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Self-Interaction Corrections in a Nonrelativistic Stochastic Theory of Quantum Mechanics

L. DE LA PEÑA-AUERBACH*
Instituto de Física, Universidad Nacional de México, México, D. F.

AND

ANA M. CETTO†
Facultad de Ciencias, Universidad Nacional de México, México, D. F.

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The radiation damping terms of classical electrodynamics are introduced into a set of stochastic equations of nonrelativistic quantum mechanics to study their effects on the energy levels of a stationary system to first order in perturbation theory. In the potential case there appear correction terms to Schrödinger's equation, one of which is calculated by solving a subsidiary equation. In particular, we obtain for the Lamb shift an expression of essentially stochastic origin, whose structure is in complete agreement with known results. According to this calculation, which has the advantage of being devoid of divergence problems, the nonrelativistic spinless contribution to the Lamb shift of the $2s$ level amounts to 60% of the total effect.

I. INTRODUCTION

DURING the last few years, different authors have been carrying out independent work with the purpose of establishing a stochastic interpretation of quantum mechanics. The common premise underlying these kinds of theories is that quantum phenomena are a result of the violent stochastic interaction between the system and the rest of the universe, i.e., the surrounding medium. This point of view has led successfully to Schrödinger quantum mechanics, as can be seen from a significant number of papers, among which we shall mention only some which are of direct interest for the present treatment, namely, those by Kershaw,[1] Nelson,[2] Santos,[3] and de la Peña.[4] Other quantum-mechanical questions have been recently approached from the stochastic standpoint, including the several-body problem and its inherent interference aspect,[5] the relativistic extension,[6,7] the treatment of particles with spin,[8,9] etc.

The results obtained up to now are encouraging and seem to indicate the convenience of exploring more complex situations in order to arrive at a final conclusion about the theory's soundness and potentialities. The present paper is oriented precisely in this direction, its purpose being to introduce the radiation damping due to the acceleration of the electron into the stochastic formulation of quantum mechanics. The origin of the radiation damping may be traced to the self-interaction of the electron, as is evident for example from a series expansion of the retarded potentials in classical electrodynamics.[10] (In this context, we recall that these radiative corrections were first obtained by Lorentz in studying the motion of an extended self-interacting particle.) This allows us to consider a first-order treat-

* Technical consultant, Comisión Nacional de Energía Nuclear, México.

† Fellow of the Ford Foundation, Facultad de Ciencias. Present address: Instituto de Física, Universidad Nacional de México, México, D. F.

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[9] T. G. Dankel, Ph.D. thesis, Princeton, 1969 (unpublished). We wish to thank T. H. Boyer and L. S. Schulman for bringing this work to our attention.

[10] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1951), Sec. 9-9.

ment of the radiation damping terms of classical electrodynamics as equivalent to a first-order calculation of the vertex corrections of quantum electrodynamics in the nonrelativistic spinless approximation. We therefore expect that the introduction of the self-interaction force into the set of stochastic equations of motion allows us to calculate the corresponding correction to the energy levels of the system under consideration. In particular, we expect to obtain in this form the nonrelativistic spinless contribution to the Lamb shift in the hydrogenlike atom.

In introducing the radiation damping, one must keep the stochastic nature of the particle's motion in mind. As has been demonstrated elsewhere,[11] the electromagnetic force acting on a stochastic particle is not given entirely by the Lorentz force, but contains additional terms of stochastic origin. Precisely one of these terms (when referred to the electron's self-field), turns out to be responsible for the major contribution to the Lamb shift of the s levels, as shown in Secs. II and III. In this context, it is interesting to note that a similar attempt[12] to calculate the self-interaction effect on the energy levels failed, essentially due to the omission of the terms giving rise to the main contribution. Further details on this subject are given in Sec. II.

We wish to remark that the procedure followed in the present paper is characterized by its mathematical and conceptual simplicity, but is at the same time naturally limited to a first-order calculation since we consider only the first relevant terms in a series expansion of the electron's retarded potentials. Such a perturbative approach guarantees, in its turn, the legitimate use of the radiation damping terms, since any runaway solution is *a priori* excluded by imposing a general behavior of the perturbed quantities characterized by their unperturbed values, as is shown in Sec. III.

II. INTRODUCTION OF RADIATION DAMPING FORCE

We refer throughout this paper to the stochastic formulation of quantum mechanics presented in Refs. 4, 8, and 11, since, as has been formerly demonstrated,[4,13] it covers the relevant aspects of similar theories such as those represented by Refs. 1–3. According to this formulation, the motion of a stochastic particle is described by the set of equations

$$\mathfrak{D}_c\mathbf{v}-\mathfrak{D}_s\mathbf{u}=\mathbf{f}^{(+)}/m, \tag{1a}$$

$$\mathfrak{D}_s\mathbf{v}+\mathfrak{D}_c\mathbf{u}=\mathbf{f}^{(-)}/m. \tag{1b}$$

Here, $\mathbf{v}$ and $\mathbf{u}$ represent the systematic and stochastic components, respectively, of the total velocity $\mathbf{c}=\mathbf{v}+\mathbf{u}$; $\mathbf{f}^{(\pm)}$ stands for the external force, the $\pm$ sign referring to the "parity" of $\mathbf{f}$ under time reversal, and $\mathfrak{D}_c$, $\mathfrak{D}_s$ are the systematic and stochastic derivative operators, given to second order, i.e., in the Markovian approximation, by

$$\mathfrak{D}_c=\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla, \tag{2a}$$

$$\mathfrak{D}_s=\mathbf{u}\cdot\nabla+D\nabla^2, \tag{2b}$$

where D is a measure of the dispersive effects of the stochastic substratum on the particle's trajectory; in the quantum-mechanical case it takes on the empirical value $\hbar/2m$.[1–4] Equations (1) can be combined and integrated to obtain Schrödinger's equation; specifically, in the potential case the integration is performed with the change of variables

$$\mathbf{v}=2D\,\mathrm{Im}\nabla\ln\psi, \tag{3a}$$

$$\mathbf{u}=2D\,\mathrm{Re}\nabla\ln\psi=D\frac{\nabla\rho}{\rho}, \tag{3b}$$

where $\rho=\psi^*\psi$.

Our present problem deals with an electron acted on by an electromagnetic field. The corresponding force terms to be introduced into Eqs. (1) have been derived in earlier papers[8,11]; however, since this is a crucial point in the present treatment, we suggest an alternative way of obtaining them in the following paragraphs.

The simplest of all methods is based on the fact that, as has been demonstrated before,[4,11] in taking the stochastic nature of the motion of the quantum-mechanical particle into consideration, the total time derivative d/dt must be replaced by the forward derivative[2,4]

$$\mathfrak{D}=\mathfrak{D}_c+\mathfrak{D}_s=\frac{\partial}{\partial t}+(\mathbf{v}+\mathbf{u})\cdot\nabla+D\nabla^2.$$

In particular, by making this substitution in the Lorentz formula

$$e\left[-\nabla\phi-\frac{1}{c}\frac{d\mathbf{A}}{dt}+\frac{1}{c}(\mathbf{v}\cdot\nabla)\mathbf{A}+\frac{1}{c}\mathbf{v}\times(\nabla\times\mathbf{A})\right],$$

where $\mathbf{v}$ is the total velocity (and hence, transforms into $\mathbf{c}=\mathbf{v}+\mathbf{u}$), we obtain for its stochastic generalization

$$\mathbf{f}=e\left[\mathbf{E}+\frac{1}{c}(\mathbf{v}+\mathbf{u})\times\mathbf{H}-\frac{D}{c}\nabla^2\mathbf{A}\right]. \tag{4}$$

This result agrees exactly with the expression obtained from first principles in Ref. 11, and proves to be correct insofar as it leads to Schrödinger's equation with minimal electromagnetic coupling, when substituted in Eqs. (1).

In this paper we wish to include in Eq. (4) the terms arising from the electron's self-interaction; in order to

[11] L. de la Peña-Auerbach and A. M. Cetto, Phys. Letters 29A, 562 (1969); Rev. Mex. Fís. 18, 253 (1969).
[12] J. E. Krizan, Phys. Rev. 165, 1725 (1968).
[13] L. de la Peña-Auerbach and R. M. Velasco, Rev. Mex. Fís. 18, 397 (1969); L. de la Peña-Auerbach, R. M. Velasco, and A. M. Cetto, Rev. Mex. Fís. 19, 193 (1970).

find the right expression for them, we use as a starting point the classical expressions[14]

$$\mathbf{E}_r=\frac{2e}{3c^3}\dot{\mathbf{w}},\quad \mathbf{A}_r=-\frac{e}{c^2}\mathbf{w},$$

where $\mathbf{w}$ is the acceleration. Once more writing $\mathfrak{D}$ instead of d/dt, we get

$$\mathbf{E}_r=-\tau\mathfrak{D}\nabla\phi, \tag{5a}$$

$$\mathbf{A}_r=\frac{e^2}{mc^2}\nabla\phi, \tag{5b}$$

with $\tau=2e^2/3mc^3$, since the acceleration is determined to first order by the external potential through $\mathbf{f}_0=-e\nabla\phi$. Since all the calculations are to be carried out to first order in perturbation theory and, hence, to first order in τ, we may use in Eqs. (5) the unperturbed values of $\mathbf{v}$ and $\mathbf{u}$; moreover, in the particular case dealt with in the present treatment, namely, the hydrogenlike atom, it is evident from Eqs. (3) that $\mathbf{u}$ is time independent and we may take $\mathbf{v}=0$ without loss of generality due to the spherical symmetry of the problem. According to Eq. (2a), we may therefore write 0 instead of $\mathfrak{D}_c$ in Eq. (5a), thus obtaining from Eqs. (4) and (5)

$$\mathbf{f}=\mathbf{f}_0+\tau\mathfrak{D}_s\mathbf{f}_0+(De^2/mc^3)\nabla^2\mathbf{f}_0. \tag{6}$$

Making use of Eqs. (1) and (6), with $\mathbf{f}^{(-)}=0$, we write the equations of motion for our problem in the form

$$m(\mathfrak{D}_c\mathbf{v}-\mathfrak{D}_s\mathbf{u})=[1+\tau\mathfrak{D}_s+(De^2/mc^3)\nabla^2]\mathbf{f}_0, \tag{7a}$$

$$\mathfrak{D}_s\mathbf{v}+\mathfrak{D}_c\mathbf{u}=0, \tag{7b}$$

and combine them to get the equation

$$m\mathfrak{D}_q\mathbf{v}_q=\mathbf{f} \tag{8}$$

and its complex conjugate, where

$$\mathfrak{D}_q=\mathfrak{D}_c-i\mathfrak{D}_s, \tag{9a}$$

$$\mathbf{v}_q=\mathbf{v}-i\mathbf{u}. \tag{9b}$$

Equation (8) may be readily integrated upon the change of variable[4,11]

$$\mathbf{v}_q=2D\nabla w-(i\tau/m)\mathbf{B}. \tag{10}$$

Indeed, in terms of the new variables w and $\mathbf{B}$ we obtain to first order in τ

$$\partial_k\left[\hbar\mathfrak{D}_q w-\hbar D(\nabla w)^2+V+\tau\mathfrak{D}_s V+\frac{De^2}{mc^3}\nabla^2 V\right]$$
$$=\tau\sum_j(-f_{0j}\partial_k u_j-u_j\partial_k B_j+\mathfrak{D}_s B_k), \tag{11}$$

where $\mathbf{f}_0=-\nabla V$. In view of this result, we propose to select $\mathbf{B}$ such that the right-hand side of Eq. (11) cancels out, i.e.,

$$\mathfrak{D}_s B_k=\sum_j(u_j\partial_k B_j+f_{0j}\partial_k u_j). \tag{12}$$

Now Eq. (11) can be immediately integrated to yield Schrödinger's equation, upon the further change of variable $w=-i\ln\psi$:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi$$
$$+\left(V+\tau\mathfrak{D}_s V+\frac{De^2}{mc^3}\nabla^2 V-2D\tau\mathbf{B}\cdot\nabla\right)\psi. \tag{13}$$

Note that there are three first-order contributions to the energy shift, namely,

$$\delta E_{nl}=\tau\langle\mathfrak{D}_s V\rangle+\tfrac{3}{2}D\tau\langle\nabla^2 V\rangle-2D\tau\langle\mathbf{B}\cdot\nabla\rangle. \tag{14}$$

It can be readily seen from an integration by parts, recalling the definition of $\mathfrak{D}_s$, that the first term in Eq. (14) vanishes—this is just the correction predicted in Ref. 12. We are therefore left with two corrections to the energy, namely,

$$\delta E_{nl}^{(1)}=\tfrac{3}{2}D\tau\langle\nabla^2 V\rangle=\epsilon_n K_1(n,l) \tag{15a}$$

and

$$\delta E_{nl}^{(2)}=-2D\tau\langle\mathbf{B}\cdot\nabla\rangle=\epsilon_n K_2(n,l), \tag{15b}$$

where we have introduced

$$\epsilon_n=\frac{8}{3\pi}\frac{\alpha^3 Z^4 R\hbar}{n^3}=\hbar L, \tag{16}$$

where R is the Rydberg constant, L is the Lamb constant,[15] and the K's are two dimensionless parameters. The correction $\delta E^{(1)}$ owes its presence to the last term in Eq. (4), while $\delta E^{(2)}$ is due to the perturbation of the stochastic velocity $\mathbf{u}$, $\delta\mathbf{u}=\tau m^{-1}\mathbf{B}$, as is easily seen from Eqs. (9b) and (10). Krizan does not obtain the first term at all, while he effectively reduces the second one to zero by neglecting the inhomogeneous term in Eq. (12), which amounts to taking $\mathbf{B}=0$.

The calculation of K_1 in Eq. (15a) is immediate, its value being

$$K_1(n,l)=\tfrac{3}{2}\pi\delta_{l0}. \tag{17}$$

On the other hand, the calculation of K_2 is less direct since it involves the solution of Eq. (12). Section III is devoted to the construction of the solutions to this subsidiary equation; we consider it convenient, however, to anticipate some of the main results in order to proceed to the general discussion.

Although K_2 is found to be in general different from zero for $n>1$ and any l, its numerical value is small

[14] D. Ivanenko and A. Sokolov, *Klassische Feldtheorie* (Akademie-Verlag, Berlin, 1953). This book includes a particularly lucid exposition of many topics related to radiation damping.

[15] See, e.g., H. A. Bethe and E. Salpeter, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 34, Chap. 1.

compared to $\frac{3}{2}\pi$, which is the value of K_1 for s states. Taking the smallness of $\delta E_{nl}^{(2)}$ into consideration, we may state that the main result of the present treatment is a shift of the levels of the hydrogenlike atom, given essentially by $\delta E_{nl}^{(1)}$. Hence, in this nonrelativistic spinless theory, the Lamb shift is a direct consequence of the additional force term $-(eD/c)\nabla^2\mathbf{A}$ in Eq. (4), and the remaining contribution $\delta E_{nl}^{(2)}$ is of secondary importance.

At this point, we wish to compare the predictions of our theory with the quantum electrodynamical results. As is well known,[15] the Lamb splitting between s and p levels is given by $K(n,0)-K(n,1)\approx 7.75$ in units of ϵ_n; therefore, our result corresponds to about 60% of the total effect, certainly a reasonable one considering that we are working in a nonrelativistic spinless approximation.

III. SOLUTION OF SUBSIDIARY EQUATION

Equation (12) may be written in our working approximation in the following more convenient form:

$$D\nabla^2\mathbf{B}=(\mathbf{f}_0\cdot\nabla)\mathbf{u}+\mathbf{u}\times(\nabla\times\mathbf{B}). \tag{18}$$

To solve this equation in the general case is a rather complex task. Quite fortunately in the case of greatest interest to us, namely, the s states, the solution turns out to be almost immediate: The fact that $\mathbf{u}$ is radial and depends only on r allows us to take $\nabla\times\mathbf{B}=0$ and write, instead of Eq. (18),

$$D\nabla^2\mathbf{B}=(\mathbf{f}_0\cdot\nabla)\mathbf{u}=-\frac{Ze^2}{r^2}\frac{\partial\mathbf{u}}{\partial r}, \tag{19}$$

where the last equality applies specifically to s states in a Coulomb field. Hence, imposing on $\mathbf{B}$ the usual homogeneous boundary conditions, and writing $\mathbf{u}=u(r)\mathbf{a}_r$, we get

$$\mathbf{B}=\frac{Ze^2}{4\pi D}\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{d\mathbf{u}(r')}{dr'}dr'd\Omega'$$

$$=\frac{Ze^2\mathbf{a}_r}{3D}\left[-\frac{1}{r^2}\int_0^r u(x)dx+2r\int_r^\infty\frac{u(x)}{x^3}dx\right], \tag{20}$$

where the last equality results from an integration by parts with respect to r' and u. The introduction of $u=2D\,\mathrm{Re}\nabla\ln\psi$, where ψ is the unperturbed hydrogenlike amplitude, yields for ns states,

$$\mathbf{B}=\tfrac{2}{3}Ze^2\beta^2\mathbf{a}_r\left[-\frac{1}{\rho^2}\ln\frac{L_{n-1}^1(\rho)}{L_{n-1}^1(0)}+2\rho I_{n0}\right], \tag{21}$$

where $\rho=\beta r$, $\beta=2Z/na_0$, a_0 is the Bohr radius, and I_{n0} is an integral involving associated Laguerre polynomials

$$I_{n0}=\int_\rho^\infty\frac{dL_{n-1}^1(x)}{dx}\frac{dx}{x^3L_{n-1}^1(x)}. \tag{22}$$

An integration by parts shows that $-2\langle\mathbf{B}\cdot\nabla\rangle=\langle\nabla\cdot\mathbf{B}\rangle$ and hence, according to Eqs. (15b) and (16), the dimensionless parameter $K_2(n,0)$ can be calculated from the divergence of $\mathbf{B}$ in Eq. (21) to yield

$$K_2(n,0)=4\pi\left\langle-\frac{1}{\rho^2L_{n-1}^1}\frac{dL_{n-1}^1}{d\rho}+2I_{n0}(\rho)\right\rangle. \tag{23}$$

In particular, for $n=1$ this correction is identically zero, and for $n=2$ we have

$$K_2(2,0)=-\pi\left\langle\frac{1}{2-\rho}+\frac{3}{\rho}+\frac{4}{\rho^2}+\ln\frac{|2-\rho|}{\rho}\right\rangle$$

$$=-\pi[\tfrac{7}{2}+\gamma+\ln 2-7e^{-2}\,\mathrm{Ei}(2)]\approx-0.08\pi, \tag{24}$$

where $\gamma=0.577\cdots$ is the Euler constant and $\mathrm{Ei}(x)$ is the exponential integral function of x. From this result we see that $K_2(2,0)$ amounts to only a few percent of $K_1(2,0)$, as stated above.

In the case $l\neq 0$, Eq. (18) is not as easily soluble because $\nabla\times\mathbf{B}\neq 0$. The calculation of $K_2(n,l)$ for this case is actually of little relevance since its contribution to the total energy shift is small, as can be inferred from the above result. Nevertheless, we sketch below a possible method of finding the solutions to the subsidiary equation for $l\neq 0$. Perhaps the main advantage of obtaining an explicit expression for $\mathbf{B}$ in this case consists of the fact that it allows us to prove the possibility of arriving at the final result (the relative shift of the s and p levels, for example) without divergence problems. We have seen that $\tau m^{-1}\mathbf{B}$ must be considered a first-order correction to $\mathbf{u}$ due to radiation damping; therefore, we must demand the general behavior of $\mathbf{B}$ to be in accordance with that of $\mathbf{u}$. Since to the particular solution to the subsidiary equation we may always add an appropriate solution to the corresponding homogeneous equation, we use the latter one to guarantee the correct behavior of $\delta\mathbf{u}=\tau m^{-1}\mathbf{B}$, as stated in the Introduction. This criterion is explicitly used in the following calculation.

In solving Eq. (18) for $l\neq 0$, we first recall that the energy shift cannot depend on the magnetic quantum number m since we are working with a central force. We may therefore take $m=0$ without loss of generality. Further, we write the solution to Eq. (18) in the form

$$\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1+\mathbf{B}_2, \tag{25}$$

where $\mathbf{B}_0$ is a function—to be determined later—satisfying the homogeneous equation corresponding to Eq. (18), namely,

$$D\nabla^2\mathbf{B}_0-\mathbf{u}\times(\nabla\times\mathbf{B}_0)=0. \tag{26}$$

Defining $\mathbf{u}_1=u_r\mathbf{a}_r$ and $\mathbf{u}_2=u_\theta\mathbf{a}_\theta$, we select $\mathbf{B}_1$ such that $\nabla\times\mathbf{B}_1=0$ and

$$D\nabla^2\mathbf{B}_1=(\mathbf{f}_0\cdot\nabla)\mathbf{u}_1. \tag{27a}$$

Consequently, the equation for $\mathbf{B}_2$ is

$$D\nabla^2\mathbf{B}_2-\mathbf{u}\times(\nabla\times\mathbf{B}_2)=(\mathbf{f}_0\cdot\nabla)\mathbf{u}_2. \tag{27b}$$

Here,

$$u_r=2D\beta\left(-\frac{1}{2}+\frac{l}{\rho}+\frac{1}{L_{n-l-1}{}^{2l+1}}\frac{dL_{n-l-1}{}^{2l+1}}{d\rho}\right)$$

$$\equiv 2D\beta F_{nl}(\rho) \tag{28a}$$

and

$$u_\theta=-\frac{2D\beta}{\rho}\sin\theta\frac{P_l'(x)}{P_l(x)},\quad x=\cos\theta. \tag{28b}$$

A procedure similar to that followed in the case $l=0$ leads to the following result for $\mathbf{B}_1$:

$$\mathbf{B}_1=-\frac{1}{4\pi D}\int\frac{(\mathbf{f}_0\cdot\nabla)\mathbf{u}_1}{|\mathbf{r}-\mathbf{r}'|}r'^2dr'd\Omega'$$

$$=\frac{Ze^2\beta\mathbf{a}_r}{3D}\left[-\frac{1}{\rho^2}\int_0^\rho u_r(x)dx+2\rho\int_\rho^\infty\frac{u_r(x)dx}{x^3}\right].$$

In performing the integration there appears the term $2D\beta l\rho^{-2}\mathbf{a}_r\ln x$ calculated for $x=0$; we therefore select $\mathbf{B}_0$ such as to cancel out this infinite term, thus assuring that $\delta\mathbf{u}$ is small compared with $\mathbf{u}$. In this form we obtain

$$\mathbf{B}_0+\mathbf{B}_1=\tfrac{2}{3}Ze^2\beta^2\mathbf{a}_r$$

$$\times\left[\frac{l}{\rho^2}(\tfrac{2}{3}-\ln\rho)-\frac{1}{\rho^2}\ln\frac{L_{n-l-1}{}^{2l+1}(\rho)}{L_{n-l-1}{}^{2l+1}(0)}+2\rho I_{nl}(\rho)\right], \tag{29a}$$

where

$$I_{nl}(\rho)=\int_\rho^\infty\frac{dL_{n-l-1}{}^{2l+1}}{dx}\frac{dx}{x^3L_{n-l-1}{}^{2l+1}}. \tag{29b}$$

The energy shift associated with this part of $\mathbf{B}$ may be written in units of ϵ_n as

$$K_2'(n,l)=4\pi\left\langle 2I_{nl}(\rho)\right.$$

$$\left.-\frac{1}{\rho^2L_{n-l-1}{}^{2l+1}}\frac{dL_{n-l-1}{}^{2l+1}}{d\rho}-\frac{l}{3\rho^3}\right\rangle, \tag{30}$$

where Eq. (23) is a particular case of this result. When $n=l+1$, Eq. (30) attains the simple form

$$K_2'(l+1,l)=-\frac{4\pi}{3}l\left\langle\frac{1}{\rho^3}\right\rangle=-\frac{\pi}{3(l+1)(2l+1)} \tag{31a}$$

and, hence, we have for the $2p$ state

$$K_2'(2,1)=-\pi/18. \tag{31b}$$

Equation (27b) may be solved as follows. According to its structure, let us propose a series development of the components of $\mathbf{B}_2$ in terms of Legendre polynomials

$$B_{2r}=\sum_{t\text{ even}}f_t(\rho)P_t(x), \tag{32a}$$

$$B_{2\theta}=\sin\theta\sum_{s\text{ odd}}g_s(\rho)P_s(x), \tag{32b}$$

where the f's and g's are to be determined. Using well-known properties of the Legendre polynomials, we obtain

$$\nabla\cdot\mathbf{B}_2=\frac{1}{\rho^2}\sum_t\left(\frac{\partial}{\partial\rho}\rho^2f_t\right)P_t$$

$$+\frac{1}{\rho}\sum_s g_s\left[\frac{(s+1)(s+2)}{2s+1}P_{s+1}-\frac{s(s-1)}{2s+1}P_{s-1}\right]. \tag{33}$$

Since, as stated above, the energy correction due to $\mathbf{B}_2$ is proportional to $\langle\nabla\cdot\mathbf{B}_2\rangle$, it follows from Eq. (33) that the only terms which may contribute to it are those for which $t\leqslant 2l$ and $s\leqslant 2l+1$.

Substitution of Eqs. (32) into Eq. (27b) separated into its r and θ components gives rise to a system of equations from which the functions f_t and g_s can be determined by demanding that the coefficient of every Legendre polynomial vanish. In the general case, it is easier to multiply each of both equations by $P_q(x)$, with q an arbitrary integer, and integrate over x, thus eliminating the angular variables and obtaining an infinite set of coupled equations, some of which are to be solved to find the f's and g's of interest. For arbitrary l, the resulting equations are so involved that we consider it worthless to write them down here; we shall henceforth limit the discussion to p states. In this case, the only relevant equations are those which contain f_t with $t=0$, 2 and g_s with $s=1$, 3, namely,

$$\rho^2f_2''+2\rho f_2'-2f_2+\tfrac{5}{2}(\rho^2f_0''+2\rho f_0'-2f_0)$$

$$=-6(\rho h_1'-h_1), \tag{34a}$$

$$\rho^2h_1''+2\rho(1+\rho F_{n1})h_1'-2(3-\rho F_{n1})h_1$$

$$=-\frac{10Ze^2\beta^2}{3\rho^2}+2(1-\rho F_{n1})f_2, \tag{34b}$$

where

$$h_j=\frac{1}{2j+1}g_j-\frac{1}{2j+5}g_{j+2}. \tag{34c}$$

The system of equations (34) is satisfied by

$$f_0=(a_0-\tfrac{4}{3}Ze^2\beta^2\ln\rho)\rho^{-2},$$
$$f_2=h_1=(5/9)Ze^2\beta^2\rho^{-2}, \tag{35}$$

with arbitrary a_0. From this result it follows that

$$\langle\nabla\cdot\mathbf{B}_2\rangle=-\tfrac{4}{3}Ze^2\beta^2\langle 1/\rho^3\rangle\langle P_0-\tfrac{5}{2}P_2\rangle=0,$$

since for p states, $\langle P_2\rangle=\frac{2}{5}\langle P_0\rangle$. Hence, for $l=1$ the total energy shift is given by Eq. (30). For the $2p$ level, in particular, the theory predicts a negative shift as given by Eq. (31b), which is in agreement with known results.[15]

IV. CONCLUSIONS

The stochastic theory of quantum mechanics has allowed us to calculate readily the nonrelativistic spinless part of the self-interaction effect in the hydrogenlike atom. According to this theory, we may consider the Lamb shift essentially as an observable effect of the stochastic electromagnetic force $-(eD/c)\nabla^2\mathbf{A}$, which has no analog in Newtonian mechanics. Clearly, from the very outset we cannot expect the numerical results to be accurate since, as is well known, relativistic and spin effects play an important role in this problem.

A remarkable feature of the calculation is the absence of divergence and renormalization difficulties. This result is in agreement with the modern formulation of the Lorentz-Dirac theory of classical electrodynamics,[16] according to which the electromagnetic mass term is zero (in other words, one might say that classical Dirac electrodynamics is a renormalized theory). On the other hand, the Lorentz-Dirac theory endows the particle with an effective radius, which manifests itself through the well-known "preacceleration" effects.[14]

Concerning this last point of view, we wish to add some speculative comments. As stated above, the stochastic process which may be associated to the quantum mechanics of point particles is Markovian. In the case of extended particles, however, the nonlocality of their interactions induces a non-Markovian behavior, a feature that may be taken into account by adding higher-order terms—with as yet unknown coefficients—to the stochastic equations. This procedure represents perhaps the simplest possible way of introducing self-interaction effects phenomenologically, by adscribing an effective radius to the particle, as conjectured in an earlier paper.[4] The above argument may be reinforced by recalling the well-known heuristic picture proposed by Welton,[17] according to which the Lamb shift is related to the effective radius of the electron due to its jiggling under the action of the electromagnetic vacuum fluctuations.

[16] F. Röhrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), Chap. 6.

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Near-Field Approximation for Strong Gravitational Fields*

THOMAS A. MORGAN†
California Institute of Technology, Pasadena, California 91109
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The essential features of the near-field approximation for strong gravitational fields are elucidated by developing the approximation (i) in first order for quasistatic systems; (ii) to arbitrary order for nonrotating systems with axial symmetry; and (iii) by sketching the approximation for rotating, axially symmetric systems. The restrictions, placed by Einstein's equations, on the time dependence of the "multipole moments" of a system which is isolated from other bodies of empty space are exhibited. These restrictions are statements of the global conservation of energy and linear momentum. In Newtonian theory they would state that, for an isolated system, $(d/dt)A_0=0$ and $(d^2/dt^2)A_1=0$ where A_0, A_1 are multipole moments. The principal assumption made in this paper is simply that a near-field zone exists for the systems which we consider (i.e., $L/\lambda<1$). We do not, in any sense, assume that the gravitational fields are weak. The contracted Bianchi identity $G_0{}^{\nu}{}_{;\nu}\equiv 0$ plays a crucial role in the analysis since it implies, for a quasistationary system, that if the empty-space field equations $G_{\mu\nu}=0$ are obeyed in order n, then $[(-g)^{1/2}G_0{}^{i}]_{,i}=0$ is obeyed in order $n+1$. This in turn implies the existence of a vanishing surface integral which restricts the time dependence of the quasistationary field in each order. It is shown that there are close similarities between strong gravitational fields and electromagnetic fields in the near-field approximation. For example, just as the first effect of a quasistatic electromagnetic field is to induce a magnetic field, so the first effect of a quasistatic gravitational field is to induce a magneticlike field, whose potentials are g_{0i}.

I. INTRODUCTION

RECENTLY, Morgan and Bondi[1] have treated the near-field transfer of energy by gravitational fields without any assumptions that the fields are weak. They posed the question of how a time sequence of static configurations which can be continuously deformed into each other differs from an arbitrary sequence of static

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† On leave from the University of Nebraska, Lincoln, Neb. 68508.

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