Lorentz Noninvariance of the Complex-Ghost Relativistic Field Theory*

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It is proved that the complex-ghost relativistic field theory with real space momenta, which was proposed by Lee and Wick and by Lee, is not invariant under real Lorentz transformations in the second-order self-energy part involving a complex-ghost-pair intermediate state.

I. INTRODUCTION

Recently, Lee and Wick[1,2] and Lee[3] renewed interest in constructing a convergent field theory by using "complex-ghost" states, namely, the zero-norm states whose energy eigenvalues are not real. Previously, it had been believed[4,5] that such a theory would almost inevitably violate the unitarity of the physical S matrix because the state consisting of a complex ghost and its conjugate could have a real eigenvalue and negative norm. Lee[3] explicitly showed, however, that this belief was groundless in relativistic theories.

According to Lee and Wick,[1] the transition matrix in the theory involving complex ghosts diverges exponentially as the time interval tends to infinity, but if one extracts the physical part of it, one obtains the correct S matrix which is defined by using in-states and out-states. Hence, in the relativistic field theory, the S matrix can be calculated by means of Feynman integrals as usual, but some modification becomes necessary owing to the above extraction. Instead of the conventional Feynman $-i\epsilon$ prescription, the energy integration has to be made along a complex contour C, which is obtained by deforming the real contour in such a way that when the imaginary parts of the complex-ghost masses gradually increase from infinitesimal to their actual finite values, no paths of the masses collide with the energy integration contour. Here, it is important to note that all the space-momentum integrations are still kept to be real. Then it can be shown that the modified Feynman integral involving a complex-ghost-pair intermediate state does not have the expected cut corresponding to the pair state along the real axis, and therefore the unitarity of the S matrix is not violated.[3] Thus, Lee and Wick[1,2] succeeded in constructing a convergent, unitary, relativistic field theory.

There arises a question on the Lee-Wick prescription, however. The conventional Feynman integral is manifestly Lorentz invariant (or covariant in general) because by means of the Feynman $-i\epsilon$ prescription both energy and space-momentum integrations can be regarded as real ones. If the energy integration is regarded as an integration along a contour instead of using $-i\epsilon$, the whole Feynman integral may be regarded as a multiple contour integral as done explicitly in the homological approach to the Feynman integral. The Lee-Wick modified Feynman integral violates this manifest Lorentz invariance, because its energy integration *must* be a contour integration while its space-momentum integrations *must* be real ones. It is, therefore, necessary to check whether or not the Lee-Wick modified Feynman integral is Lorentz invariant. Possible violation of the Lorentz invariance[6] was implied by the work of Cutkosky, Landshoff, Olive, and Polkinghorne,[7] who showed that in a certain complicated graph at least two different results could emerge from the Lee-Wick modified Feynman integral if one considers various Lorentz frames having *complex* space momenta.

The purpose of the present paper is to prove that the Lee-Wick prescription violates the invariance under real Lorentz transformations for *real* values of the total 4-momentum in the second-order self-energy part involving a complex-ghost-pair intermediate state. The proof is given in Sec. II. In Sec. III, some remarks and comments are made.

II. PROOF OF LORENTZ NONINVARIANCE

We consider the Lee-Wick modified Feynman integral corresponding to the second-order self-energy

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[1] T. D. Lee and G. C. Wick, Nucl. Phys. **B9**, 209 (1969).

[2] T. D. Lee and G. C. Wick, Phys. Rev. D **2**, 1033 (1970). In its unpublished version, Lee and Wick had referred to the paper of Cutkosky *et al.* (Ref. 7) merely as an alternative method in a footnote (without touching on the question of Lorentz invariance), but in the publication they proposed to adopt the prescription of Ref. 7 explicitly whenever Lee's original prescription (Ref. 3) might violate Lorentz invariance in complicated graphs.

[3] T. D. Lee, *Quanta: Essays in Theoretical Physics Dedicated to Gregor Wentzel* (Chicago U.P., Chicago, 1970), p. 260.

[4] R. Ascoli and E. Minardi, Nuovo Cimento **14**, 1254 (1959); K. Nagy, *State Vector Spaces with Indefinite Metric in Quantum Field Theory* (P. Noordhoff, Groningen, The Netherlands, 1966).

[5] S. Tanaka, Progr. Theoret. Phys. (Kyoto) **29**, 104 (1963). (His relativistic theory will lead us to a non-self-adjoint Hamiltonian.)

[6] The present author believes that the reasoning given in the Appendix A of Ref. 7 does not constitute a proof of the noninvariance. Some drawbacks of it are as follows. (1) Cutkosky *et al.* tacitly assume the existence of the triangle singularity, but it is dubious because the Lee-Wick modified Feynman integral does not define an analytic function (see Sec. III). (2) They consider *two* different complex masses aside from the complex conjugate, and therefore the one-complex-mass theory is excluded. (3) Their consideration does not exclude a possible cancellation of the noninvariant contribution by some other graphs of the same order.

[7] R. E. Cutkosky, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, Nucl. Phys. **B12**, 281 (1969).

graph in a model theory in which all particles are spinless.[3] Each internal line corresponds to a linear combination of Feynman propagators of various physical and ghost particles in such a way that the integral is convergent. As is easily seen and was explicitly shown by Lee,[3] we can extract the controversial part from the integral without losing the essential features of the problem. This part is exactly the modified Feynman integral involving the two propagators of a complex ghost and its conjugate. It is explicitly written as

$$I(p_0)\equiv\int_{-\infty}^{+\infty}d^3q\int_C dq_0 \times\left\{\frac{1}{(q_0{}^2-E_{\mathbf{q}}{}^2)[(p_0-q_0)^2-(E_{\mathbf{p}-\mathbf{q}}{}^*)^2]}-(p_0=p_0{}^{(0)},\mathbf{p}=\mathbf{p}^{(0)})\right\}. \quad (2.1)$$

Here p_0 and $\mathbf{p}$ are the external energy and space momentum, respectively (the three components of $\mathbf{p}$ are regarded as real parameters);

$$E_{\mathbf{q}}\equiv(M^2+\mathbf{q}^2)^{1/2}, \quad (2.2)$$

where M stands for a complex mass, that is, $\mathrm{Im}M\neq 0$ [8]; an asterisk denotes complex conjugation; C is a contour shown in Fig. 1; $(p_0=p_0{}^{(0)}, \mathbf{p}=\mathbf{p}^{(0)})$ indicates a subtraction term for avoiding ultraviolet divergence.

By means of Cauchy's theorem it is straightforward to carry out the contour integration over q_0. After some manipulation, we find

$$I(p_0)=-\tfrac{1}{2}\pi i[F(p_0)+F(-p_0)], \quad (2.3)$$

where

$$F(p_0)\equiv\int_{-\infty}^{+\infty}d^3q\left[\frac{1}{E_{\mathbf{q}}E_{\mathbf{p}-\mathbf{q}}{}^*\Phi(p_0,\mathbf{q})}-(p_0=p_0{}^{(0)},\mathbf{p}=\mathbf{p}^{(0)})\right], \quad (2.4)$$

with

$$\Phi(p_0,\mathbf{q})\equiv p_0-E_{\mathbf{q}}-E_{\mathbf{p}-\mathbf{q}}{}^*. \quad (2.5)$$

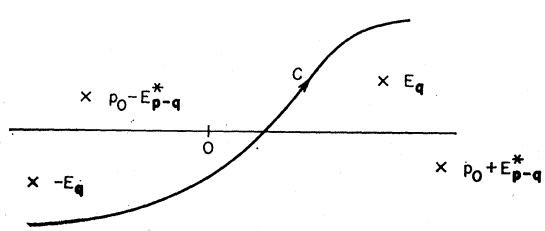

FIG. 1. Contour C in the q_0 plane, where we suppose $\mathrm{Im}M>0$.

[8] The imaginary part of M may be an intrinsic one or may be due to radiative corrections.

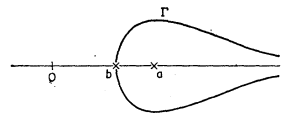

FIG. 2. Nonanalytic barrier Γ in the p_0 plane, where its shape is merely qualitative.

It is evident that (2.4) is holomorphic in p_0 whenever $\Phi(p_0,\mathbf{q})$ does not vanish anywhere in the integration range of $\mathbf{q}$. Therefore, if (2.4) were a contour integral, for which contour deformation is allowed, it would have a singularity only at

$$p_0=a\equiv[(M+M^*)^2+\mathbf{p}^2]^{1/2}, \quad (2.6)$$

which would be accompanied with a cut along the real axis. According to Lee's prescription,[3] however, (2.4) has to be regarded as an integral of *real* variables. Then $\Phi(p_0,\mathbf{q})$ can vanish if p_0 lies in a domain D, whose bounding curve Γ is shown in Fig. 2. The curve Γ intersects the real axis at only one point,

$$p_0=b\equiv E_{\mathbf{p}/2}+E_{\mathbf{p}/2}{}^*=\mathrm{Re}(4M^2+\mathbf{p}^2)^{1/2}. \quad (2.7)$$

It is important to note that

$$b<a \quad \text{for} \quad \mathbf{p}\neq 0 \quad (2.8)$$

because of the nonreality of M. Though $\Phi(p_0,\mathbf{q})$ can vanish in D, as was shown by Lee,[3] $F(p_0)$ is well defined for all values of p_0 and continuous in p_0 for $\mathbf{p}\neq 0$,[9] because for each $p_0\in D$, $\Phi(p_0,\mathbf{q})$ vanishes only on a *one*-dimensional manifold in the $\mathbf{q}$ space and therefore the contribution from its neighborhood is infinitesimal.

Lemma. The first derivative of $F(p_0)$ is discontinuous at $p_0=b$.

Proof. By means of Feynman parametrization, we can write

$$\frac{F(b+\Delta p_0)-F(b)}{\Delta p_0}=\int d^3q\frac{-1}{E_{\mathbf{q}}E_{\mathbf{p}-\mathbf{q}}{}^*[\Phi(b,\mathbf{q})+\Delta p_0]\Phi(b,\mathbf{q})}=-\int_0^1 d\alpha\int d^3q\frac{1}{E_{\mathbf{q}}E_{\mathbf{p}-\mathbf{q}}{}^*[\Phi(b,\mathbf{q})+\alpha\Delta p_0]^2}. \quad (2.9)$$

Suppose $\mathbf{p}\neq 0$, and let

$$\mathbf{k}\equiv\mathbf{q}-\tfrac{1}{2}\mathbf{p}. \quad (2.10)$$

If the direction of $\mathbf{p}$ is chosen as the z axis of the coordinate system, we have

$$p_1=p_2=0, \quad p_3\neq 0. \quad (2.11)$$

Then, for $|\mathbf{k}|$ infinitesimal, it is straightforward to show that

$$\Phi(b,\tfrac{1}{2}\mathbf{p}+\mathbf{k})\simeq -A(k_1{}^2+k_2{}^2)-iBk_3, \quad (2.12)$$

[9] According to Lee (Ref. 3), the $\mathbf{p}=0$ case should be considered only as a limit $\mathbf{p}\to 0$.

where

$$A \equiv \mathrm{Re}E_{\mathbf{p}/2}^{-1} \neq 0, \quad B \equiv p_3 \,\mathrm{Im}E_{\mathbf{p}/2}^{-1} \neq 0. \quad (2.13)$$

Now, we calculate the difference $\Delta F'(b)$ of the right and left derivatives of $F(p_0)$ at $p_0=b$:

$$\Delta F'(b) \equiv \lim_{\Delta p_0 \to +0} \frac{F(b+\Delta p_0)-F(b)}{\Delta p_0} - \lim_{\Delta p_0 \to -0} \frac{F(b+\Delta p_0)-F(b)}{\Delta p_0}. \quad (2.14)$$

Since it is obvious that the contribution to $\Delta F'(b)$ from $|\mathbf{k}|$ noninfinitesimal exactly vanishes, we may confine ourselves to the region of $|\mathbf{k}|$ infinitesimal. Then with $r=k_1^2+k_2^2$, we find

$$\begin{aligned} \Delta F'(b) &= -\frac{\pi}{|E_{\mathbf{p}/2}|^2} \lim_{\epsilon \to +0} \int_0^1 d\alpha \int_{-\infty}^{+\infty} dk_3 \int_0^{+\infty} dr \\ &\quad \times \left[\frac{1}{(Ar+iBk_3-\alpha\epsilon)^2} - \frac{1}{(Ar+iBk_3+\alpha\epsilon)^2} \right] \\ &= \frac{\pi i}{A|E_{\mathbf{p}/2}|^2} \lim_{\epsilon \to +0} \int_0^1 d\alpha \int_{-\infty}^{+\infty} dk_3 \\ &\quad \times \left(\frac{1}{Bk_3+i\alpha\epsilon} - \frac{1}{Bk_3-i\alpha\epsilon} \right) \\ &= \frac{2\pi^2}{A|B|\,|E_{\mathbf{p}/2}|^2} = \frac{2\pi^2 |E_{\mathbf{p}/2}|^2}{|\mathbf{p}|\ \mathrm{Re}E_{\mathbf{p}/2} |\mathrm{Im}E_{\mathbf{p}/2}|}, \end{aligned} \quad (2.15)$$

where we have used an identity

$$\lim_{\epsilon \to +0} \left[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} \right] = -2\pi i \delta(x). \quad (2.16)$$

Thus $\Delta F'(b)$ is nonvanishing for $\mathbf{p} \neq 0$. (We can likewise show that it is infinite for $\mathbf{p}=0$.[3]) Q.E.D.

Let $\tilde{I}(p_0)$ be the analytic continuation of $I(p_0)$ from $|p_0|<b$; that is, let $\tilde{I}(p_0)$ be the integral obtained from (2.1) by deforming the $\mathbf{q}$ contours. Then $\tilde{I}(p_0)$ is manifestly Lorentz invariant and holomorphic except for two cuts $p_0 \geq a$ and $p_0 \leq -a$. Let

$$D(p_0) \equiv I(p_0) - \tilde{I}(p_0). \quad (2.17)$$

Then $D(p_0)$ has the following properties. (1) It is an even function of p_0. (2) It identically vanishes for $|p_0|<b$; in particular,

$$D(p_0)=0 \quad \text{for} \quad -b<p_0<b. \quad (2.18)$$

(3) It is continuous in p_0; in particular,

$$D(\pm b)=0. \quad (2.19)$$

(4) Its first derivative (along the real axis) is discontinuous at $p_0=\pm b$.

Now, we *assume* that $I(p_0)$ is Lorentz invariant at least for p_0 *real*, and show that this leads to an absurdity. If $I(p_0)$ is Lorentz invariant, then the function

$$\bar{D}(z) \equiv D((z+\mathbf{p}^2)^{1/2}) \quad (2.20)$$

is Lorentz invariant, where $z=p_0^2-\mathbf{p}^2$. From (2.19) we have

$$\bar{D}(\beta)=0, \quad (2.21)$$

with

$$\beta \equiv b^2 - \mathbf{p}^2. \quad (2.22)$$

Under finite, real Lorentz transformations, β can move throughout the interval

$$2(M^2+M^{*2}) < \beta \leq (M+M^*)^2. \quad (2.23)$$

The Lorentz invariance of $\bar{D}(z)$ implies that (2.21) holds for any value of β satisfying (2.23). Using (2.18) and (2.20), therefore, we have

$$D(p_0)=0 \quad \text{for} \quad -a \leq p_0 \leq a. \quad (2.24)$$

For $\mathbf{p} \neq 0$, (2.24) contradicts property (4) of $D(p_0)$ because of (2.8). Thus $I(p_0)$ cannot be Lorentz invariant.

Remark. If we consider the case in which the number of spatial dimensions is one, then the proof of the Lorentz noninvariance of $I(p_0)$ becomes much simpler. In this case, $F(p_0)$ is holomorphic except on Γ and divergent on Γ. In particular, $I(p_0)$ is divergent at $p_0=b$. Since the image β of b in the z plane is not invariant under Lorentz transformations, $I(p_0)$ cannot be Lorentz invariant.

III. DISCUSSION

First, we should make a comment on Lee's proof[3] of the Lorentz invariance of $I(p_0)$. He proved that the violation of the Lorentz invariance of $I(p_0)$ is *at most infinitesimal* under any *infinitesimal* Lorentz transformation. This result is indeed true, but it by no means guarantees that the violation is infinitesimal under any *finite* Lorentz transformation. As seen from our proof presented in Sec. II, the violation occurs only for finite Lorentz transformations, as long as the space momentum has three dimensions. Therefore, our conclusion is not inconsistent with Lee's proof.

In Sec. II, we have shown that $F(p_0)$ is not holomorphic at $p_0=b$. By the same reasoning, we can show that $F(p_0)$ is singular at every point on Γ. Thus the singularity of $F(p_0)$ entirely encloses a domain D. We call Γ the "nonanalytic barrier," which is qualitatively different from the usual cut appearing in the theory of analytic functions. The nonanalytic barrier is artificial in the mathematical sense; that is, it shrinks if analytic continuation is enforced mathematically. However, it is physically very significant. It is important to observe that $E_{\mathbf{q}}+E_{\mathbf{p}-\mathbf{q}}^*$ is the total energy of the system consisting of a complex ghost and its conjugate and having the total space momentum $\mathbf{p}$. Therefore, D is nothing but the range of the energy eigenvalues of the complex-ghost-pair states. We note that almost every point (in Lebesgue's sense) of D is nonreal. As was pointed out by Ascoli and Minardi[10] and emphasized by Lee and Wick,[1,2] any state having a nonreal energy eigenvalue cannot contribute to the final state if the initial state consists of physical particles alone. Thus the nonreality of the energy eigenvalues of the pair states makes the S matrix unitary. Conversely, if the S matrix is unitary, $F(p_0)$ should not have the usual cut $p_0 \geq a$. Hence there should exist a nonanalytic barrier enclosing it in order to forbid us to continue $F(p_0)$ analytically. This barrier should have a definite location and a definite shape[11] if the theory has a Hamiltonian, because as seen from (5.11) and (5.12) of Ref. 1 the S-matrix elements are expected not to be continued analytically to the region of the eigenvalue spectrum of the Hamiltonian. Thus the existence of the nonanalytic barrier is vital to the unitarity of the S matrix, as long as the theory has a Hamiltonian.

Cutkosky, Landshoff, Olive, and Polkinghorne[7] proposed a modified version of the Lee-Wick modified Feynman integral in the framework of the S-matrix theory. According to their prescription, one first supposes that every complex-ghost propagator involved in the integral has an imaginary part of the mass different from that of any other propagator. Allowing space-momentum contour deformation, one can analytically calculate the integral. Since any pair of complex ghosts cannot yield a branch point on the real axis, the unitarity of the S matrix is guaranteed. Finally, one takes the limit in which all complex-ghost masses (whose imaginary parts have the same sign) coincide. Though this limiting procedure is ambiguous in higher-order graphs, their prescription always yields a manifestly Lorentz-invariant, unitary S matrix. This recipe should not, however, be regarded as a substitute for the Lee-Wick modified Feynman integral, because the former has not yet been derived from a Lagrangian field theory. It is quite plausible that it cannot be derived from a Lagrangian field theory, as is seen from the following reasoning.

(1) The contour deformation of the space momentum, which is vital to the Lorentz invariance, has never been justified physically (i.e., not in the sense of a purely mathematical device) in the field-theoretical formulation.

(2) Because of the absence of the nonanalytic barrier, there will not exist a Hamiltonian as discussed above.

(3) Before taking the limit, higher-order Feynman graphs require more complex ghosts of different masses; that is, the interaction Lagrangian, if we assume it, has to contain infinitely many terms. Furthermore, as long as we start with a Lagrangian, we cannot forbid the appearance of the Feynman graphs which have an intermediate state involving both complex ghost and its own conjugate.

We may thus conjecture that the complex-ghost theory is unitary at the sacrifice of *either* the Lorentz invariance *or* the Lagrangian field-theoretical formulation.

Though the complex-ghost relativistic field theory is not Lorentz invariant,[12] this fact should not be regarded as a serious defect of it. If ReM is large while ImM is small, then the Lorentz invariance is violated very slightly[13] only in high energies. Such a slight violation of the Lorentz invariance is rather natural because it is experimentally known that the space- and time-inversion invariances are violated. This theory provides an interesting example of Lorentz-noninvariant theories without losing the formal Lorentz invariance as a guiding principle in the formulation. This feature is somewhat analogous to the so-called spontaneous breakdown of symmetry in the sense that the basic formalism is invariant but the noninvariance emerges from state vectors. Complex-ghost states having a real space momentum[14] cannot span a Lorentz-invariant subspace of Hilbert space,[15] so that the extraction of the S matrix stated in Sec. I is not made in a Lorentz-invariant way.

A difficulty of the complex-ghost relativistic field theory may happen when one carries out the renormalization procedure according to Dyson's prescription.[16] (The renormalization is mathematically unnecessary in the convergent field theory, but it is necessary from the physical point of view.) Then propagators involve radiative corrections, and therefore their analyticity domains are bordered by nonanalytic barriers. In an integral involving such modified propagators, two nonanalytic barriers of the opposite sides may collide if the total energy is higher than 4 ReM. If this situation happens, we can no longer choose any adequate contour in that integral.

ACKNOWLEDGMENT

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[10] R. Ascoli and E. Minardi, Nuovo Cimento 8, 951 (1958).

[11] Therefore, it is different from the nonanalytic cut encountered in Ref. 7.

[12] Rigorously speaking, if ImM is due to radiative corrections as in Ref. 2, then we cannot exclude a possibility that the Lorentz-noninvariant part found in the present paper might be canceled by a similar part in higher-order modified Feynman integrals. Of course, there is no reason why such a cancellation should happen.

[13] Note that $a-b$ is of order $(\text{Im}M)^2$ for ImM small.

[14] To secure Lorentz invariance, one might try to construct a theory in which the velocity vector is real instead of the space momentum. In such a theory, however, it seems to be very difficult to deal with many-particle systems and to define Feynman integrals.

[15] A similar remark was made from a different standpoint by K. L. Nagy, ITP-Budapest Report No. 267, 1969 (unpublished).

[16] F. J. Dyson, Phys. Rev. 75, 1736 (1949).