appendix B.

At this point we introduce two quantities which we will need later. The vector $\boldsymbol{\delta}$ is defined by

$$\boldsymbol{\delta}=\tfrac{1}{2}\sum_{\alpha>0}\boldsymbol{\alpha}\,, \qquad \text{(A4)}$$

where the summation extends only over the positive roots of the algebra, i.e., $\boldsymbol{\alpha}$'s whose first nonzero component is positive. The quantity $\xi_S$ is defined by

$$\xi_S=\begin{cases}+1 & \text{if} \quad S\in\mathcal{W} \text{ is a rotation}\\ -1 & \text{if} \quad S\in\mathcal{W} \text{ is a rotation-reflection.}\end{cases}$$

We will also need the concept of the *weight diagram.* To each unitary irreducible representation there corresponds a unique weight diagram, i.e., a collection of points (called weights) in an $l$-dimensional space, and a positive integer called the multiplicity $\gamma_{\mathbf{M}}$ assigned to each weight $\mathbf{M}$. The $l$ components of the weights may be taken as the simultaneous eigenvalues of the $n\times n$ matrices which represent the $H$'s. Thus if $\psi$ is an $n$-component eigenvector of all the $H_i$ simultaneously, then

$$H_i\psi=m_i\psi \qquad \text{(A5)}$$

and the weight associated with $\psi$ may be taken to be $\mathbf{M}=(m_1,m_2,\ldots,m_l)$. The multiplicity $\gamma_{\mathbf{M}}$ is the number of linearly independent $\psi$'s which have weight $\mathbf{M}$.[32]

The weight diagram of an irreducible representation has the following properties: (1) It is invariant under the Weyl group $\mathcal{W}$, so that $\gamma_{\mathbf{M}}=\gamma_{S\mathbf{M}}$ for any $S\in\mathcal{W}$.

[26] D. R. Speiser, Helv. Phys. Acta 38, 73 (1965); G. Racah, in *Group Theoretical Concepts and Methods In Elementary Particle Physics*, edited by F. Gürsey (Gordon and Breach, New York, 1964). We follow the nomenclature of A. J. Macfarlane, L. O'Raifeartaigh, and P. S. Rao, J. Math. Phys. **8**, 536 (1967). The latter reference contains a rather complete summary and bibliography of mathematical investigations in this field through 1966.

[27] Compare R. C. Hwa and S. H. Patil, Phys. Rev. **138**, B933 (1965).

[28] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962).

[29] G. Racah, Ergeb. Exakt. Naturw. **37**, 28 (1965).

[30] The dot product is given as usual by $\alpha\cdot\beta=\alpha_1\beta_1+\alpha_2\beta_2+\cdots+\alpha_l\beta_l$.

[31] N. Jacobson, *Lie Algebras* (Interscience, New York, 1962), p. 119.

[32] For example, in the octet (adjoint) representation of $SU(3)$, where the eight $\psi$'s can be thought of as representing the usual set of eight baryons, $\Sigma^0$ and $\Lambda^0$ both have weight $(I_z,Y)=(0,0)$, and so the multiplicity of (0,0) is two. The general rule for adjoint representations is that all weights have multiplicity one, except for the central point $(0,0,\ldots,0)$, which has multiplicity $l$. This follows directly from (A1)–(A3).

(2) If $\mathbf{\Lambda}$ is the *highest weight*[33] of the representation, then $\gamma_\Lambda = 1$. (3) The *contracted weight diagram*[34] contains one point only, namely, the point $\mathbf{\Lambda}$ with multiplicity one.

By contraction we mean that each weight $\mathbf{M}$, with multiplicity $\gamma_\mathbf{M}$, is replaced by a vector $\mathbf{N}=S(\mathbf{M}+\boldsymbol{\delta})-\boldsymbol{\delta}$ with multiplicity $\gamma_\mathbf{N}=\xi_S\gamma_\mathbf{M}$, where $S\in\mathcal{W}$ is chosen so that $\mathbf{N}$ is dominant.[33] If $\mathbf{M}=\mathbf{N}$ for some $S\in\mathcal{W}$, then that point is simply discarded. Often several points $\mathbf{M}$ may contract to the same point $\mathbf{N}$, in which case $\gamma_\mathbf{N}$ is the sum over the various $\xi_S(\mathbf{M})\gamma_\mathbf{M}$. The contraction process will be explicitly illustrated below in Appendix B.

The three properties above serve to determine all the $\gamma_\mathbf{M}$, given $\mathbf{\Lambda}$, and to delineate the allowed values of $\mathbf{\Lambda}$. A highest-weight $\mathbf{\Lambda}$, in turn, uniquely labels the corresponding irreducible representation. It turns out that for each given algebra (of rank $l$) there exist $l$ weights $\mathbf{\Lambda}^{(i)}$ called *fundamental dominant weights* such that for any highest weight $\mathbf{\Lambda}$, we have[29]

$$\mathbf{\Lambda}=\sum_{i=1}^{l} p_i\mathbf{\Lambda}^{(i)}, \tag{A6}$$

where the *Dynkin indices* $p_i$ are non-negative integers. (In fact, any such linear combination is the highest weight of some representation.) The set $\langle p_1,p_2,\ldots,p_l\rangle$ therefore specifies, just as the components of $\mathbf{\Lambda}$ do, an irreducible representation, and we will give in Appendix B the final form of our results in terms of them. The $p_i$ are unique up to ordering and are independent of the coordinate system used for the weight space.

Reducible representations have weight diagrams which are the sums, in the sense of adding multiplicities, of weight diagrams (with centers superimposed) for irreducible representations. The contraction of the weight diagram for a reducible representation yields a set of points which are the highest weights of the representations which it is a direct sum of [cf. property (3) above]. The multiplicity of each such highest weight is the number of times the corresponding representation appears in the direct sum, and this makes the contraction process particularly convenient for actual computation.

### Kronecker Products

We are now ready to state the method for determining the Clebsch-Gordan series for an arbitrary Kronecker product representation.

Let $\mathfrak{D}^{(1)}$ and $\mathfrak{D}^{(2)}$ be finite-dimensional unitary irreducible representations of a semisimple Lie algebra. (The restriction to irreducible representations is only for simplicity of discussion.) Then the direct-product representation $\mathfrak{D}^{(1)}\otimes\mathfrak{D}^{(2)}$ has a Clebsch-Gordan series which is, in effect, its contracted weight diagram. We can construct this diagram by the following technique.[35] Shift all weights $\mathbf{M}$ in the weight diagram for $\mathfrak{D}^{(1)}$ by the highest weight $\mathbf{\Lambda}$ for $\mathfrak{D}^{(2)}$ and thus obtain a new set of multiplicities given by $\gamma_\mathbf{M}=\gamma_{\mathbf{M}-\Lambda}{}^{(1)}$, where the $\gamma_\mathbf{M}{}^{(1)}$'s are multiplicities for $\mathfrak{D}^{(1)}$. The resulting diagram is not the weight diagram for the product representation since, for example, it is not invariant under $\mathcal{W}$. However, it is much simpler than the full weight diagram, and its contraction can be shown to be the contracted weight diagram for the product representation.

The dimensionality $N(\mathbf{\Lambda})$ of irreducible representations which appear in this diagram may be evaluated by the use of Weyl's formula[36]

$$N(\mathbf{\Lambda})=\prod_{\alpha>0}\left(1+\frac{\mathbf{\Lambda}\cdot\boldsymbol{\alpha}}{\boldsymbol{\delta}\cdot\boldsymbol{\alpha}}\right), \tag{A7}$$

where the product extends only over positive roots[33] and $\mathbf{\Lambda}$ is the highest weight of the irreducible representation; or indirectly by Freudenthal's[37] or Kostant's formula[38] for the multiplicities[39] of the weights. The dimensionality of any representation is simply the sum over the multiplicities of all its weights.

The results for the case $\mathfrak{D}^{(1)}=\mathfrak{D}^{(2)}=\mathfrak{D}^{(\mathrm{adj})}$ are given in Sec. II. They are presented in terms of the dimensionalities of the relevant irreducible representations.

### Symmetrization

By adopting specific coordinate systems and by applying the procedure just outlined to the adjoint representations, one can arrive at all the results presented in Sec. II except for the assignments of the various representations to the symmetric or the antisymmetric part of the Kronecker square $\mathfrak{D}\otimes\mathfrak{D}$. In many cases this partitioning of the Clebsch-Gordan

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[33] A weight $\mathbf{M}$ is said to be *higher* than a weight $\mathbf{M}'$ if the first nonvanishing component of $\mathbf{M}-\mathbf{M}'$ is a positive number. $\mathbf{M}$ is said to be *positive* if $\mathbf{M}$ is higher than zero. A weight $\mathbf{M}$ is said to be *equivalent* to a weight $\mathbf{M}'$ if there exists an $S\in\mathcal{W}$ such that $\mathbf{M}'=S\mathbf{M}$. A *dominant weight* is the highest weight of a set of equivalent weights. The *highest weight* of a representation is the dominant weight which is higher than any other dominant weight.

[34] Our contracted diagrams correspond directly to the girdle diagrams of Ref. 28. However, each one of our points corresponds to many points there.

[35] This procedure is the analog of multiplying the character $\chi^{(1)}$ of $\mathfrak{D}^{(1)}$ by the girdle $\xi^{(2)}$ of $\mathfrak{D}^{(2)}$ to obtain the girdle of the product representation as discussed in Ref. 28. An alternative method of construction is given by D. Radhakrishnan, J. Math. Phys. **9**, 2061 (1968). See also B. Gruber, *ibid.* **11**, 1783 (1970). Remarks regarding the general mathematical situation will be found in the latter work and in Macfarlane *et al.*, Ref. 26. In the important special case of the unitary algebras $A_l$, *Young shapes* can be utilized and this results in considerably less work than with the method discussed in the present paper. Furthermore there exist diagrammatic rules (cf. Ref. 42) which help in the partitioning of Kronecker squares into symmetric and antisymmetric parts [see R. C. King, *ibid.* **11**, 280 (1970)].

[36] H. Weyl, Math. Z. **24**, 377 (1926), in particular, p. 389.

[37] N. Jacobson, Ref. 31, p. 247; H. Freudenthal, Ned. Akad. Wetensch. Indag. Math. **57**, 369 (1954).

[38] N. Jacobson, Ref. 31, p. 261: B. Kostant, Transac. Am. Math. Soc. **93**, 53 (1959).

[39] See J. G. Belinfante and B. Kolman, SIAM Rev. **11**, 510 (1969), for a discussion of all three formulas. Do not confuse the "internal" multiplicity of the weights with the multiplicity of a particular irreducible representation in a Kronecker product, the so-called "external" multiplicity. The two are related but not identical. See Macfarlane *et al.*, Ref. 26.

series can be simply accomplished by examining the dimensionalities of the various representations. However, occasionally two inequivalent representations with the same dimensionality will appear in the series, or there will otherwise be more than one way of satisfying the dimensionality requirements. We therefore need to invoke a more general method for the partitioning. [Examples of two inequivalent representations of the same dimensionality appearing in the same Clebsch-Gordan series occur for $\mathfrak{D}^{(\mathrm{adj})}\otimes\mathfrak{D}^{(\mathrm{adj})}$ in $B_2$ ($\sim C_2$) and $G_2$.] In any case, an explicit calculation of this partitioning is not at all difficult and provides a useful check on the work. We now discuss the method for accomplishing this.

In group theory a representation is uniquely labeled by the functional form of its character function[40] $\chi$ or by its weight diagram. Operations on characters thus should have corresponding operations on weight diagrams. Now we know that the characters for the symmetric and antisymmetric Kronecker squares of a representation $\mathfrak{D}$ are given in terms of the character $\chi(r)$, where $r$ is a group element, by[41]

$$\begin{aligned}\chi_S(r)&=\tfrac{1}{2}[\chi^2(r)+\chi(r^2)],\\ \chi_A(r)&=\tfrac{1}{2}[\chi^2(r)-\chi(r^2)].\end{aligned}\tag{A8}$$

These formulas need to be translated into the corresponding operations on weight diagrams, which operations will then be applicable to Lie algebras. We have already discussed how to find the contracted weight diagram for the Kronecker-square representation, that is for the analog of $\chi^2$; now we discuss the same procedure for $\chi_2$ where $\chi_2(r)\equiv\chi(r^2)$. $\chi_2$ has a diagram which is simply the weight diagram for $\chi$ except that the coordinates for each weight are doubled.[42] The multiplicities are left unchanged. To properly include symmetrization, we simply contract the diagram for $\chi_2$, add it to (or subtract it from) the contracted diagram for $\chi^2$, and then divide all multiplicities by 2; this yields the contracted weight diagram corresponding to $\chi_S$ (or $\chi_A$). (Addition and subtraction here refer to the addition or subtraction of the multiplicities of points with the same coordinates.) The contracted diagram for $\chi_2$, unlike those for $\chi$ and $\chi^2$, will in general have negative multiplicities as well as positive ones. The Clebsch-Gordan series for the symmetrized and antisymmetrized Kronecker squares are given by the contracted weight diagrams for $\chi_S$ and $\chi_A$. Explicit examples are given in a previous paper[42] by one of us and below in Appendix B.

[40] The character $\chi(r)$ of the group element $r$ is given by $\chi(r)=\mathrm{tr}D(r)$, where $D(r)$ is the matrix representing $r$. In terms of the algebraic language employed here, the character function can be computed by use of Weyl's character formula:

$$\chi(\mathbf{\Lambda})=\sum_{S\in\mathcal{W}}[\xi_S]e^{S(\delta+\Lambda)}\Big/\sum_{S\in\mathcal{W}}[\xi_S]e^{S\delta},$$

where the sum runs over the Weyl group, $\xi_S$ is as defined above, and $\mathbf{\Lambda}$ is as above the highest weight of the representation of interest. For explicit examples and discussion see Secs. III and IV of Ref. 28.

[41] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley, Reading, Mass., 1962), p. 134.

[42] C. M. Andersen, J. Math. Phys. **8**, 988 (1967).

## APPENDIX B

Using the rather abstract rules given in Appendix A, we now want to derive the results of Sec. II. Because we are dealing with adjoint representations, we have the immense simplification that their weight diagrams are identical to the root diagrams for the algebras. Furthermore, from (A1) and (A2) we see that the multiplicity of the point at the origin in the root diagram is $l$, and all other points have multiplicity one.

Our calculations will proceed in two steps. First we pick for each algebra a convenient coordinate system in which to describe the root diagram, the vector $\boldsymbol{\delta}$, and the operations of the Weyl group. Any results we now obtain will be dependent on our choice of coordinate system. Later we will proceed to transform to the unique (up to ordering) specification $\langle p_1,p_2,\ldots,p_l\rangle$ using the Dynkin indices of (A5).

We let the basis in each case be a system of orthogonal unit vectors $\hat{e}_i$. In Table I we list for each algebra of the Cartan classification the nonzero roots as given by Racah[29] and the vector $\boldsymbol{\delta}$ in terms of the $\hat{e}_i$. For the algebras $A_l$, $G_2$, $E_6$, and $E_7$, the $l$-dimensional root and weight diagrams are imbedded in an $(l+1)$-dimensional space[42] and a constraint is introduced among the $l+1$ components of any root or weight. For $A_l$, $G_2$, and $E_7$, the constraint is that the sum of all the components of any root or weight is zero; for $E_6$ the sum over the last six components is zero. This imbedding is purely for convenience in specifying the components of the weights and in performing the operations of the Weyl group. In Table I the $\pm$ signs are to be taken independent of one another unless otherwise specified. Thus for $B_l$ we have roots $\hat{e}_1+\hat{e}_2$, $\hat{e}_1-\hat{e}_2$, $-\hat{e}_1+\hat{e}_2$, $-\hat{e}_1-\hat{e}_2$, $\hat{e}_1+\hat{e}_3$, etc. Since the origin has weight $l$, the number $l$ plus the number of roots listed in column 3 equals in each case the dimensionality of the adjoint representation as listed in column 2.

For each algebra, the Weyl group contains the permutations of the root or weight components labeled 1, 2, 3, .... To see this we note that any such permutation is the result of successive transpositions of two components at a time, and it is easily seen that transposition of the $i$th and $j$th components corresponds to reflection in the Weyl plane normal to $\hat{e}_i-\hat{e}_j$. Since, as seen in Table I, all the algebras contain roots of the form $\hat{e}_i-\hat{e}_j$, $i$, $j=1, 2, \ldots$, the above statement follows. Note, however, that the algebra $E_6$ does not contain roots of the form $\pm(\hat{e}_i-\hat{e}_0)$. Thus permutations involving the zeroth component of a weight are not elements of the Weyl group for $E_6$. For $A_l$ permutations of the weight components generate the entire Weyl group.

For the algebras $B_l$ and $C_l$ the Weyl group contains, in addition to the permutations discussed above, the

TABLE I. List of the algebras of the Cartan classification, their dimensionalities, and their nonzero roots. The vector $\boldsymbol{\delta}$ is defined as one-half the sum over the positive roots (Ref. 32). If an expression contains more than one set of $\pm$ signs, they are independent of one another unless a restriction is specified.

| Algebra | Dimensionality | Nonzero roots | $\boldsymbol{\delta}=\frac{1}{2}\sum_{\alpha>0}\boldsymbol{\alpha}$ |
|---|---|---|---|
| $A_l$ | $l(l+2)$ | $\hat{e}_i-\hat{e}_j$ $(i, j=1, 2, \ldots, l+1$ but $i\neq j)$ | $\frac{1}{2}l\hat{e}_1+\frac{1}{2}(l-2)\hat{e}_2+\cdots-\frac{1}{2}(l-2)\hat{e}_l-\frac{1}{2}l\hat{e}_{l+1}$ |
| $B_l$ | $l(2l+1)$ | $\pm\hat{e}_i\pm\hat{e}_j$, $\pm\hat{e}_i$ $(i, j=1, 2, \ldots, l$ but $i\neq j)$ | $(l-\frac{1}{2})\hat{e}_1+(l-\frac{3}{2})\hat{e}_2+\cdots+\frac{3}{2}\hat{e}_{l-1}+\hat{e}_l$ |
| $C_l$ | $l(2l+1)$ | $\pm\hat{e}_i\pm\hat{e}_j$, $\pm2\hat{e}_i$ $(i, j=1, 2, \ldots, l$ but $i\neq j)$ | $l\hat{e}_1+(l-1)\hat{e}_2+\cdots+2\hat{e}_{l-1}+\hat{e}_l$ |
| $D_l$ | $l(2l-1)$ | $\pm\hat{e}_i\pm\hat{e}_j$ $(i, j=1, 2, \ldots, l$ but $i\neq j)$ | $(l-1)\hat{e}_1+(l-2)\hat{e}_2+\cdots+\hat{e}_{l-1}$ |
| $G_2$ | 14 | $\hat{e}_i-\hat{e}_j$; $+(2\hat{e}_i-\hat{e}_j-\hat{e}_k)$ $(i, j, k=1, 2, 3$ but $i\neq j\neq k\neq i)$ | $3\hat{e}_1-\hat{e}_2-2\hat{e}_3$ |
| $F_4$ | 52 | $\pm\hat{e}_i\pm\hat{e}_j$ $(i, j=1, 2, 3, 4$ but $i\neq j)$; $\frac{1}{2}(\pm\hat{e}_1\pm\hat{e}_2\pm\hat{e}_3\pm\hat{e}_4)$ | $\frac{11}{2}\hat{e}_1+\frac{5}{2}\hat{e}_2+\frac{3}{2}\hat{e}_3+\frac{1}{2}\hat{e}_4$ |
| $E_6$ | 78 | $\hat{e}_i-\hat{e}_j$ $(i, j=1, 2, \ldots, 6$ but $i\neq j)$; $\pm\sqrt{2}\hat{e}_0$; $\frac{1}{2}(\pm\sqrt{2}\hat{e}_0\pm\hat{e}_1\pm\hat{e}_2\pm\hat{e}_3\pm\hat{e}_4\pm\hat{e}_5\pm\hat{e}_6)$ with 3 plus signs and 3 minus signs on last six terms | $(11/\sqrt{2})\hat{e}_0+\frac{5}{2}\hat{e}_1+\frac{3}{2}\hat{e}_2+\frac{1}{2}\hat{e}_3+\frac{1}{2}\hat{e}_4+\frac{3}{2}\hat{e}_5+\frac{5}{2}\hat{e}_6$ |
| $E_7$ | 133 | $\hat{e}_i-\hat{e}_j$ $(i, j=1, 2, \ldots, 8$ but $i\neq j)$; $\frac{1}{2}(\pm\hat{e}_1\pm\hat{e}_2\pm\hat{e}_3\pm\hat{e}_4\pm\hat{e}_5\pm\hat{e}_6\pm\hat{e}_7\pm\hat{e}_8)$ with 4 plus signs and 4 minus signs | $(49/4)\hat{e}_1+(5/4)\hat{e}_2+\frac{1}{4}\hat{e}_3-\frac{3}{4}\hat{e}_4-(7/4)\hat{e}_5-(11/4)\hat{e}_6-(15/4)\hat{e}_7-(17/4)\hat{e}_8$ |
| $E_8$ | 248 | $\pm\hat{e}_i\pm\hat{e}_j$ $(i, j=1, 2, \ldots, 8$ but $i\neq j)$; $\frac{1}{2}(\pm\hat{e}_1\pm\hat{e}_2\pm\hat{e}_3\pm\hat{e}_4\pm\hat{e}_5\pm\hat{e}_6\pm\hat{e}_7\pm\hat{e}_8)$ with even number of plus signs and even number of minus signs | $23\hat{e}_1+6\hat{e}_2+5\hat{e}_3+4\hat{e}_4+3\hat{e}_5+2\hat{e}_6+\hat{e}_7$ |

TABLE II. Steps used to calculate the Clebsch-Gordan series for the Kronecker square of the octet representation of $SU(3)$. See explanation in text.

| $\mathbf{M}$ | $\mathbf{M}+\boldsymbol{\Lambda}+\boldsymbol{\delta}$ | $S(\mathbf{M}+\boldsymbol{\Lambda}+\boldsymbol{\delta})$ | $S(\mathbf{M}+\boldsymbol{\Lambda}+\boldsymbol{\delta})-\boldsymbol{\delta}$ |
|---|---|---|---|
| [2](0,0,0) | [2](2, 0, −2) | [2](2, 0, −2) | [2](1, 0, −1) |
| (1, −1, 0) | (3, −1, −2) | (3, −1, −2) | (2, −1, −1) |
| (1, 0, −1) | (3, 0, −3) | (3, 0, −3) | (2, 0, −2) |
| (0, 1, −1) | (2, 1, −3) | (2, 1, −3) | (1, 1, −2) |
| (−1, 1, 0) | (1, 1, −2) | ··· | ··· |
| (−1, 0, 1) | (1, 0, −1) | (1, 0, −1) | (0,0,0) |
| (0, −1, 1) | (2, −1, −1) | ··· | ··· |

TABLE III. Steps used to obtain the contracted weight diagram for $\chi_2$ for the octet representation of $SU(3)$. The symmetrized and antisymmetrized Kronecker squares are found by combining the terms in the fourth column here with the terms in the fourth column of Table II.

| $\mathbf{M}'$ | $\mathbf{M}'+\boldsymbol{\delta}$ | $S(\mathbf{M}'+\boldsymbol{\delta})$ | $S(\mathbf{M}'+\boldsymbol{\delta})-\boldsymbol{\delta}$ |
|---|---|---|---|
| [2](0,0,0) | [2](1, 0, −1) | [2](1, 0, −1) | [2](0,0,0) |
| (2, −2, 0) | (3, −2, −1) | [−1](3, −1, −2) | [−1](2, −1, −1) |
| (2, 0, −2) | (3, 0, −3) | (3, 0, −3) | (2, 0, −2) |
| (0, 2, −2) | (1, 2, −3) | [−1](2, 1, −3) | [−1](1, 1, −2) |
| (−2, 2, 0) | (−1, 2, −1) | ··· | ··· |
| (−2, 0, 2) | (−1, 0, 1) | [−1](1, 0, −1) | [−1](0,0,0) |
| (0, −2, 2) | (1, −2, 1) | ··· | ··· |

operation of replacing any number of components by their negatives. Replacing the $i$th component by its negative is the result of reflection in the Weyl plane normal to $\hat{e}_i$. The Weyl group for $E_6$ includes replacing the zeroth component by its negative.

For the algebras $D_l$, $F_4$, and $E_8$, the generators of the Weyl group include replacing pairs of coordinates by their negatives. Reflection in the Weyl plane normal to $\hat{e}_i+\hat{e}_j$ followed by reflection in the Weyl plane normal to $\hat{e}_i-\hat{e}_j$ results in changing the sign of the $i$th and $j$th components.

For $G_2$ the Weyl group has 12 elements, the six permutations of the three components and these six permutations combined with inversion, whereby the sign of all three components is changed. For the other exceptional algebras $F_4$, $E_6$, $E_7$, and $E_8$, additional generators are required as can be seen from Table I.

Let us illustrate the procedures of Appendix A by calculating the Clebsch-Gordan series for $\mathfrak{D}^{(\mathrm{adj})}\otimes\mathfrak{D}^{(\mathrm{adj})}$ for the algebra $A_2$. Here the highest weight of the adjoint (octet) representation is given by $\boldsymbol{\Lambda}=\boldsymbol{\delta}=(1, 0, -1)$. In the first column of Table II we list the

TABLE IV. Steps used to calculate the Clebsch-Gordan series for the Kronecker square of the adjoint representation of $E_6$. The symbol $(0)=(0;\,0,0,0,0,0,0)$.

| $\mathbf{M}$ | $\mathbf{M}+\mathbf{\Lambda}+\boldsymbol{\delta}$ | $S(\mathbf{M}+\mathbf{\Lambda}+\boldsymbol{\delta})$ | $S(\mathbf{M}+\mathbf{\Lambda}+\boldsymbol{\delta})-\boldsymbol{\delta}$ |
|---|---|---|---|
| $[6](0;\,0,0,0,0,0,0)$ | $[6](\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $[6](\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(\sqrt{2};\,0,0,0,0,0,0)$ |
| $(0;\,1,0,0,0,0,-1)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{7}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{7}{2})$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{7}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{7}{2})$ | $(\sqrt{2};\,1,0,0,0,0,-1)$ |
| $(0;\,-1,1,0,0,0,0)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{3}{2},\tfrac{5}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $[-1](\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | ... |
| $(0;\,0,-1,1,0,0,0)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{1}{2},\tfrac{3}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | same as column to left | ... |
| $(0;\,0,0,-1,1,0,0)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | same as column to left | ... |
| $(0;\,0,0,0,-1,1,0)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{1}{2},-\tfrac{5}{2})$ | same as column to left | ... |
| $(0;\,0,0,0,0,-1,1)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{5}{2},-\tfrac{3}{2})$ | same as column to left | ... |
| $(\sqrt{2};\,0,0,0,0,0,0)$ | $(\tfrac{15}{2}\sqrt{2};\,-\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(\tfrac{15}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(2\sqrt{2};\,0,0,0,0,0,0)$ |
| $(-\sqrt{2};\,0,0,0,0,0,0)$ | $(\tfrac{11}{2}\sqrt{2};\,-\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(\tfrac{11}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(0)$ |
| $(2^{-1/2};\,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$ | $(14/\sqrt{2};\,-3,2,1,-1,-2,-3)$ | $(14/\sqrt{2};\,-3,2,1,-1,-2,-3)$ | $(\tfrac{3}{2}\sqrt{2};\,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$ |
| $(-2^{-1/2};\,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$ | $(12/\sqrt{2};\,3,2,1,-1,-2,-3)$ | ... | ... |
| $(+2^{-1/2};\,\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2})$ | $(14/\sqrt{2};\,3,2,0,-1,-2,-2)$ | ... | ... |

TABLE V. Steps used to obtain the contracted weight diagram for $\chi_2$ for the adjoint representation of $E_6$, just as for **8** of $SU(3)$ in Table III. The symbol $(0)=(0;\,0,0,0,0,0,0)$.

| $\mathbf{M}'$ | $\mathbf{M}'+\boldsymbol{\delta}$ | $S(\mathbf{M}'+\boldsymbol{\delta})$ | $S(\mathbf{M}'+\boldsymbol{\delta})-\boldsymbol{\delta}$ |
|---|---|---|---|
| $[6](0)$ | $[6](\tfrac{11}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $[6](\tfrac{11}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(0)$ |
| $(2\sqrt{2};\,0,0,0,0,0,0)$ | $(\tfrac{15}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(\tfrac{15}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $(2\sqrt{2};\,0,0,0,0,0,0)$ |
| $(0;\,-2,0,2,0,0,0)$ | $(\tfrac{11}{2}\sqrt{2};\,\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $[-1](\tfrac{11}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | ... |
| $(0;\,0,-2,0,2,0,0)$ | $(\tfrac{11}{2}\sqrt{2};\,\tfrac{5}{2},-\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | same as above | ... |
| $(0;\,0,0,-2,0,2,0)$ | $(\tfrac{11}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{5}{2})$ | same as above | ... |
| $(0;\,0,0,0,-2,0,2)$ | $(\tfrac{11}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{5}{2},-\tfrac{3}{2},-\tfrac{1}{2})$ | same as above | ... |
| $(-\sqrt{2};\,1,1,-1,1,-1,-1)$ | $(\tfrac{9}{2}\sqrt{2};\,\tfrac{7}{2},\tfrac{5}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{5}{2},-\tfrac{7}{2})$ | same as above | ... |
| $(\sqrt{2};\,1,1,1,-1,-1,-1)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{7}{2},\tfrac{5}{2},\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{5}{2},-\tfrac{7}{2})$ | $[-1](7\sqrt{2};\,3,2,1,-1,-2,-3)$ | $[-1](\tfrac{3}{2}\sqrt{2};\,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})$ |
| $(\sqrt{2};\,1,-1,1,-1,1,-1)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{7}{2},\tfrac{1}{2},\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{1}{2},-\tfrac{7}{2})$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{7}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{7}{2})$ | $(\sqrt{2};\,1,0,0,0,0,-1)$ |
| $(\sqrt{2};\,-1,1,-1,1,-1,1)$ | $(\tfrac{13}{2}\sqrt{2};\,\tfrac{3}{2},\tfrac{5}{2},-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{5}{2},-\tfrac{3}{2})$ | $[-1](\tfrac{13}{2}\sqrt{2};\,\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2})$ | $[-1](\sqrt{2};\,0,0,0,0,0,0)$ |
| $(-\sqrt{2};\,1,-1,1,-1,1,-1)$ | $(\tfrac{9}{2}\sqrt{2};\,\tfrac{7}{2},\tfrac{1}{2},\tfrac{3}{2},-\tfrac{3}{2},-\tfrac{1}{2},-\tfrac{7}{2})$ | ... | ... |

weights $\mathbf{M}$ of the adjoint representation. The entries in square brackets indicate multiplicities; where there is no such entry unit multiplicity is implied. We need to shift the entries in column 1 by $\mathbf{\Lambda}^{(\mathrm{adj})}$ and then contract. However, since the first step of the contraction process is to shift by $\boldsymbol{\delta}$, we show in column 2 the weights of column 1 shifted by $\mathbf{\Lambda}+\boldsymbol{\delta}$. In the next step we discard those weights which now lie on the Weyl planes. We are also instructed to perform Weyl reflections (in this case permutations of the weight components) until all weights are dominant, which in the case of the unitary algebras means that each component is smaller than the preceding one. However, for this particular calculation no permutations are needed. The final step of the contraction process is to shift back by $-\boldsymbol{\delta}$. The terms of the Clebsch-Gordan series appear in column 4 of Table II. Interpreted in terms of dimensionalities [cf. Eq. (A7)], they yield the familiar result

$$\{8\}\otimes\{8\}=2\{8\}\oplus\{10\}\oplus\{27\}\oplus\{\overline{10}\}\oplus\{1\}. \quad \text{(B1)}$$

The second part of the procedure given in Appendix A will enable us to partition the Clebsch-Gordan series just obtained into its symmetric and antisymmetric parts. In column 1 of Table III we list weights of the diagram for $\chi_2$. Their components are double those of the corresponding weights of the adjoint representation as given in column 1 of Table II. We perform the contraction procedure in three steps. We shift the weights of $\chi_2$ by $\boldsymbol{\delta}$ to get the terms of column 2 of Table III. We obtain column 3 by performing Weyl reflections and changing the signs of the multiplicity whenever the permutation of the components is odd. Again we drop those weights which lie on Weyl planes. Finally we shift back by $-\boldsymbol{\delta}$ to obtain column 4. Referring to Eq. (A8) we find that the contracted weight diagram for $\chi_S$ contains $(1,0,-1)$, $(0,0,0)$, and $(2,0,-2)$, while the corresponding diagram for $\chi_A$ contains $(1,0,-1)$, $(2,-1,-1)$, and $(1,1,-2)$. By referring to Eq. (A7) we find

$$\{8\}\otimes\{8\}=[\{1\}\oplus\{8\}\oplus\{27\}]_{\mathrm{symm}} \oplus[\{8\}\oplus\{10\}\oplus\{10\}^*]_{\mathrm{antisym}}. \quad \text{(B2)}$$

Now let us illustrate the procedure of Appendix A on a somewhat more complicated example. We calculate the Clebsch-Gordan series for the Kronecker square of the adjoint representation of $E_6$. The coordinate systems of Table I can always be trivially modified so that the entire calculation for any given representation may be performed with integers only. This could be accomplished in the case of $E_6$ by using different changes of scale for the zeroth component and for the remaining six components. However, for simplicity of discussion we will not do that here. Table IV is the counterpart of Table II in the discussion above. However in column 1 we have listed only some of the weights of the adjoint representation. The 61 omitted weights when shifted by $\mathbf{\Lambda}+\boldsymbol{\delta}$ lie on Weyl planes and thus do not contribute. The same is true for the last two entries of column 2. The first of these lies on a Weyl plane normal to $\hat{e}_5-\hat{e}_6$, and the last entry lies on a Weyl plane normal to the root $(-2^{-1/2};\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$.

In Table V we show the calculations for the second part of the problem, finding the symmetric and antisymmetric terms. This table is constructed in the same manner as Table III. Just as in Table IV we have left out many weights which do not contribute to column 3. Those terms which appear with positive (negative) multiplicity in column 4 of Table V belong to the symmetrized (antisymmetrized) Kronecker square. In column 4 of Tables IV and V we have collected like terms, and combined with the top entry.

In this manner we obtain the Clebsch-Gordan series for the Kronecker squares of the adjoint representations in terms of the coordinates of the highest weights of the irreducible representations (we list the terms here in the same order as they are given in Sec. II; however, for brevity we have left out the special low-rank cases):

$$\begin{aligned}
&A_l,\ l\geq4:\ (1,0,\ldots,0,-1)\otimes(1,0,\ldots,0,-1)\\
&\qquad=[(0,\ldots,0)\oplus(1,0,\ldots,0,-1)\oplus(2,0,\ldots,0,-2)\oplus(1,1,0,\ldots,0,-1,-1)]_{\mathrm{symm}}\\
&\qquad\quad\oplus[(1,0,\ldots,0,-1)\oplus(2,0,\ldots,0,-1,-1)\oplus(1,1,0,\ldots,0,-2)]_{\mathrm{antisym}},\\
&B_l \text{ and } D_l,\ l\geq4:\ (1,1,0,\ldots,0)\otimes(1,1,0,\ldots,0)\\
&\qquad=[(0,\ldots,0)\oplus(2,0,\ldots,0)\oplus(2,2,0,\ldots,0)\oplus(1,1,1,1,0,\ldots,0)]_{\mathrm{symm}}\\
&\qquad\quad\oplus[(1,1,0,\ldots,0)\oplus(2,1,1,0,\ldots,0)]_{\mathrm{antisym}},\\
&C_l,\ l\geq3:\ (2,0,\ldots,0)\otimes(2,0,\ldots,0)\\
&\qquad=[(0,0,0,\ldots,0)\oplus(1,1,0,\ldots,0)\oplus(2,2,0,\ldots,0)\oplus(4,0,0,\ldots,0)]_{\mathrm{symm}}\\
&\qquad\quad\oplus[(2,0,0,\ldots,0)\oplus(3,1,0,\ldots,0)]_{\mathrm{antisym}},\\
&G_2:\ (2,-1,-1)\otimes(2,-1,-1)=[(0,0,0)\oplus(2,0,-2)\oplus(4,-2,-2)]_{\mathrm{symm}}\\
&\qquad\quad\oplus[(2,-1,-1)\oplus(3,0,-3)]_{\mathrm{antisym}},\\
&F_4:\ (1,1,0,0)\otimes(1,1,0,0)=[(0,0,0,0)\oplus(2,0,0,0)\oplus(2,2,0,0)]_{\mathrm{symm}}\oplus[(1,1,0,0)\oplus(2,1,1,0)]_{\mathrm{antisym}},\\
&E_6:\ (\sqrt{2};0,0,0,0,0,0)\otimes(\sqrt{2};0,0,0,0,0,0)\\
&\qquad=[(0;0,0,0,0,0,0)\oplus(\sqrt{2};1,0,0,0,0,-1)\oplus(2\sqrt{2};0,0,0,0,0,0)]_{\mathrm{symm}}\\
&\qquad\quad\oplus[(\sqrt{2};0,0,0,0,0,0)\oplus(3/\sqrt{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})]_{\mathrm{antisym}},
\end{aligned}$$

TABLE VI. List of highest weights and dimensionalities for the fundamental irreducible representations $\mathbf{\Lambda}^{(k)}$, $k=1,\ldots,l$ for the simple Lie algebras. For $A_l$, $B_l$, $C_l$, and $D_l$, the entry in the square bracket on the right gives the general formula for the dimensionality of $\mathbf{\Lambda}^{(k)}$, with the dimensionality of spinor representations of $B_l$ and $D_l$ listed separately. A bar over the dimensionality of a representation indicates that that representation is the complex conjugate of the representation listed above it.

| Algebra | Fundamental representation | Dimensionality | Formula for dimensionality |
|---|---|---|---|
| $A_l$ | $\mathbf{\Lambda}^{(1)}=\left(\frac{l}{l+1},\frac{-1}{l+1},\frac{-1}{l+1},\ldots,\frac{-1}{l+1}\right)$ | $l+1$ | $\left[\frac{(l+1)!}{k!(l+1-k)!}\right]$; $(k=1,2,\ldots,l)$ |
| | $\mathbf{\Lambda}^{(2)}=\left(\frac{l-1}{l+1},\frac{l-1}{l+1},\frac{-2}{l+1},\ldots,\frac{-2}{l+1}\right)$ | $\frac{l(l+1)}{2}$ | |
| | $\vdots$ | $\vdots$ | |
| | $\mathbf{\Lambda}^{(l-1)}=\left(\frac{2}{l+1},\frac{2}{l+1},\ldots,\frac{2}{l+1},\frac{-l+1}{l+1},\frac{-l-1}{l+1}\right)$ | $\frac{l(l+1)}{2}$ | |
| | $\mathbf{\Lambda}^{(l)}=\left(\frac{1}{l+1},\frac{1}{l+1},\ldots,\frac{1}{l+1},\frac{1}{l+1},\frac{-l}{l+1}\right)$ | $l+1$ | |
| $B_l$ | $\mathbf{\Lambda}^{(1)}=(1,0,0,\ldots,0,0)$ | $2l+1$ | $\left[\frac{(2l+1)!}{k!(2l+1-k)!}\right]$; $(k=1,2,\ldots,l-1)$ |
| | $\mathbf{\Lambda}^{(2)}=(1,1,0,\ldots,0,0)$ | $l(2l+1)$ | |
| | $\vdots$ | $\vdots$ | |
| | $\mathbf{\Lambda}^{(l-1)}=(1,1,1,\ldots,1,0)$ | $\frac{(2l+1)!}{(l-1)!(l+2)!}$ | |
| | $\mathbf{\Lambda}^{(l)}=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2})$ | $2^l$ | $2^l$ |
| $C_l$ | $\mathbf{\Lambda}^{(1)}=(1,0,0,\ldots,0,0)$ | $2^l$ | $2^l$ |
| | $\mathbf{\Lambda}^{(2)}=(1,1,0,\ldots,0,0)$ | $(2l+1)\ (l-1)$ | $\left[\frac{(2l+1)!(2l+2-2k)}{k!(2l+2-k)!}\right]$; $(k=1,2,\ldots,l)$ |
| | $\vdots$ | $\vdots$ | |
| | $\mathbf{\Lambda}^{(l)}=(1,1,1,\ldots,1,1)$ | $\frac{2(2l+1)!}{l!(l+2)!}$ | |
| $D_l$ | $\mathbf{\Lambda}^{(1)}=(1,0,0,\ldots,0,0,0)$ | $2^l$ | $2^l$ |
| | $\mathbf{\Lambda}^{(2)}=(1,1,0,\ldots,0,0,0)$ | $l(2l-1)$ | $\left[\frac{(2l)!}{k!(2l-k)!}\right]$; $(k=1,2,\ldots,l-2)$ |
| | $\vdots$ | $\vdots$ | |
| | $\mathbf{\Lambda}^{(l-2)}=(1,1,1,\ldots,1,0,0)$ | $\frac{(2l)!}{(l-2)!(l+2)!}$ | |
| | $\mathbf{\Lambda}^{(l-1)}=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $2^{l-1}$ | $2^{l-1}$ |
| | $\mathbf{\Lambda}^{(l)}=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $2^{l-1}$ | $2^{l-1}$ |
| $G_2$ | $\mathbf{\Lambda}^{(1)}=(1,0,-1)$ | 7 | |
| | $\mathbf{\Lambda}^{(2)}=(2,-1,-1)$ | 14 | |
| $F_4$ | $\mathbf{\Lambda}^{(1)}=(1,0,0,0)$ | 26 | |
| | $\mathbf{\Lambda}^{(2)}=(1,1,0,0)$ | 52 | |
| | $\mathbf{\Lambda}^{(3)}=(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | 273 | |
| | $\mathbf{\Lambda}^{(4)}=(2,1,1,0)$ | 1274 | |
| $E_6$ | $\mathbf{\Lambda}^{(1)}=(1/\sqrt{2};\ \frac{5}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6})$ | 27 | |
| | $\mathbf{\Lambda}^{(2)}=(1/\sqrt{2};\ \frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6})$ | $\overline{27}$ | |
| | $\mathbf{\Lambda}^{(3)}=(\sqrt{2};\ 0,0,0,0,0,0)$ | 78 | |
| | $\mathbf{\Lambda}^{(4)}=(\sqrt{2};\ \frac{2}{3},\frac{2}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$ | 351 | |
| | $\mathbf{\Lambda}^{(5)}=(\sqrt{2};\ \frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{2}{3},-\frac{2}{3})$ | $\overline{351}$ | |
| | $\mathbf{\Lambda}^{(6)}=(3/\sqrt{2};\ \frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ | 2925 | |
| $E_7$ | $\mathbf{\Lambda}^{(1)}=(\frac{3}{4},\frac{3}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},\frac{1}{4})$ | $7\times 8$ | |
| | $\mathbf{\Lambda}^{(2)}=(1,0,0,0,0,0,0,-1)$ | $7\times 19$ | |
| | $\mathbf{\Lambda}^{(3)}=(7/4,-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4},-\frac{1}{4})$ | $2^4\times 3\times 19$ | |
| | $\mathbf{\Lambda}^{(4)}=(\frac{3}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{6})$ | $3^4\times 19$ | |
| | $\mathbf{\Lambda}^{(5)}=(2,0,0,0,0,0,-1,-1)$ | $5\times 7\times 13\times 19$ | |
| | $\mathbf{\Lambda}^{(6)}=(9/4,\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4},-\frac{3}{4},-\frac{3}{4},-\frac{3}{4})$ | $2^4\times 7\times 13\times 19$ | |
| | $\mathbf{\Lambda}^{(7)}=(3,0,0,0,0,-1,-1,-1)$ | $2\times 5^3\times 7\times 11\times 19$ | |
| $E_8$ | $\mathbf{\Lambda}^{(1)}=(1,1,0,0,0,0,0,0)$ | $2^3\times 31$ | |
| | $\mathbf{\Lambda}^{(2)}=(2,0,0,0,0,0,0,0)$ | $5^3\times 31$ | |
| | $\mathbf{\Lambda}^{(3)}=(2,1,1,0,0,0,0,0)$ | $2^2\times 5\times 7^2\times 31$ | |
| | $\mathbf{\Lambda}^{(4)}=(\frac{5}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ | $2\times 5^3\times 19\times 31$ | |
| | $\mathbf{\Lambda}^{(5)}=(3,1,1,1,0,0,0,0)$ | $2^6\times 5\times 13\times 19\times 31$ | |
| | $\mathbf{\Lambda}^{(6)}=(\frac{7}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ | $2^6\times 3^3\times 5^3\times 31$ | |
| | $\mathbf{\Lambda}^{(7)}=(4,1,1,1,1,0,0,0)$ | $2\times 3\times 5\times 7^2\times 13^2\times 19\times 31$ | |
| | $\mathbf{\Lambda}^{(8)}=(5,1,1,1,1,1,0,0)$ | $2^5\times 3\times 7^2\times 11^2\times 17\times 23\times 31$ | |

$$E_7:\ (1,0,0,0,0,0,0,-1)\otimes(1,0,0,0,0,0,0,-1)$$
$$=[(0,0,0,0,0,0,0,0)\oplus(\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2})]_{\text{symm}}$$
$$\oplus[(2,0,0,0,0,0,0,-2)\oplus(1,0,0,0,0,0,0,-1)\oplus(2,0,0,0,0,0,-1,-1)]_{\text{antisym}},$$
$$E_8:\ (1,1,0,0,0,0,0,0)\otimes(1,1,0,0,0,0,0,0)$$
$$=[(0,0,0,0,0,0,0,0)\oplus(2,0,0,0,0,0,0,0)\oplus(2,2,0,0,0,0,0,0)]_{\text{symm}}$$
$$\oplus[(1,1,0,0,0,0,0,0)\oplus(2,1,1,0,0,0,0,0)]_{\text{antisym}}. \quad \text{(B3)}$$

This description of the Clebsch-Gordan series for Kronecker squares conveys much more information than the formulas of Sec. III, which give only the dimensionalities of the representations. Though adequate from the point of view of proving the conjecture, it is not a unique description as the choice of coordinate system is arbitrary. A somewhat better description is in terms of the coefficients $\langle p_1,p_2,\ldots,p_l\rangle$ of Eq. (A6) which are unique except for ordering.

In order to write Eqs. (B3) in terms of $\langle p_1,p_2,\ldots,p_l\rangle$, we need to know the components of the fundamental dominant weights in the coordinate systems we have been using. This information is given in Table VI. Important checks on the entries of this table are that the vectors $\mathbf{\Lambda}^{(i)}$ $(i=1,\ldots,l)$ be linearly independent and that the relation

$$\sum_{i=1}^{l}\mathbf{\Lambda}^{(i)}=\boldsymbol{\delta} \quad \text{(B4)}$$

holds. For $A_l$, $B_l$, and $C_l$ it is particularly simple to go from the components of a highest weight $\mathbf{\Lambda}$ to the coefficients $p_i$ of $\mathbf{\Lambda}^{(i)}$ in Eq. (A6) since we have $p_i=\Lambda_i-\Lambda_{i+1}$, $i=1,\ldots,l$ for $A_l$; $p_i=\Lambda_i-\Lambda_{i+1}$, $i=1,\ldots,l-1$ and $p_l=2\Lambda_l$ for $B_l$; and $p_i=\Lambda_i-\Lambda_{i+1}$, $i=1,\ldots,l-1$ and $p_l=\Lambda_l$ for $C_l$. In Table VI the highest weights for representations of $A_l$ have $l+1$ components and are restricted by

$$\sum_{i=1}^{l+1}\mathbf{\Lambda}_i=0. \quad \text{(B5)}$$

The highest weights for $B_l$, $C_l$, and $D_l$ have $l$ independent components.

We now write the equations of Sec. II in their final form keeping the same ordering of terms:

For $A_l$,

$$l=1:\ \langle1\rangle\otimes\langle1\rangle=[\langle0\rangle\oplus\langle2\rangle]_{\text{symm}}\oplus[\langle1\rangle]_{\text{antisym}},$$
$$l=2:\ \langle1,1\rangle\otimes\langle1,1\rangle=[\langle0,0\rangle\oplus\langle1,1\rangle\oplus\langle2,2\rangle]\oplus[\langle1,1\rangle\oplus\langle3,0\rangle\oplus\langle0,3\rangle]_{\text{antisym}},$$
$$l=3:\ \langle1,0,1\rangle\otimes\langle1,0,1\rangle=[\langle0,0,0\rangle\oplus\langle1,0,1\rangle\oplus\langle2,0,2\rangle\oplus\langle0,2,0\rangle]_{\text{symm}}\oplus[\langle1,0,1\rangle\oplus\langle2,1,0\rangle\oplus\langle0,1,2\rangle]_{\text{antisym}}, \quad \text{(B6)}$$
$$l\geq4:\ \langle1,0,\ldots,0,1\rangle\otimes\langle1,0,\ldots,0,1\rangle$$
$$=[\langle0,\ldots,0\rangle\oplus\langle1,0,\ldots,0,1\rangle\oplus\langle2,0,\ldots,0,2\rangle\oplus\langle0,1,0,\ldots,0,1,0\rangle]_{\text{symm}}$$
$$\oplus[\langle1,0,\ldots,0,1\rangle\oplus\langle2,0,\ldots,0,1,0\rangle\oplus\langle0,1,0,\ldots,0,2\rangle]_{\text{antisym}}.$$

For $B_l$ and $D_l$ for $l\geq4$,

$$l\geq4:\ \langle0,1,0,\ldots,0\rangle\otimes\langle0,1,0,\ldots,0\rangle$$
$$=[\langle0,0,\ldots,0\rangle\oplus\langle2,0,\ldots,0\rangle\oplus\langle0,2,0,\ldots,0\rangle\oplus\langle0,0,0,1,0,\ldots,0\rangle]_{\text{symm}}$$
$$\oplus[\langle0,1,0,\ldots,0\rangle\oplus\langle1,0,1,0,\ldots,0\rangle]_{\text{antisym}}. \quad \text{(B7)}$$

The special cases are

$$B_2:\ \langle0,2\rangle\otimes\langle0,2\rangle=[\langle0,0\rangle\oplus\langle2,0\rangle\oplus\langle0,4\rangle\oplus\langle1,0\rangle]\oplus[\langle0,2\rangle\oplus\langle1,2\rangle]_{\text{antisym}},$$
$$B_3:\ \langle0,1,0\rangle\otimes\langle0,1,0\rangle=[\langle0,0,0\rangle\oplus\langle2,0,0\rangle\oplus\langle0,2,0\rangle\oplus\langle0,0,2\rangle]_{\text{symm}}\oplus[\langle0,1,0\rangle\oplus\langle1,0,2\rangle]_{\text{antisym}}, \quad \text{(B8)}$$
$$D_3:\ \langle0,1,1\rangle\otimes\langle0,1,1\rangle=[\langle0,0,0\rangle\oplus\langle2,0,0\rangle\oplus\langle0,2,2\rangle\oplus\langle0,1,1\rangle]_{\text{symm}}\oplus[\langle0,1,1\rangle\oplus\langle1,2,0\rangle\oplus\langle1,0,2\rangle]_{\text{antisym}}.$$

For $C_l$,

$$l=2:\ \langle2,0\rangle\otimes\langle2,0\rangle=[\langle0,0\rangle\oplus\langle0,1\rangle\oplus\langle0,2\rangle\oplus\langle4,0\rangle]\oplus[\langle2,0\rangle\oplus\langle2,1\rangle]_{\text{antisym}},$$
$$l\geq3:\ \langle2,0,\ldots,0\rangle\otimes\langle2,0,\ldots,0\rangle \quad \text{(B9)}$$
$$=[\langle0,0,\ldots,0\rangle\oplus\langle0,1,0,\ldots,0\rangle\oplus\langle0,2,0,\ldots,0\rangle\oplus\langle4,0,\ldots,0\rangle]_{\text{symm}}$$
$$\oplus[\langle2,0,\ldots,0\rangle\oplus\langle2,1,0,\ldots,0\rangle]_{\text{antisym}}.$$

For the exceptional algebras,

$$\begin{aligned}
G_2:\ &\langle 0,1\rangle\otimes\langle 0,1\rangle=[\langle 0,0\rangle\oplus\langle 2,0\rangle\oplus\langle 0,2\rangle]_{\rm symm}\oplus[\langle 0,1\rangle\oplus\langle 3,0\rangle]_{\rm antisym}\,,\\
F_4:\ &\langle 0,1,0,0\rangle\otimes\langle 0,1,0,0\rangle=[\langle 0,0,0,0\rangle\oplus\langle 2,0,0,0\rangle\oplus\langle 0,2,0,0\rangle]_{\rm symm}\oplus[\langle 0,1,0,0\rangle\oplus\langle 1,0,1,0\rangle]_{\rm antisym}\,,\\
E_6:\ &\langle 0,0,1,0,0,0\rangle\otimes\langle 0,0,1,0,0,0\rangle\\
&=[\langle 0,0,0,0,0,0\rangle\oplus\langle 0,0,2,0,0,0\rangle\oplus\langle 1,1,0,0,0,0\rangle]_{\rm symm}\oplus[\langle 0,0,1,0,0,0\rangle\oplus\langle 0,0,0,0,0,1\rangle]_{\rm antisym}\,,\\
E_7:\ &\langle 0,1,0,0,0,0,0\rangle\otimes\langle 0,1,0,0,0,0,0\rangle\\
&=[\langle 0,0,0,0,0,0,0\rangle\oplus\langle 0,0,0,1,0,0,0\rangle\oplus\langle 0,2,0,0,0,0,0\rangle]_{\rm symm}\oplus[\langle 0,1,0,0,0,0,0\rangle\oplus\langle 0,0,0,0,1,0,0\rangle]_{\rm antisym}\,,\\
E_8:\ &\langle 1,0,0,0,0,0,0,0\rangle\otimes\langle 1,0,0,0,0,0,0,0\rangle\\
&=[\langle 0,0,0,0,0,0,0,0\rangle\oplus\langle 0,1,0,0,0,0,0,0\rangle\oplus\langle 2,0,0,0,0,0,0,0\rangle]_{\rm symm}\\
&\quad\oplus[\langle 1,0,0,0,0,0,0,0\rangle\oplus\langle 0,0,1,0,0,0,0,0\rangle]_{\rm antisym}\,.
\end{aligned}\tag{B10}$$

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# Relativistic Hydrodynamics in One Dimension*

MONTGOMERY H. JOHNSON†
*Department of Applied Science, University of California, Davis–Livermore and Lawrence Radiation Laboratory, University of California, Livermore, California 94550*

AND

CHRISTOPHER F. MCKEE‡
*Lawrence Radiation Laboratory, University of California, Livermore, California 94550*
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Hydrodynamic equations for one-dimensional motion, of interest in supernova explosions, are integrated in the relativistic limit. A simple solution is found for free expansion into a vacuum. The propagation of a shock into a medium of decreasing density is determined, and the solution for the subsequent flow behind the shock is also obtained.

## I. INTRODUCTION

EXTREME relativistic motions of a fluid can occur in supernova explosions as the result of a strong shock propagating through the outermost mantle of the star. It has been proposed that cosmic radiation is matter ejected from the surface of the star in this manner.[1]

In this paper the hydrodynamic equations for one-dimensional motion are integrated in the relativistic limit. A simple solution is found for free expansion into a vacuum. The propagation of a shock into a medium of decreasing density is also determined, and the solution for the subsequent flow behind the shock front is obtained.

## II. HYDRODYNAMIC EQUATIONS

The equations for the motion of a fluid in the absence of external forces are obtained by setting the divergence of the energy-momentum tensor equal to zero.[2] Let $p$ be the pressure, $E$ the proper energy density, $c$ the speed of light, and $\beta c$ the fluid speed. For one-dimensional motion in the $x$ direction, the vanishing of the divergence gives

$$\frac{\partial}{\partial x}\left(\frac{p+\beta^2E}{1-\beta^2}\right)+\frac{\partial}{\partial ct}\left(\frac{\beta(p+E)}{1-\beta^2}\right)=0\,,\tag{1}$$

$$\frac{\partial}{\partial x}\left(\frac{\beta(p+E)}{1-\beta^2}\right)+\frac{\partial}{\partial ct}\left(\frac{E+\beta^2p}{1-\beta^2}\right)=0\,.\tag{2}$$

There is also a conservation law for the nucleon number density,

$$\frac{\partial}{\partial x}\frac{n\beta}{(1-\beta^2)^{1/2}}+\frac{\partial}{\partial ct}\frac{n}{(1-\beta^2)^{1/2}}=0\,,\tag{3}$$

where $n$ is the nucleon number density in the proper frame of reference. In the nonrelativistic limit, (1)–(3) reduce to the classical forms of momentum, energy, and mass conservation.

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† Permanent address: Philco-Ford, Newport Beach, Calif.
‡ Present address: Hale Observatories, Carnegie Institution of Washington, California Institute of Technology, Pasadena, Calif. 91109.
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