Models for Meson-Deuteron Scattering in the Resonant Region*

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A model is presented which extends the Glauber-theory description of meson scattering from deuterons to regions in which the meson-nucleon cross section varies rapidly or resonates. A Watson form of multiple-scattering theory is employed to derive a model which includes effects caused by the nucleons' Fermi momenta, nonforward intermediate scattering, and off-energy-shell scattering. It is concluded, after a numerical analysis, that the Glauber theory itself could generate the large oscillations in $\langle r^{-2}\rangle_d$ reported in the analysis of π^+d data around 1 GeV/c and could introduce small peaks into the deduced $I=0$ channel of the K^+n cross section. Nevertheless, conclusions drawn by following the usual folding procedures are not altered significantly by this model, for the K^+d data of Cool *et al.*

I. INTRODUCTION

HISTORICALLY, scattering from deuterons has been studied both as an investigation of nature's only strongly bound two-body state and as a substitute for scattering from neutrons. Since the deuteron is loosely bound with a large neutron-proton separation, the total deuteron scattering can be represented at high energies—with an error of less than 10%—as the sum of free neutron and proton scattering. For this reason, any improvements of the existing detailed description of the scattering process or of the procedure for deducing precise neutron cross sections from those of deuterons and protons must necessarily involve the calculation of relatively small correction terms and the accurate measurement of total cross sections.

Usually the Glauber multiple-scattering theory[1] is employed to describe scattering from deuterons. The simplest form for the meson-deuteron total cross section in terms of proton and neutron cross sections is

$$\sigma^d=\sigma^p+\sigma^n-(1/4\pi)\langle r^{-2}\rangle_d\sigma^p\sigma^n. \qquad (1)$$

This paper describes an attempt to examine and extend specific aspects of this theory which may be of importance in the calculation of meson-deuteron total cross sections in the meson-nucleon resonant region.[2–5]

The motivation for this study was provided by two interesting and important effects, deduced from K^+d and π^+d total cross sections, which call into question the validity of the present theory of meson-deuteron scattering. In the analysis of their $\pi^\pm p$ and $\pi^\pm d$ data, Carter *et al.*[5] solved the appropriate form of Eq. (1) for the parameter $\langle r^{-2}\rangle_d$. All other quantities are known if charge independence is assumed. In disagreement with the customarily employed Glauber picture, which predicts this parameter to be constant, Carter *et al.*'s determination of $\langle r^{-2}\rangle_d$ as a function of momentum oscillated considerably (at one point becoming negative), with structure very similar to the measured resonant structure in the $\pi^\pm p$ total cross sections.

In an experiment by Cool *et al.*,[2] a 7% peak in $\sigma^{\text{tot}}(K^+p)$, whose interpretation is uncertain,[6,7] and an 8% peak in $\sigma^{\text{tot}}(K^+d)$ were measured at ~1-GeV/c kaon lab momentum. When $\langle r^{-2}\rangle_d$ was assumed constant, these peaks were amplified by the scattering model and by the unfolding of Fermi motion into a resonancelike 25% peak in the deduced $I=0$ part of the K^+n cross section—the Z^0 particle or resonance. There is reluctance, however, to accept the Z^0 as a resonance both because it would be an exotic, five-quark state,[6] and also because the reliability of the deuteron scattering model has been questioned, especially in the resonance region.[5,8,9]

A source of error in the Glauber model which may cause some of these results is its treatment of the effects of the momentum of each nucleon within the deuteron, the Fermi momentum, upon the c.m. energy dependence of the scattering amplitudes. This seemingly small (~40 MeV/c) momentum is quite important, however, as it causes a spread in the effective beam momentum which is often comparable to the range over which meson-nucleon cross sections vary rapidly. In this regard, Fäldt and Ericson[10] have examined the $\pi^\pm d$

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[1] R. J. Glauber, Phys. Rev. **99**, 630 (1955); **100**, 242 (1955); in *Lectures in Theoretical Physics*, edited by W. Brittin and L. Dunham (Interscience, New York, 1959).

[2] R. L. Cool, G. Giacomelli, T. F. Kycia, B. A. Leontic, K. K. Li, A. Lundby, and J. Teiger, Phys. Rev. Letters **17**, 102 (1966).

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[5] A. A. Carter, K. F. Riley, R. J. Tapper, D. V. Bugg, R. S. Gilmore, K. M. Knight, D. C. Salter, G. H. Stafford, E. J. N. Wilson, J. D. Davies, J. D. Dowell, P. M. Hattersley, R. J. Homer, and A. W. O'Dell, Phys. Rev. **168**, 1457 (1968).

[6] R. D. Tripp, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 174, and references therein; M. Krammer and E. L. Lomon, Phys. Rev. Letters 20, 71 (1968).

[7] S. Kato, P. Koehler, T. Novey, A. Yokosawa, and G. Burleson, Phys. Rev. Letters **24**, 615 (1970), and references therein.

[8] L. R. Price, N. Barash-Schmidt, O. Benary, R. W. Bland, A. H. Rosenfeld, and C. G. Wohl, LRL Report No. UCRL-20000K^+N 1969 (unpublished).

[9] J. Pumplin, Phys. Rev. **173**, 1651 (1968).

[10] G. Fäldt and T. E. O. Ericson, Nucl. Phys. **B8**, 1 (1968).

and $\pi^{\pm}p$ total cross-section data and have calculated more carefully the effects of Fermi motion in the predominant, and rather model-independent, single-scattering terms. Included in the folding procedure were both S- and D-wave parts to the deuteron wave function and certain kinematical factors. In their calculation of double-scattering terms, however, Fäldt and Ericson employed the usual Glauber theory, which does not take into account the effects of Fermi motion on the energy dependence of the amplitudes, or of nonforward scattering. They obtained results which indicate that the oscillations of $\langle r^{-2}\rangle_d$ do seem to be decreased in amplitude somewhat but are still present. In a more recent work by Alberi and Bertocchi,[11] the importance of including Fermi motion in the calculation of double scattering is discussed, but not evaluated.

A further source of error in Glauber theory, when the meson-nucleon amplitude resonates, arises from the use of the eikonal approximation and the assumption of forward scattering. While both assumptions require that many partial waves add together coherently to cause scattering, in the resonant region only a small number of waves will dominate and scattering will not be restricted solely to the forward direction. Consequently, a multi-partial-wave theory may be inapplicable, especially for multiple scattering where two particles resonating back and forth with each other could be the dominant contribution.

We will now develop a model in which the calculation of double-scattering terms includes both the possibility of a more general angular variation of the scattering amplitudes and some of the effects of Fermi motion, for the case where the total amplitude can be separated into resonant and diffractive parts. This model indicates that for some realistic phenomena Glauber theory may be so inaccurate as to be misleading. In the analysis of K^+d data around 1 GeV/c, however, conclusions drawn by following the experimentalists' folding procedure[3,12] are not altered significantly by our model. We believe this to be the first quantitative calculation of these effects in the literature.

II. DEVELOPMENT OF MODEL

Our model for the description of meson-deuteron scattering is derived from formal multiple-scattering (MS) theory.[13] In this theory the operator for scattering from a deuteron $\mathbf{T}^d$ can be expanded as a MS series in the scattering operators for bound neutrons and protons $\boldsymbol{\tau}^n$ and $\boldsymbol{\tau}^p$ and in the propagator $\mathbf{G}$ for the entire system:

$$\mathbf{T}^d=\boldsymbol{\tau}^p+\boldsymbol{\tau}^n+\boldsymbol{\tau}^p\mathbf{G}\boldsymbol{\tau}^n+\boldsymbol{\tau}^n\mathbf{G}\boldsymbol{\tau}^p+\boldsymbol{\tau}^p\mathbf{G}\boldsymbol{\tau}^n\mathbf{G}\boldsymbol{\tau}^p+\boldsymbol{\tau}^n\mathbf{G}\boldsymbol{\tau}^p\mathbf{G}\boldsymbol{\tau}^n+\cdots. \quad (2)$$

Because our concern is with high-energy scattering from a weakly bound system, we will assume that the scattering amplitude for bound and free nucleons are equal (the impulse approximation) and that the recoil and binding of the nucleons do not contribute significantly to the propagator but may be significant in the energy dependence of the amplitude. Furthermore, since we will employ the optical theorem to calculate the total cross section, only forward scattering need be considered, and therefore we will assume that spin effects and inelastic intermediate states are not significant. The importance of these approximations with relevant references is discussed in the recent review by Glauber.[14]

We also assume that triple and higher orders of multiple scattering do not contribute; only at low energies or for large-angle scattering are the contributions from these terms significant.[9,15] As shown by the small size of the measured MS contribution,[2–5] of order 10%, and the smallness of the calculated double-scattering contribution, for energies in the resonant region the MS expansion of the forward amplitude is rapidly convergent and consequently can be approximated by just the single- and double-scattering terms.

With the above assumptions, the MS expansion of the T matrix element for forward scattering of mesons of 3-momentum $\mathbf{p}$ from stationary deuterons can be written as

$$\begin{aligned}&T^d(\mathbf{P}_d=0,\mathbf{P}_M=\mathbf{p}\rightarrow\mathbf{P}_d=0,\mathbf{P}_M=\mathbf{p})\\&=\int d\mathbf{q}\ \varphi^*(\mathbf{q})T^n(-\mathbf{q},\mathbf{p}\rightarrow-\mathbf{q},\mathbf{p})\varphi(\mathbf{q})\\&\quad+\int d\mathbf{q}_1d\mathbf{q}_2\varphi^*(\mathbf{q}_1-\tfrac{1}{2}\mathbf{q}_2)\\&\quad\times T^n(-\mathbf{q}_1-\tfrac{1}{2}\mathbf{q}_2,\mathbf{p}\rightarrow-\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_2,\mathbf{p}-\mathbf{q}_2)g_p(\mathbf{p}-\mathbf{q}_2)\\&\quad\times T^p(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_2,\mathbf{p}-\mathbf{q}_2\rightarrow\mathbf{q}_1-\tfrac{1}{2}\mathbf{q}_2,\mathbf{p})\varphi(\mathbf{q}_1+\tfrac{1}{2}\mathbf{q}_2)\\&\quad+(\text{terms with }p\leftrightarrow n). \quad (3)\end{aligned}$$

Here T^d, T^p, and T^n are the T-matrix elements for meson scattering from free d's, p's, and n's,

$$g_p(\mathbf{k})=[(p^2+m_M{}^2)^{1/2}-(k^2+m_M{}^2)^{1/2}+i\epsilon]^{-1}$$

is the propagator for free relativistic mesons of mass m_M and 3-momentum $\mathbf{k}$, and $\varphi(\mathbf{q})$ is the deuteron wave function for relative proton-neutron 3-momentum $\mathbf{q}$. This equation is represented schematically in Fig. 1, where the double line represents a deuteron at rest, the dashed line a meson, a box a T-matrix element, and

[11] G. Alberi and L. Bertocchi, Phys. Letters **28B**, 186 (1968); Nuovo Cimento **63A**, 285 (1969).

[12] D. V. Bugg, D. C. Salter, G. H. Stafford, R. F. George, K. F. Riley, and R. J. Tapper, Phys. Rev. **146**, 980 (1966).

[13] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N. Y.) **8**, 551 (1959); G. Chew and M. Goldberger, Phys. Rev. **87**, 778 (1952); M. Goldberger and K. Watson, *Collision Theory* (Wiley, New York, 1964).

[14] R. J. Glauber, in *Proceedings of the Third International Conference on High-Energy Physics and Nuclear Structure, New York, 1961*, edited by S. Devons (Plenum, New York, 1970).

[15] J. H. Hetherington and L. H. Schick, Phys. Rev. **138**, B1411 (1965).

a circle the deuteron wave function, and where all internal momenta (**q**'s) are integrated over.

In order to apply Eq. (3) the simplifying assumption is made that the meson-nucleon scattering amplitude can be separated into two parts. The first part is assumed to depend, except for kinematical factors, only on momentum transfer and is called the "diffractive" amplitude. We assume it to have the form

$$T_{\mathrm{diff}}{}^{p(n)}(s,t)=A^{p(n)}(s)e^{\beta^{p(n)}t/2}. \quad (4)$$

Equation (14) determines a differential cross section proportional to $e^{\beta t}$, typical for small-angle K^+N scattering above ~ 1 GeV/c[8] and π^+N above ~ 2 GeV/c.[16] The remaining part of the amplitude is assumed to depend only on the c.m. energy and is called the "resonant" amplitude. The form employed is

$$T_{\mathrm{res}}{}^{p(n)}(s)=G^{p(n)}(s)/[\sqrt{s}-E_{\mathrm{res}}{}^{p(n)}+\tfrac{1}{2}i\Gamma_{p(n)}(s)]. \quad (5)$$

We do not claim that this is an exact decomposition of the scattering amplitude, especially at low energies where diffractive scattering may not have set in, but rather a theoretical model which permits a convenient determination of the importance of a rapidly varying contribution to the magnitude, phase, and angular dependence of the amplitude. There is in fact some evidence for the validity of this decomposition in the resonant region,[17] although in any case it is more appropriate than the purely diffractive Glauber model presently used there.

s and t are the usual Mandelstam variables; A and G are kinematic and normalization factors which, with the normalization of states and definition of T matrix implied by Eq. (3), are given by

$$A^{p(n)}(s)=-\sigma_{\mathrm{diff}}{}^{p(n)}(s)v^{MN}(\alpha_{p(n)}{}^{\mathrm{diff}}+i)/16\pi^3, \quad (6)$$

$$G^{p(n)}(s)=\Gamma(s)\frac{\Gamma_{\mathrm{el}\,0}}{\Gamma_0}(J+\tfrac{1}{2})v^{MN}\frac{P_l(1+t/2k_{\mathrm{c.m.}}{}^2)}{8\pi^2k_{\mathrm{c.m.}}{}^2}. \quad (7)$$

Here v^{MN} is the magnitude of the relative meson-nucleon velocity; $\alpha_{p(n)}{}^{\mathrm{diff}}$ is the ratio of real to imaginary part of the pure diffractive (without resonances) meson-nucleon scattering amplitude, assumed constant for zero and small scattering angles but permitted to vary with energy; σ_{diff} is the diffractive, or background, part of the total cross section; and $k_{\mathrm{c.m.}}$ is the meson-nucleon c.m. momentum. The s dependence of $\Gamma(s)$ is that given by Jackson[18]:

$$\Gamma(s)=\Gamma_0\left(\frac{k_{\mathrm{c.m.}}}{k_{\mathrm{c.m.}}{}^{\mathrm{res}}}\right)^{2l+1}\left[\frac{E_{\mathrm{res}}{}^{p(n)}[A^2+(k_{\mathrm{c.m.}}{}^{\mathrm{res}})^2]}{(\sqrt{s})(A^2+k_{\mathrm{c.m.}}{}^2)}\right]^l, \quad (8)$$

with $A=0.350$ GeV/c.[3]

[16] G. C. Fox and C. Quigg, LRL Report No. UCRL-20001, 1970 (unpublished).

[17] R. Levi-Setti and E. Predazzi, in *Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966* (University of California Press, Berkeley, 1967).

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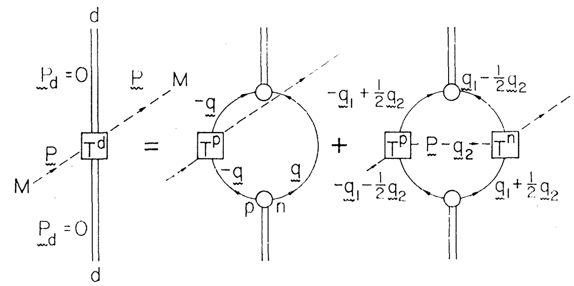

FIG. 1. Schematic representation of the single- and double-scattering contributions to meson-deuteron scattering as calculated with multiple-scattering theory, Eq. (3). Double line represents a deuteron at rest, dashed line a meson, box a T-matrix element, circle the deuteron wave function, and **q**'s internal 3-momenta which are integrated over. 3-momentum but not energy is conserved at each vertex.

The single-scattering term in Eq. (3) is at worst a three-dimensional numerical integral and consequently can be computed with any form of the deuteron wave function. For ease in calculating the double-scattering term, however, a Gaussian wave function

$$\varphi(q)=(\gamma/\pi)^{3/4}e^{-\gamma q^2/2} \quad (9)$$

is selected. Even though a Gaussian does not describe the deuteron very accurately, we know that at least the diffractive part of the forward double-scattering term is not very sensitive to the form of the wave function,[19] and assume (9) still affords a good enough description of the Fermi motion to calculate the size of the expected correction to Glauber theory for the nondiffractive parts. Because of increased high-momentum components, a Hulthén wave function,[20] or one which also contains a D-wave part, would accentuate the importance of Fermi motion to a greater extent than a Gaussian and therefore increase any correction.

When the assumed amplitudes and wave function are substituted into Eq. (3) and the optical theorem,

$$\sigma_{\mathrm{tot}}{}^d(s)=-16\pi^3\,\mathrm{Im}T^d(\theta_{\mathrm{scat}}=0)/v^{Md}, \quad (10)$$

is employed, an expression for the total cross section in terms of definite integrals is obtained. The single-scattering contribution is

$$\sigma^{\mathrm{ss}}(s[p])=\int d\mathbf{q}\,|\varphi(\mathbf{q})|^2 \times[\sigma_{\mathrm{res}}{}^p(s_0)+\sigma_{\mathrm{res}}{}^n(s_0)+\sigma_{\mathrm{diff}}{}^p(s_0)+\sigma_{\mathrm{diff}}{}^n(s_0)] \times v^{MN}(\mathbf{p},\mathbf{q})/v^{Md}(p) \quad (11a)$$

or just simply

$$\sigma^{\mathrm{ss}}(s[p])=\int d\mathbf{q}\,|\varphi(\mathbf{q})|^2 \times[\sigma_{\mathrm{tot}}{}^p(s_0)+\sigma_{\mathrm{tot}}{}^n(s_0)]v^{MN}(\mathbf{p},\mathbf{q})/v^{Md}(p). \quad (11b)$$

[19] V. Franco and R. J. Glauber, Phys. Rev. **142**, 1195 (1966).

[20] M. Moravcsik, Nucl. Phys. **7**, 113 (1958).

The double-scattering contribution is

$$\sigma^{\text{ds}}(p)=-\frac{16\pi^3(\gamma/\pi)^{3/2}}{v^{Md}(p)}\,\text{Im}\int d\mathbf{q}_1 d\mathbf{q}_2\,\exp[-\gamma(q_1^2+\tfrac{1}{4}q_2^2)]\,g_p(\mathbf{p}-\mathbf{q}_2)$$

$$\times\{T_{\text{diff}}^n(s_1,t_1)T_{\text{diff}}^p(s_2,t_2)+T_{\text{diff}}^n(s_1,t_1)T_{\text{res}}^p(s_2)+T_{\text{res}}^n(s_1)T_{\text{res}}^p(s_2)\}+(\text{terms with } p\leftrightarrow n) \quad (12\text{a})$$

$$=-\frac{16\pi^3(\gamma/\pi)^{3/2}}{v^{Md}(p)}\,\text{Im}\int\frac{d\mathbf{q}_1 d\mathbf{q}_2\,\exp[-\gamma(q_1^2+\frac{1}{4}q_2^2)]}{(p^2+m_M^2)^{1/2}-[(\mathbf{p}-\mathbf{q}_2)^2+m_M^2]^{1/2}+i\epsilon}$$

$$\times\left\{A^n(s_1)e^{\beta^n t_1/2}A^p(s_2)e^{\beta^p t_2/2}+\frac{A^n(s_1)e^{\beta^n t_2/2}G^p(s_2)}{\sqrt{s_2}-E_{\text{res}}^p+\frac{1}{2}i\Gamma_p(s_2)}\right.$$

$$\left.+\frac{G^n(s_1)G^p(s_2)}{[\sqrt{s_1}-E_{\text{res}}^n+\frac{1}{2}i\Gamma_n(s_1)][\sqrt{s_2}-E_{\text{res}}^p+\frac{1}{2}i\Gamma_p(s_2)]}\right\}+(\text{terms with } p\leftrightarrow n)\,. \quad (12\text{b})$$

The double-scattering contribution actually calculated contains three times as many terms because of the inclusion of double-charge-exchange (isospin-flip) scattering.[21]

Equation (11b) is the complete and proper definition of a folded cross section even when no resonant and diffractive separation of the amplitude is made. For single scattering s_0 as represented in Fig. 2(a),

$$s_0=m_N^2+m_M^2+2[(m_N^2+q^2)(m_M^2+p^2)]^{1/2}+2\mathbf{p}\cdot\mathbf{q}\,, \quad (13)$$

is equal to the c.m. energy squared of either the incoming or outgoing meson-nucleon system and is the appropriate c.m. energy at which to evaluate the meson-nucleon cross section. The velocity factor arises from the difference in the relative velocity of the meson with respect to a nucleon, $v^{MN}(\mathbf{p},\mathbf{q})$, and with respect to the static deuteron, $v^{Md}(p)$. In the analysis of experimental data the small changes caused by this are often ignored.[10]

As opposed to the choice of s_0 for single scattering, the correct choice of the invariant s's and t's for double scattering is ambiguous; Eq. (3) represents a noncovariant three-dimensional, and not a covariant four-dimensional, theory. Since these double-scattering terms describe off-energy-shell, but on-mass-shell, scattering, particular choices of s and t, as shown in Fig. 2(b), e.g., are equivalent to specific off-shell behaviors of the scattering amplitude. Though it would be best to compare the results of this calculation performed with each possible set of s and t, only the choices of the incoming s for the first scattering,

$$s_1=m_N^2+m_M^2+2\{(m_M^2+p^2)[m_N^2+(-\mathbf{q}_1-\tfrac{1}{2}\mathbf{q}_2)^2]\}^{1/2}+2\mathbf{p}\cdot\mathbf{q}_1+\mathbf{p}\cdot\mathbf{q}_2\,, \quad (14\text{a})$$

the outgoing s for the second scattering,

$$s_2=m_N^2+m_M^2+2\{(m_M^2+p^2)[m_N^2+(\mathbf{q}_1-\tfrac{1}{2}\mathbf{q}_2)^2]\}^{1/2}-2\mathbf{p}\cdot\mathbf{q}_1+\mathbf{p}\cdot\mathbf{q}_2\,, \quad (14\text{b})$$

and t defined as the 3-momentum transfer,

$$t_1=t_2=-q_2^2\,, \quad (14\text{c})$$

are used. The importance of a specific choice of off-shell behavior has been investigated for nonforward single scattering[22] where, typically, different choices of s result in folded amplitudes which agree at 0-momentum transfer but differ by 20% at a momentum transfer of 200 MeV/c and by greater amounts at larger momentum transfer. Since in this paper we consider only forward meson-deuteron scattering with correspondingly small intermediate momentum transfers of ~40 MeV/c, this effect is not very important here.

After choosing the above definitions of s and t, we make a judicious change of variables and use the high-energy approximation (in the denominators only) that the meson momentum p is much greater than the internal momenta q_1 and q_2. In this way the double-diffractive-scattering integral, the first term in Eq. (12), can be evaluated analytically, and the diffractive-resonant and double-resonant scattering integrals, the second and third terms, can be reduced to three-dimensional numerical integrations. Because of their complexity and length, these integrals are not listed here but rather in Appendix A. For simplicity, in the double-scattering calculation the dependence of v^{MN} and $k_{\text{c.m.}}$ on the nucleon's Fermi momentum are ignored as the more sensitive dependence of the denominators is assumed to dominate the cross-section behavior. Also, for simplicity, the resonance's $2J+1$ statistical factor is retained but the integrals are only calculated for an $l=0$ resonance. The oscillatory be-

[21] The most transparent method to treat charge independence and isospin flip is to consider the MS expansion for T^d an operator equation in isospin space and then take its matrix element between singlet deuteron states. The use of isospin identity operators in the intermediate states automatically includes the spin-flip contribution properly. A more intuitive and simple method is given by C. Wilkin, Phys. Rev. Letters **17**, 501 (1966).

[22] R. H. Landau (unpublished).

havior of P_l for higher l values would give less double scattering.

Equation (11) and the set of expressions obtained by approximating Eq. (12) form our multiple-scattering-theory (MST) model for calculating high-energy meson-deuteron total cross sections when either, or both, a neutron and proton resonance exist(s). If only the double-diffractive part of the double-scattering contribution had been considered, i.e., the amplitudes are assumed to depend only on the momentum transfer, a model would be obtained which is similar to the one derived by Pumplin.[9] This model makes a correction to Glauber theory for nonzero angles, and especially for large angles where triple scattering may enter, but reduces to it for forward scattering. By extending this purely diffractive model so as to include some of the effects of (1) the nucleons' Fermi motion upon the s dependence of the amplitudes, (2) the sensitive phase variation of the amplitudes due to a resonance in one partial wave, (3) off-energy shell scattering in intermediate states, and (4) nonforward double scattering, the MST model differs from the generally successful Glauber-theory (GT) model.

The appropriateness of using the MST as a means of calculating corrections to GT, such as the ones we are calculating here, has recently been questioned[14] on the basis of a conclusion drawn by Harrington[23] that off-energy-shell corrections to GT's double-scattering term must be canceled by higher orders of multiple scattering. This conclusion is based on the assumption that GT is exact at high enough energies and, as Harrington points out, is only valid if the same approximations used in deriving GT are used in the application of MST. Since our concern is with situations in which GT is not exact, and since the model developed here differs from GT not just by the inclusion of off-energy shell scattering but also by the inclusion of Fermi motion and nonforward scattering, the conclusions drawn by Harrington are not relevant to the work reported here.

In practice, most applications of GT to total cross-section measurements employ a "doctored-up" form of the theory[3,12] in which Fermi-motion effects are introduced, and in a sense doubly counted, by folding the nucleon cross sections before inserting them into the Glauber formula. Since theoretically the two approaches are quite different, with the MST model expected to be an improvement, it should be extremely interesting to compare the folded GT and MST models as a test of the experimental procedure. In Sec. III we make this comparison.

III. APPLICATION OF MODEL

A. Analysis of Hypothetical Pion-Deuteron Experiment

The first application of the MST model is a hypothetical experiment in which we use a folded GT model

[23] D. R. Harrington, Phys. Rev. **184**, 1745 (1969).

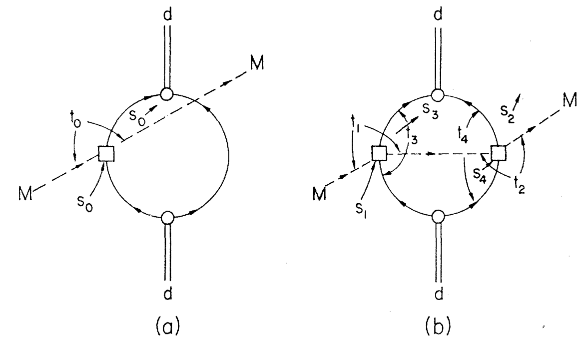

FIG. 2. Different possible choices of s and t for (a) single scattering and (b) double scattering. Each s is the c.m. energy for the two particles on the side of the vertex indicated; each t is the 4-momentum transferred between the two particles indicated. The momentum labels are given in Fig. 1.

to analyze π^+d total-cross-section data that have been calculated from assumed πN data with the MST model. This procedure permits a convenient comparison of the two models by simply comparing "output" deduced by the GT model with "input" used in the MST model.

The data analysis employs sets of π^+d, π^+p, and π^-p total cross sections with each set at a different pion momentum. The π^+p and π^-p cross sections are folded to obtain σ^p and σ^n and the GT equation

$$\sigma^{\text{tot}}(\pi^+d)=\sigma^p+\sigma^n-(1/4\pi)\langle r^{-2}\rangle_d[\tfrac{3}{2}\sigma^p\sigma^n(1-\alpha_p\alpha_n)-\tfrac{1}{4}\sigma^p\sigma^p(1-\alpha_p\alpha_p)-\tfrac{1}{4}\sigma^n\sigma^n(1-\alpha_n\alpha_n)] \quad (15)$$

is solved for the parameter $\langle r^{-2}\rangle_d$. This parameter enters as a measure of the strength of double scattering. We have used the isospin decomposition

$$\sigma^{\text{tot}}(\pi^+p)=\sigma(I=\tfrac{3}{2})\,, \quad (16)$$

$$\sigma^{\text{tot}}(\pi^+n)=\sigma^{\text{tot}}(\pi^-p)=\tfrac{1}{3}[\sigma(I=\tfrac{3}{2})+2\sigma(I=\tfrac{1}{2})]\,. \quad (17)$$

When Carter *et al.*[5] analyzed their experimental $\pi^\pm p$ and π^+d total-cross-section data in this way, rather large oscillations were found in the deduced $\langle r^{-2}\rangle_d$ as the pion momentum varied from 0.5 to 2.65 GeV/c. These oscillations were only slightly reduced in the reanalysis of this data by Fäldt and Ericson.[10] GT predicts the parameter $\langle r^{-2}\rangle_d$ to be equal to the energy-independent expectation value of $1/r^2$ in the deuteron ground state only if the deuteron form factor is an extremely rapidly varying function of momentum transfer. The variation of $\langle r^{-2}\rangle_d$ predicted by a general form of GT, such as Eq. (B3), however, is only a few percent—considerably smaller than calculated by Carter *et al.* The specific purpose of this first application of the MST model is to gain some understanding of the cause of these oscillations without performing detailed fits to the highly structured $\pi^\pm p$ cross sections.

Convenient forms for $\sigma(I=\frac{1}{2})$ and $\sigma(I=\frac{3}{2})$ were chosen and by using these cross sections as input to our MST model the complementary $\sigma^{\text{tot}}(\pi^+d)$ was obtained. The chosen scattering parameters are listed

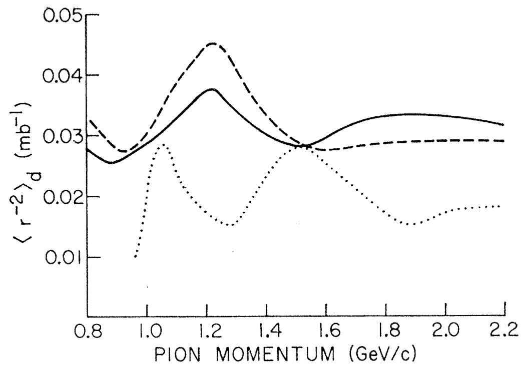

FIG. 3. Oscillations of $\langle r^{-2}\rangle_d$ deduced from π^+d scattering with Glauber theory. Dashed and solid curves from hypothetical experiment, using diffractive and total values of α, respectively. Dotted curve from experiment of Ref. 5.

in column 1 of Table I. The resonances are only representative of the behavior of π^+N scattering since, for simplicity, just a single resonance in each isospin channel has been assumed. The value of γ was chosen to make the quantity $\langle r^{-2}\rangle_g$, defined by Eq. (B2), take on the value of 0.03 mb^{-1}. This is a typical value of $\langle r^{-2}\rangle_g$ obtained with the Hulthén wave function for this region, and is the appropriate way of determining the size of the wave function used to calculate double scattering.[24]

When the GT model, Eq. (20), was used to analyze these data, it deduced the oscillations of $\langle r^{-2}\rangle_d$ shown in Fig. 3. Since the exact variation is rather sensitive to the energy dependence of α_p and α_n, which is rarely known precisely,[25] the dependence of $\langle r^{-2}\rangle_d$ on meson momentum for two different but reasonable choices of α's is displayed along the Carter *et al.*'s results[5] (dotted curve). Energy-independent α's characteristic of the diffractive part of the amplitudes were used for the dashed curve and values of α determined by the energy dependence of the entire amplitudes were used to calculate the solid curve.

The oscillations from the hypothetical data are seen to be of the same magnitude as those obtained from the actual data of Carter *et al.* These curves confirm our suspicion that the GT description of double scattering is not complete, and simultaneously provide a possible explanation of Carter *et al.*'s $\langle r^{-2}\rangle_d$ variation. That is, the GT description actually generates oscillations in $\langle r^{-2}\rangle_d$ if the π^+d cross section from which $\langle r^{-2}\rangle_d$ is deduced is calculated with the more nearly correct MST description.

There is a difference in the position and number of oscillations since only two resonances influenced the theoretical determination, whereas in the experimental determination many more resonances along with their appropriately varying α's are present. The somewhat low average value of $\langle r^{-2}\rangle_d$ calculated from experimental data, which was noted by Carter *et al.*, is not explained by our model. Fäldt and Ericson[10] have suggested that it is caused by a small normalization error in the experimental data.

TABLE I. Meson-deuteron scattering parameters.

Meson	π^+	K^+
$\sigma_{\rm diff}{}^p$ (mb)	20[a]	10[b]
$\sigma_{\rm diff}{}^n$ (mb)	27[a]	10[c]
$\alpha_p{}^{\rm diff}$	−0.25[c]	−0.65[d]
$\alpha_n{}^{\rm diff}$	−0.17[c]	−0.35[d]
β^p (GeV^{-2})	9[c]	2.0[e]
β^n (GeV^{-2})	10[c]	2.0[e]
γ (GeV^{-2})	134	102
$E_{\rm res\,1}$ (GeV)	1.950	1.865
I	$\frac{3}{2}$	1
$E_{\rm res\,2}$ (GeV)	1.672	...
I	$\frac{1}{2}$	0
$\Gamma_0{}^1$ (GeV)	0.18	0.15
$\Gamma_0{}^2$ (GeV)	0.17	...
$(J+\frac{1}{2})\Gamma_{\rm el\,0}/\Gamma_0\|_1$	1.8	1,2
$(J+\frac{1}{2})\Gamma_{\rm el\,0}/\Gamma_0\|_2$	3.2	...

[a] Reference 5. [b] Reference 4. [c] Reference 27. [d] Reference 26.
[e] Reference 8.

B. Analysis of Hypothetical Kaon-Deuteron Experiment

The second application of our model is a somewhat different hypothetical experiment. "Output" K^+n total cross sections, deduced by a folded GT model which uses MST-generated K^+d cross sections, are compared to the K^+n cross sections which were used as "input" to the MST model. The same K^+p cross sections are used throughout. If GT affords a good description of the scattering process, it should deduce neutron cross sections which are very similar to those used as input to the more correct MST model.

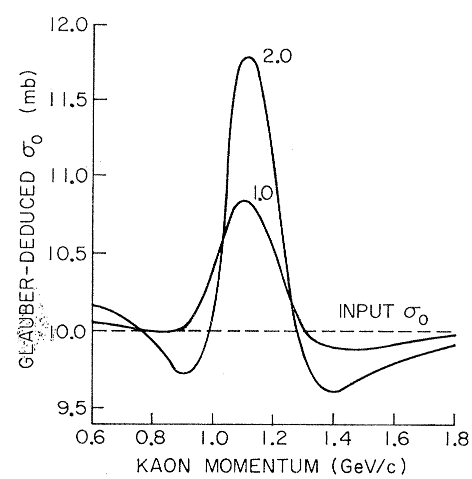

FIG. 4. Glauber-theory-deduced $I=0$ part of the folded K^+n total cross section for a proton resonance of $(J+\frac{1}{2})\ \Gamma_{\rm el\,0}/\Gamma_0=1$, 2 (solid curves). Dashed curve is the multiple-scattering-theory input $I=0$ cross section.

[24] Different, somewhat empirical values of $\langle r^{-2}\rangle_d$ are, in fact, frequently used in different experiments.

[25] D. Hywel White, BNL Report No. BNL 50212, C-58 (unpublished).

The appropriate form of GT for K^+N scattering is

$$\sigma^d=\sigma^p+\sigma^n-(1/4\pi)\langle r^{-2}\rangle_d[2\sigma^p\sigma^n(1-\alpha_p\alpha_n)$$
$$-\tfrac{1}{2}\sigma^p\sigma^p(1-\alpha_p\alpha_p)-\tfrac{1}{2}\sigma^n\sigma^n(1-\alpha_n\alpha_n)], \quad (18)$$

with σ^p and σ^n denoting folded cross sections. The appropriate isospin decomposition of the K^+N total cross section is

$$\sigma^{\text{tot}}(K^+p)=\sigma(I=1)=\sigma_1, \quad (19)$$

$$\sigma^{\text{tot}}(K^+n)=\tfrac{1}{2}[\sigma(I=1)+\sigma(I=0)]=\tfrac{1}{2}(\sigma_1+\sigma_0). \quad (20)$$

When Cool *et al.*[2] analyzed their data with these equations, a 7% peak in $\sigma^{\text{tot}}(K^+p)$ and an 8% peak in $\sigma^{\text{tot}}(K^+d)$ were amplified into a 25% peak in the deduced σ_0—the Z^0. The specific purpose of this second application is to determine how much structure the GT itself introduces into deduced cross sections. When comparing GT and MST, the *un*folding of Fermi motion will deliberately be avoided, however, since it is a delicate process[3,12] which would magnify any differences.

Several different energy dependences for σ_0 and σ_1 were chosen, and the corresponding $\sigma^{\text{tot}}(K^+d)$ was calculated with the MST model. Some typical K^+N scattering parameters used are listed in column 2 of Table I. The resonant parameters are characteristic of the structure seen in K^+N scattering around 1 GeV/c and the value 102 GeV^{-2} for γ was chosen to give the same $\langle r^{-2}\rangle_g$ in this region as does the usual Hulthén wave function.

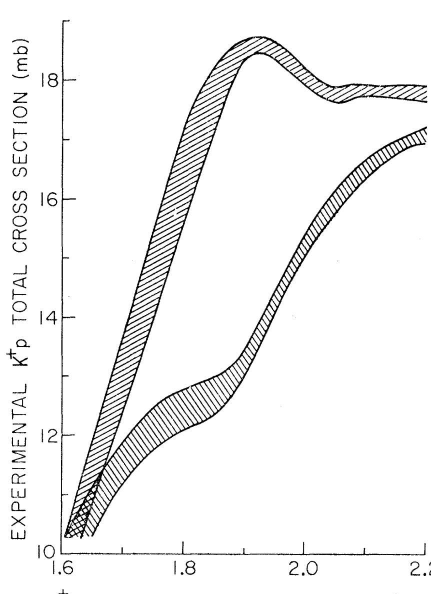

FIG. 5. Error band for K^+p total cross section (upper band) and for "diffractive" background. Data from Ref. 2.

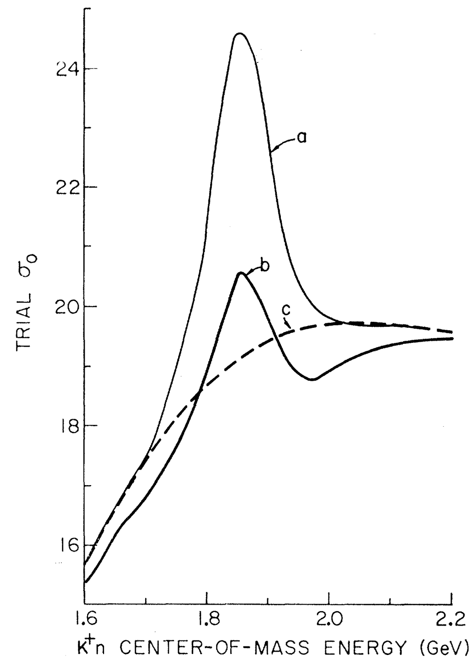

FIG. 6. Three of the shapes used as trial $I=0$ parts to the K^+n total cross section: (a) resonance of the size found in Ref. 2, (b) resonance of half this size, and (c) thresholdlike shoulder.

When the input σ_0 contained a resonant peak, the folded GT model would deduce a reasonably accurate peak if the constant $\langle r^{-2}\rangle_d$ in GT was assumed slightly adjustable, though energy independent.[24] However, if the input σ_1 resonated while the input σ_0 remained constant, the GT model would deduce an output σ_0 with a small but fictitious bump. In Fig. 4 this bump can be seen for both large and small values of the elasticity of the proton resonance. The larger the proton resonance the larger this fictitious bump. Even though these peaks are relatively small, they would be enlarged by unfolding the Fermi motion and, since all structures in K^+N total cross sections are small, possibly misinterpreted as significant. That is, if there is a peak in the known σ_1 the GT description will actually generate a small "coupled" peak in the deduced σ_0, at least when the MST description is used to calculate $\sigma^{\text{tot}}(K^+d)$.

C. Analysis of Actual Kaon-Deuteron Experiment

The final application of the MST model is a detailed analysis of the K^+d and K^+p total cross sections measured by Cool *et al.*[2] Our purpose is to compare the MST and folded GT models' analysis of experimental data and especially to determine to what extent the σ_0 peak deduced in this experiment is related to the fictitious σ_0 peak deduced in the hypothetical experiments of Sec. III B.[26]

[26] Any conclusions derived from studying these simpler, singly peaked data should also apply to the doubly peaked data of Ref. 3.

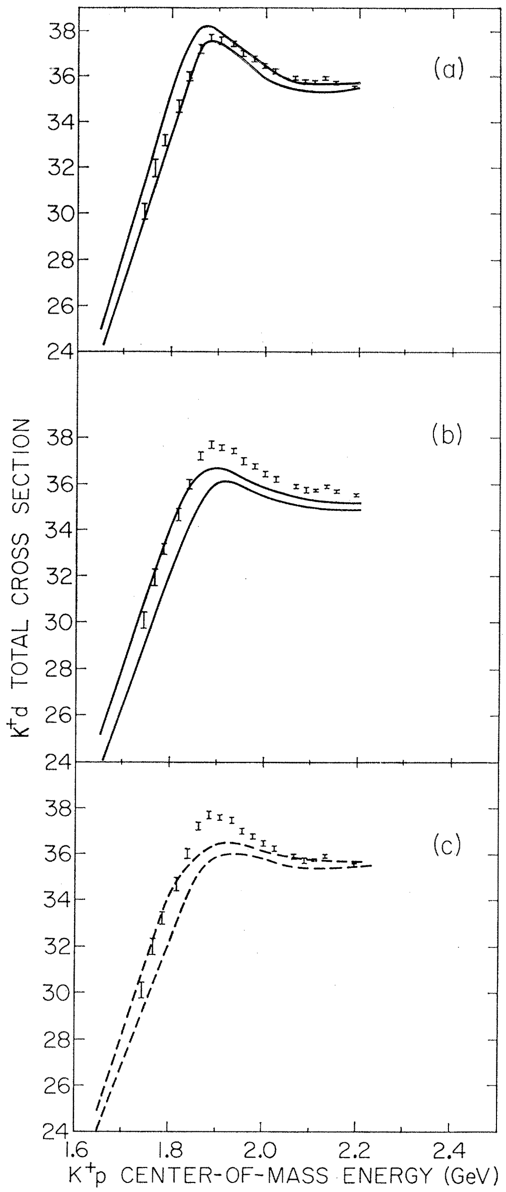

FIG. 7. K^+d total-cross-section data of Ref. 2 and bands predicted for this cross section using the MST model: (a) for a resonance of the size found in Ref. 2, (b) for a resonance of half this size, and (c) for the thresholdlike shoulder of Fig. 6.

Inasmuch as this application deals with fits to experimental data, we chose the most realistic values possible for the scattering parameters. β^p and β^n were assumed equal and taken from an energy-dependent fit to experimental data[8]; α_p^{diff} and α_n^{diff} were calculated from values of α_p^{tot} and α_n^{tot} given by dispersion relations[27]; γ was chosen as 102 GeV^{-2}; the K^+p total-cross-section data were fitted with a "resonance" of mass 1.900 GeV/c^2, width 0.25 GeV, and $(J+\frac{1}{2})\Gamma_{\text{el}\,0}/\Gamma_0=0.40$, and a smoothly varying "diffractive" background, $\sigma^{\text{diff}}(K^+p)$. The error bands thus obtained for $\sigma^{\text{tot}}(K^+p)$ and $\sigma^{\text{diff}}(K^+p)$ are shown in Fig. 5.

We deduced σ_0 indirectly by looking at the output $\sigma^{\text{tot}}(K^+d)$ from the MST and folded GT models when the above fit to the K^+p data (σ_1) and a stepwise continuous range of shapes for σ_0 were used as input. These shapes varied from a thresholdlike shoulder to a large resonance ($1\frac{1}{2}$ times the size deduced by Cool *et al.*[2] using a folded GT) on top of another diffractive background. Figure 6 shows three of these assumed σ_0 shapes: (a) a resonance of Cool *et al.*'s size, (b) a resonance of half this size, and (c) a thresholdlike shoulder. Since the diffractive parts of σ_0 and σ_1 were not constant, they too were folded as were the total cross sections before being used to predict $\sigma^{\text{tot}}(K^+d)$. Figure 7 displays the corresponding MST model's predicted band of $\sigma^{\text{tot}}(K^+d)$ for the three σ_0 shapes of Fig. 6.

As Fig. 7(a) displays, Cool *et al.*'s GT-deduced σ_0 as input to the MST model yields a good fit to $\sigma^{\text{tot}}(K^+d)$. Though at first this may seem somewhat surprising, we have found that several different σ_0 shapes are capable of predicting reasonably good fits to $\sigma^{\text{tot}}(K^+d)$. This ambiguity is caused by the uncertainties in (1) the fitting of data, (2) the procedure for and results of folding cross sections, (3) the energy dependences of α_p and α_n, and (4) the actual data themselves. In particular, an acceptable prediction of $\sigma^{\text{tot}}(K^+d)$ is still possible if the strength of the $I=0$ resonance is reduced by more than 25%, regardless of the MST used. Of course GT's algebraic deduction procedure only determines one σ_0 shape.

Even more unexpected than the MST and GT agreement, within a general level of uncertainty, on $\sigma^{\text{tot}}(K^+d)$ is the observation that when all of the MST model's separate double-scattering parts are added together a total contribution is obtained which is rather close to the total double-scattering contribution predicted by a folded (but not unfolded) GT. This is demonstrated in Fig. 8 where the lower curves are the separate parts of a typical MST's double-scattering contribution, the dashed upper curve is their sum, and the solid upper curve is the complete folded GT contribution. The curve labeled "RES" includes both double-resonant and resonant-diffractive scattering. The complete double-scattering terms do differ, but only by ~10%.

Aside from the general level of uncertainty in the data, there seems to be three causes for the two models' similarity of output in this section. First, no sensitive inversion or algebraic deduction procedures which tend to magnify differences are being performed. Second, a major part of the double-scattering contribution is the double-diffractive scattering terms, which both theories calculate similarly, and not the resonant terms which are calculated differently. Third, in contrast to the hypothetical experiments of Secs. III A and III B, the diffractive parts of the total cross sections are rapidly varying. Consequently, their treatment is only approxi-

[27] A. A. Carter, HEP Report No. HEP68-10, 1968 (unpublished).

mate and their variation tends to mask any model-generated bumps which a constant background does not.

IV. SUMMARY AND DISCUSSION

A model has been presented which extends the Glauber description of diffractive multiple scattering of mesons from deuterons[1,9,10,14] to regions of energy in which meson-nucleon resonances exist. A Watson form of multiple scattering theory has been employed to account inherently for some effects of the nucleons' internal momenta, nonforward intermediate scattering, and off-energy shell scattering, upon the energy dependence of the total meson-deuteron cross section. Since this model separates the scattering amplitude into so-called resonant and diffractive parts, it leads to more complex expressions than found in pure diffractive models. However, it determines for the first time the significance of the above effects in the important resonant region and establishes some initial limits on the reliability of GT-deduced neutron cross sections there.

A series of calculations employing hypothetical but realistic data was performed in which a folded GT analyzed meson-deuteron cross sections which were calculated with the MST model. We conclude from these calculations that GT could introduce small "coupled" peaks into the $I=0$ channel of deduced K^+n cross sections and would generate fairly large oscillations in the deduced π^+d scattering parameter $\langle r^{-2}\rangle_d$ when the π^+p and π^+n $(=\pi^-p)$ amplitudes contained resonances. Since all structures in K^+N scattering are rather small, it is quite possible for a fictitious peak to be magnified by unfolding, and misinterpreted. The $\langle r^{-2}\rangle_d$ oscillations are of the same size as found experimentally[5] and provide an explanation of this phenomenon as a breakdown of simple GT.

In order to determine if the small structure introduced into deduced neutron cross sections by the deduction procedure may partially account for the Z^0's found experimentally, an analysis of Cool *et al.*'s[2] K^+d, p data around 1 GeV/c was performed. We conclude from these calculations that there is considerable uncertainty in the exact size of the $I=0$ structure but that the strength of any resonance present is not large enough, and the diffractive background too rapidly varying for the GT and MST models to be significantly different.

Further improvement and extension of the MST model are both possible and in some cases worthwhile. In particular it appears desirable to modify our model so as to also calculate nonforward scattering. Since recent phase-shift analyses[7] of differential cross-section and polarization measurements yield several possible K^+p resonances not clearly seen in total cross sections, it may be profitable to search for neutron resonances by making measurements of this type on deuterons. To analyze these data a theory of wider applicability than GT is required. Furthermore, since it has been

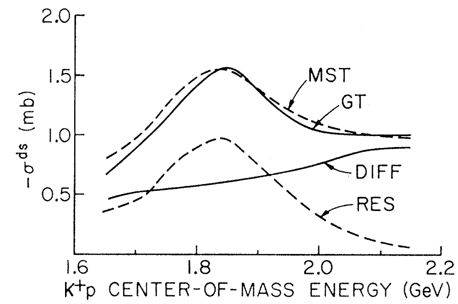

FIG. 8. Comparison of the double-scattering contribution as calculated by Glauber theory (upper solid curve) and the MST model (upper dashed curve). Also displayed is the diffractive part of the MST model contribution (lower solid curve) and the resonant part (lower dashed curve).

found necessary[11,14] to include many effects which are relatively unimportant at small angles to predict precisely large-angle pd and πd scattering with GT, we also expect an increased difference between a GT and a MST model at large angles.

In order to perform a complete calculation of large-angle scattering from deuterons, it would be necessary to investigate the importance of (1) other choices of off-shell behavior,[9,28] (2) inelastic intermediate states,[29] (3) spin-flip scattering,[14] (4) more general angular dependence of the amplitudes, (5) corrections to the propagator,[30] (6) higher orders of multiple scattering,[9,14] and (7) relativistic and resonant parts to the deuteron wave function.[31] Some of this investigation is now in progress.[22]

Also of worth would be the application of a model of the type derived here to calculations which describe meson-nucleon scattering in terms of multiple quark-quark scattering.[32] Since the quarks are very tightly bound in an extremely small well, very large Fermi momenta are possible. It is again expected that some of the relatively small corrections to GT already calculated in the scattering from deuterons will, in this case considerably, increase in magnitude and possibly alter the entire scattering picture.

ACKNOWLEDGMENTS

The author wishes to thank Professor A. Hendry for initially suggesting this problem, Professor D. G. Ravenhall for his always useful advice and comments, and many colleagues, but especially Dr. L. R. Mather, for helpful discussions.

[28] E. S. Abers, H. Burkhardt, V. L. Teplitz, and C. Wilkin, Nuovo Cimento **42**, 365 (1966); L. Bertocchi and A. Capella, *ibid.* **51A**, 369 (1967).
[29] G. Alberi and L. Bertocchi, Nuovo Cimento **61A**, 203 (1969); J. Pumplin and M. Ross, Phys. Rev. Letters **21**, 1778 (1968).
[30] E. A. Remler, Phys. Rev. **176**, 2108 (1968).
[31] A. K. Kerman and L. S. Kisslinger, Phys. Rev. **180**, 1483 (1969).
[32] D. R. Harrington and A. Pagnamenta, Phys. Rev. **173**, 1599 (1968).

APPENDIX A: DOUBLE-SCATTERING INTEGRALS

Once the scattering amplitude is decomposed into diffractive and resonant parts, the double-scattering contribution, Eq. (12b), can be decomposed into double-diffractive, double-resonant, and diffractive-resonant (or resonant-diffractive) scattering parts. As discussed in the text, the dependence of v^{MN} and $k_{\text{c.m.}}$ on the nucleons' Fermi momenta and the factor $P_l(1+t/2k_{\text{c.m.}}^2)$ are ignored, as the more sensitive energy dependence of the denominators is assumed to dominate the $t=0$ behavior. In this way the following decomposition is obtained:

$$\sigma^{\text{ds}}(p)\,|_{\text{diff-diff}}=-\frac{16\pi^3(\gamma/\pi)^{3/2}}{v^{Md}(p)}\,\text{Im}\left\{A^n(s_{\text{st}})A^p(s_{\text{st}})\int d\mathbf{q}_1 d\mathbf{q}_2 g_p(\mathbf{p}-\mathbf{q}_2)e^{-\gamma(q_1^2+q_2^2/4)}e^{-\beta^n q_2^2/2}e^{-\beta^p q_2^2/2}\right\}+p\leftrightarrow n\,, \tag{A1}$$

$$\sigma^{\text{ds}}(p)\,|_{\text{diff-res}}=-\frac{16\pi^3(\gamma/\pi)^{3/2}}{v^{Md}(p)}G^p(s_{\text{st}})\,\text{Im}\left\{A^n(s_{\text{st}})\int d\mathbf{q}_1 d\mathbf{q}_2 \times e^{-\beta^n q_2^2/2}e^{-\gamma(q_1^2+q_2^2/4)}g_p(\mathbf{p}-\mathbf{q}_2)[\sqrt{s_2}-E_{\text{res}}{}^p+\tfrac{1}{2}i\Gamma_p(s_{\text{st}})]^{-1}\right\}+p\leftrightarrow n\,, \tag{A2}$$

and

$$\sigma^{\text{ds}}(p)\,|_{\text{res-res}}=-\frac{16\pi^3(\gamma/\pi)^{3/2}}{v^{Md}(p)}G^n(s_{\text{st}})G^p(s_{\text{st}})\,\text{Im}\left\{\int d\mathbf{q}_1 d\mathbf{q}_2 \times g_p(\mathbf{p}-\mathbf{q}_2)e^{-\gamma(q_1^2+q_2^2/4)}[\sqrt{s_1}-E_{\text{res}}{}^n+\tfrac{1}{2}i\Gamma_n(s_{\text{st}})]^{-1}[\sqrt{s_2}-E_{\text{res}}{}^p+\tfrac{1}{2}i\Gamma_p(s_{\text{st}})]^{-1}\right\}+p\leftrightarrow n\,, \tag{A3}$$

$$g_p(\mathbf{k})=[(p^2+m_M{}^2)^{1/2}-(k^2+m_M{}^2)^{1/2}+i\epsilon]^{-1}\,, \tag{A4}$$

$$s_{\text{st}}=m_N{}^2+m_M{}^2+2m_N(p^2+m_M{}^2)^{1/2}. \tag{A5}$$

$A(s)$ and $G(s)$ have been defined in Eqs. (6) and (7), s_1 and s_2 in Eq. (14), and s_{st} is the static-nucleon limit of s_1 or s_2.

Next, the high-energy approximation that the meson momentum p is much greater than the internal momenta q_1 and q_2 is made and the roots in the denominator are expanded with only first-order terms kept. This is a good approximation as the deuteron wave function inhibits very large Fermi momenta.

In this way the double-diffractive integral can be performed analytically, with the simple result

$$\sigma^{\text{ds}}(p)\,|_{\text{diff-diff}}=-\{2\pi[\gamma+2(\beta^p+\beta^n)]\}^{-1}\sigma_{\text{diff}}{}^p(s_{\text{st}})\sigma_{\text{diff}}{}^n(s_{\text{st}})(1-\alpha_p{}^{\text{diff}}\alpha_n{}^{\text{diff}})\,. \tag{A6}$$

As evidenced by the diffractive slopes β^p and β^n, this equation represents a form of Glauber theory modified for nonforward scattering. In practice some of the approximations made in deriving this equation are improved by the individual folding of $\sigma_{\text{diff}}{}^p$ and $\sigma_{\text{diff}}{}^n$ before computing the double-scattering contribution.

The calculation of the six-dimensional resonant-diffractive and resonant-resonant scattering integrals is considerably more complicated. By use of the above approximations for the special choice of s and t given in Eq. (14), these terms can be reduced to three-dimensional numerical integrals if care is taken to handle properly the logarithm's branch cut. In this way the following expressions are obtained:

$$\sigma^{\text{ds}}(p)\,|_{\text{res-diff}}=-\frac{(\gamma/\pi)^{3/2}\sigma_{\text{diff}}{}^n(s_{\text{st}})\Gamma_p(s_{\text{st}})(J+\frac{1}{2})_p(\Gamma_{\text{el}\,0}/\Gamma_0)_p}{2k_{\text{c.m.}}{}^2}\int_0^\infty\int_0^\infty q_1dq_1q_2dq_2 \times e^{-\gamma q_1^2/2}e^{-\gamma q_2^2/2}[\alpha_n{}^{\text{diff}}\,\text{Im}F(q_1,q_2)+\text{Re}F(q_1,q_2)]+p\leftrightarrow n\,, \tag{A7}$$

$$\text{Re}F(q_1,q_2)=\left(\frac{1}{2}\int_{-q_1}^{q_1}\left\{\frac{dy}{D_1{}^2+\frac{1}{4}\Gamma_p{}^2}\left[D_1\ln\left(\frac{D_4{}^2+\frac{1}{4}\Gamma_p{}^2}{D_5{}^2+\frac{1}{4}\Gamma_p{}^2}\right)\Gamma_p\tan^{-1}\left(\frac{\frac{1}{2}\Gamma_p}{D_4}\right)-\Gamma_p\tan^{-1}\left(\frac{\frac{1}{2}\Gamma_p}{D_5}\right)\right]\right\} -\frac{\pi\sqrt{s_{\text{st}}}}{p}\left[\tan^{-1}\left(\frac{\frac{1}{2}\Gamma_p}{D_2}\right)-\tan^{-1}\left(\frac{\frac{1}{2}\Gamma_p}{D_3}\right)\right]\right)e^{-\beta^n q_2^2}, \tag{A8}$$

$$\text{Im}F(q_1,q_2)=\left[\int_{-q_1}^{q_1}\left(\frac{dy}{D_1{}^2+\frac{1}{4}\Gamma_p{}^2}\left\{D_1\left[\tan^{-1}\left(\frac{\frac{1}{2}\Gamma_p}{D_4}\right)-\tan^{-1}\left(\frac{\frac{1}{2}\Gamma_p}{D_5}\right)\right]-\tfrac{1}{4}\Gamma_p\ln\frac{D_4{}^2+\frac{1}{4}\Gamma_p{}^2}{D_5{}^2+\frac{1}{4}\Gamma_p{}^2}\right\}\right) -\frac{\pi\sqrt{s_{\text{st}}}}{2p}\ln\left(\frac{D_3{}^2+\frac{1}{4}\Gamma_p{}^2}{D_2{}^2+\frac{1}{4}\Gamma_p{}^2}\right)\right]e^{-\beta^n q_2^2}\quad(-\pi<\tan^{-1}x<\pi)\,, \tag{A9}$$

$$D_1=\sqrt{s_{\rm st}}-E_{\rm res}{}^p+py/(\sqrt{2s_{\rm st}})\,, \tag{A10}$$

$$\begin{Bmatrix}D_2\\D_3\end{Bmatrix}=\sqrt{s_{\rm st}}-E_{\rm res}{}^p\pm pq_1/(\sqrt{2s_{\rm st}})\,, \tag{A11}$$

$$\begin{Bmatrix}D_4\\D_5\end{Bmatrix}=D_1\pm pq_2/(\sqrt{2s_{\rm st}})\,, \tag{A12}$$

$$\sigma^{\rm ds}(p)\,|_{\text{res-res}}=-2\pi\frac{(\gamma/\pi)^{3/2}}{k_{\rm c.m.}{}^4}\Gamma_p(s_{\rm st})\Gamma_n(s_{\rm st})(J+\tfrac{1}{2})_p\left(\frac{\Gamma_{\rm el\,0}}{\Gamma_0}\right)_p(J+\tfrac{1}{2})_n\left(\frac{\Gamma_{\rm el\,0}}{\Gamma_0}\right)_n \times\int_0^\infty\int_0^\infty q_1dq_1q_2dq_2e^{-\frac{1}{2}\gamma(q_1^2+q_2^2)}\,{\rm Im}J(q_1,q_2)\,. \tag{A13}$$

For $q_1>q_2$,

$${\rm Im}J(q_1,q_2)={\rm Im}T+\int_{q_1+q_2}^{q_1-q_2}dy\left\{\tfrac{1}{2}\ln|y|\left[\frac{D_{n2}\Gamma_p+D_{p1}\Gamma_n}{(D_{n2}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p1}{}^2+\frac{1}{4}\Gamma_p{}^2)}-\frac{D_{n1}\Gamma_p+D_{p2}\Gamma_n}{(D_{n1}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p2}{}^2+\frac{1}{4}\Gamma_p{}^2)}\right]\right\}, \tag{A14}$$

and for $q_1\leq q_2$,

$${\rm Im}J(q_1,q_2)={\rm Im}T+\int_0^{q_1-q_2}dy\left\{\tfrac{1}{2}\ln|y|\left[\frac{D_{n2}\Gamma_p+D_{p1}\Gamma_n}{(D_{n2}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p1}{}^2+\frac{1}{4}\Gamma_p{}^2)}-\frac{D_{n1}\Gamma_p+D_{p2}\Gamma_n}{(D_{n1}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p2}{}^2+\frac{1}{4}\Gamma_p{}^2)}\right]\right\} +\int_{q_1+q_2}^0 dy\left\{\tfrac{1}{2}\ln|y|\left[\frac{D_{n2}\Gamma_p+D_{p1}\Gamma_n}{(D_{n2}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p1}{}^2+\frac{1}{4}\Gamma_p{}^2)}-\frac{D_{n1}\Gamma_p+D_{p2}\Gamma_n}{(D_{n1}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p2}{}^2+\frac{1}{4}\Gamma_p{}^2)}\right]\right\}, \tag{A15}$$

$$\begin{aligned}{\rm Im}T=&\left(\tan^{-1}\frac{\frac{1}{2}\Gamma_p}{\sqrt{S_1}-E_{\rm res}{}^p-pq_1/\sqrt{S_1}}-\tan^{-1}\frac{\frac{1}{2}\Gamma_n}{\sqrt{S_1}-E_{\rm res}{}^p+pq_1/\sqrt{S_1}}\right)\int_{q_1+q_2}^{q_1-q_2}dy\frac{D_{n2}D_{p1}-\frac{1}{4}\Gamma_p\Gamma_n}{(D_{n2}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p1}{}^2+\frac{1}{4}\Gamma_p{}^2)}\\ &-\tfrac{1}{4}\ln\left[\frac{(\sqrt{S_1}-E_{\rm res}{}^p+pq_1/\sqrt{S_1})^2+\frac{1}{4}\Gamma_p{}^2}{(\sqrt{S_1}-E_{\rm res}{}^p-pq_1/\sqrt{S_1})^2+\frac{1}{4}\Gamma_p{}^2}\right]\int_{q_1+q_2}^{q_1-q_2}dy\frac{D_{n2}\Gamma_p+D_{p1}\Gamma_n}{(D_{n2}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p1}{}^2+\frac{1}{4}\Gamma_p{}^2)}\\ &+\pi\int_{q_1+q_2}^{q_1-q_2}dy\left[\frac{D_{n1}D_{p2}-\frac{1}{4}\Gamma_p\Gamma_n}{(D_{n1}{}^2+\frac{1}{4}\Gamma_n{}^2)(D_{p2}{}^2+\frac{1}{4}\Gamma_p{}^2)}\right],\end{aligned} \tag{A16}$$

$$\begin{Bmatrix}D_{p1}\\D_{p2}\end{Bmatrix}=\sqrt{S_1}-E_{\rm res}{}^p\begin{Bmatrix}+\\-\end{Bmatrix}\frac{pq_1}{\sqrt{S_1}}\begin{Bmatrix}-\\+\end{Bmatrix}\frac{py}{\sqrt{S_1}}, \tag{A17}$$

$$\begin{Bmatrix}D_{n1}\\D_{n2}\end{Bmatrix}=\sqrt{S_2}-E_{\rm res}{}^n\begin{Bmatrix}+\\-\end{Bmatrix}\frac{pq_1}{\sqrt{S_2}}\begin{Bmatrix}-\\+\end{Bmatrix}\frac{py}{\sqrt{S_2}}, \tag{A18}$$

and

$$\begin{Bmatrix}S_1\\S_2\end{Bmatrix}=m_M{}^2+m_N{}^2+2[(m_M{}^2+p^2)(m_N{}^2+q_{1(2)}{}^2)]^{1/2}. \tag{A19}$$

APPENDIX B: MEANING OF $\langle r^{-2}\rangle_g$ IN GT

The formulation of the Glauber approximation given in Eqs. (1), (17), and (20), although widely used, is actually only a limit of the theory valid if the deuteron form factor varies much more rapidly with momentum transfer than do the nucleon scattering amplitudes. The actual meaning of $\langle r^{-2}\rangle_d$ in GT is[14]

$$\langle r^{-2}\rangle_g=\frac{1}{{\rm Im}T^p(\theta=0)\,{\rm Im}T^n(\theta=0)}\int_0^\infty S(q) \times{\rm Im}T^p(q)\,{\rm Im}T^n(q)\,qdq\,, \tag{B1}$$

where $S(q)$ is the deuteron form factor and the variation of the amplitudes' phase with momentum transfer has been ignored.

If a Gaussian deuteron wave function, $\varphi(q)\propto e^{-\gamma q^2/2}$, is assumed and if the nucleon scattering amplitudes are assumed purely diffractive, $T^{p(n)}\propto e^{-\beta^{p(n)}q^2/2}$, $\langle r^{-2}\rangle_g$ takes an especially simple and energy-independent form:

$$\langle r^{-2}\rangle_g=2/[\gamma+2(\beta^p+\beta^n)]. \tag{B2}$$

In general, only small values of q are important in Eq. (B1) and $\langle r^{-2}\rangle_g$ is expected to be independent of energy for most forms of the amplitude, although, as found empirically,[2,4,5,12] dependent upon the particular particle which is being scattered.

If the scattering amplitudes consist of the combination of diffractive and resonant parts given by Eqs. (4) and (5) and if the deuteron is again described by a Gaussian wave function, $\langle r^{-2}\rangle_g$ takes a more complicated and weakly energy-dependent form:

$$\langle r^{-2}\rangle_g=\left[\sigma_{\rm diff}{}^p\sigma_{\rm diff}{}^n\left(\frac{2}{\gamma+2(\beta^p+\beta^n)}\right)+\sigma_{\rm res}{}^p\sigma_{\rm res}{}^n\left(\frac{2}{\gamma}\right)+\sigma_{\rm diff}{}^p\sigma_{\rm res}{}^n\left(\frac{2}{\gamma+2\beta^p}\right)+\sigma_{\rm res}{}^p\sigma_{\rm diff}{}^n\left(\frac{2}{\gamma+2\beta^p}\right)\right]\bigg/\sigma_p{}^{\rm tot}\sigma_n{}^{\rm tot}. \quad \text{(B3)}$$

Equation (B3) exhibits a quite natural weighting of the different possible limits of $\langle r^{-2}\rangle_g$.

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Double-Peripheral-Model Analysis of the Reaction* $K^+p\rightarrow K^+\pi^-\Delta^{++}{}_{1236}$ at 9 GeV/c

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Using a double-Regge-pole-exchange model, we study the low-$\Delta^{++}\pi^-$-mass enhancement in the reaction $K^+p\rightarrow K^+\pi^-\Delta^{++}{}_{1236}$ at 9 GeV/c. We find that P and π double exchange dominate the process. In general the model agrees with the data in the region where $M(K^+\pi^-)\geqslant 1.54$ GeV, $-t_{KK}<0.5$ (GeV/c)^2, and $-t_{p\Delta}<0.5$ (GeV/c)^2. The possibility of extending the model into the large-t region and problems involved in the extrapolation of the model to the $K\pi$ threshold are investigated. The importance of the contribution from the double-peripheral process in the low-$M(K^+\pi^-)$ region and its implications for the analysis of the $K\pi$ system are discussed.

I. INTRODUCTION

THE general features of the reaction $K^+p\rightarrow K^+\pi^-\Delta^{++}{}_{1236}$ at 9 GeV/c were discussed in an earlier paper.[1] In this paper we study the reaction in the high-$K\pi$-mass region $[M(K^+\pi^-)\geqslant 1.54$ GeV$]$ on the basis of a double-Regge-pole-exchange model. The advantage of this model is that it has the same simple form as a single-Regge-pole-exchange model and theoretically the Regge parameters (except the coupling at the internal vertex) used here can be wholly taken from those that were determined by the data from two-body or quasi-two-body final states. It is well known that a double-Regge-pole model can usually describe the data from three-body or quasi-three-body final states at high energies fairly well. However, in applying the model there are still some unsolved problems.

(a) The commonly used Regge parameters are known only in their order of magnitude. The exact values are not well determined. Hence when one finds that the fits of the model to the data are insensitive to the variation of the parameters, one cannot distinguish whether this is due to the effect of a collective change of the many Regge parameters or due to an incomplete study of the data. Poor statistics of the data and unclean samples can also contribute to the sources of uncertainty.

(b) There is no evidence for Toller-angle dependence at the internal vertex. By the same argument given in (a), it is not clear at all whether there should be a Toller-angle dependence for the Reggeon-Reggeon-particle coupling.

(c) Over how large a range in momentum transfer variables (t's) a peripheral model can extend is not well known.

(d) Granted that duality is a valid concept,[2] how would one extrapolate the model to small subinvariant energies s? Would the extrapolation also be insensitive to a variation of Regge parameters? Answers to these questions are not known either.

In an attempt to understand these problems, we analyze our data in an exhaustive manner. The method and the results of the analysis are presented in Secs. II and III. Section IV discusses the extrapolation of the model to small subinvariant energies. Section V gives our conclusions.

This experiment was carried out in the Brookhaven National Laboratory 80-in. hydrogen bubble chamber,

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