Quark-Model Predictions for Form Factors in Inelastic Electron Scattering

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The three form factors associated with inelastic electron scattering are discussed in the framework of the symmetric quark model. Predictions are presented for each of ten nucleon resonances; a comparison with the available coincidence data is made.

WITH the advent of coincidence experiments in inelastic electron scattering,[1] one can obtain more experimental information on nucleon resonances than was possible when only the final electron was detected. The cross section[2,3] now involves the magnitudes and relative phases of the three inelastic form factors f_c, f_+, and f_-, rather than just $|f_c|^2$ and $(|f_+|^2+|f_-|^2)$.[4] Thus theoretical predictions for each of f_c, f_+, and f_- separately are desirable.

In previous publications[5–7] we have presented results for $|f_c|^2$ and $(|f_+|^2+|f_-|^2)$ for each of ten different nucleon resonances N^* in the reaction $e+p\to e+N^*$, using the symmetric quark model. In this note we shall present the predictions for f_c, f_+, and f_- separately; these results are presented here for the first time and can in general not be deduced from a knowledge of only $|f_c|^2$ and $(|f_+|^2+|f_-|^2)$. For notation and a discussion of the model used, the reader is referred to Refs. 5 and 6.

Using a different model, predictions for $|f_c|^2$ and $(|f_+|^2+|f_-|^2)$ have been obtained by Walecka *et al.*[2]; for a general discussion of electroproduction with implications for coincidence experiments, and a presentation of f_c, f_+, and f_- in their model, the reader is referred to Ref. 1.

We shall now derive the expressions for f_c, f_+, and f_- in terms of multipole operators. These form factors are defined by[4]

$$f_c=\left(\frac{m}{8\pi M}\right)^{1/2}\left(\frac{EE'\Omega^2}{mM}\right)^{1/2}\langle\pi_R J_f\|J_0(0)\|q^*\pi J_f\rangle\,,$$

$$f_\rho=\left(\frac{m}{8\pi M}\right)^{1/2}\left(\frac{EE'\Omega^2}{mM}\right)^{1/2}\sum_j\left(\frac{2j+1}{2J_f+1}\right)^{1/2}\times\langle j,\tfrac{1}{2};1,\rho|J_f,\tfrac{1}{2}+\rho\rangle\langle\pi_R J_f\|\mathbf{J}(0)\|q^*\pi J_f\rangle\quad(\rho=\pm1)\,,\qquad(1)$$

where E and E' (m and M) are the nucleon and resonance energies (masses), respectively, Ω is the volume, $J_\mu(0)$ is the current operator at the space-time point $x_\mu=0$, and π and π_R are the parities of the proton state and nucleon resonance, respectively. Expressing the nucleon state in terms of states of definite helicity and linear momentum,[4] writing the current in terms of the appropriate multipole operators,[8] and doing the resulting integration over solid angle,[9] we obtain the following results (m/M has been set equal to 1):

$$f_c=(2\pi)^{1/2}(-i)^J\langle J_f\|\hat{M}_J{}^{\text{Coul}}\|\tfrac{1}{2}\rangle\,,$$

$$f_\rho=+(2\pi)^{1/2}\sum_J(-i)^J\left(\frac{2J+1}{2J_f+1}\right)^{1/2}\langle\tfrac{1}{2},\tfrac{1}{2};J,\rho|J_f,\tfrac{1}{2}+\rho\rangle\times\langle J_f\|-\hat{T}_J{}^{\text{el}}+\rho\hat{T}_J{}^{\text{mag}}\|\tfrac{1}{2}\rangle\quad(\rho=\pm1)\,.\qquad(2)$$

Here $\hat{M}_J{}^{\text{Coul}}$, $\hat{T}_J{}^{\text{el}}$, and $\hat{T}_J{}^{\text{mag}}$ are the Coulomb, electric, and magnetic multipole operators. [In passing from Eq. (1) to Eq. (2), an extra $\pm1=\pi_R$ has been incorporated into the final-state wave function. This phase is, of course, physically unobservable.] We note that

$$|f_c|^2=2\pi|\langle J_f\|\hat{M}_J{}^{\text{Coul}}\|\tfrac{1}{2}\rangle|^2\,,$$
$$|f_+|^2+|f_-|^2=2\pi[|\langle J_f\|\hat{T}_J{}^{\text{el}}\|\tfrac{1}{2}\rangle|^2+|\langle J_f\|\hat{T}_{J'}{}^{\text{mag}}\|\tfrac{1}{2}\rangle|^2]\,,$$

as they should.[8,10]

Thus

$$f_c=A(2\pi)^{1/2}|\langle J_f\|\hat{M}_J{}^{\text{Coul}}\|\tfrac{1}{2}\rangle|\,,$$
$$f_\rho=B_\rho(2\pi)^{1/2}|\langle J_f\|\hat{T}_J{}^{\text{el}}\|\tfrac{1}{2}\rangle|+C_\rho(2\pi)^{1/2}|\langle J_f\|\hat{T}_{J'}{}^{\text{mag}}\|\tfrac{1}{2}\rangle|\,,\qquad(3)$$

where A, B_ρ, and C_ρ are constants which can easily be computed from Eq. (2) for each resonance, if the phases of the reduced multipole matrix elements are known. The symmetric quark model, of course, yields phases as well as magnitudes of reduced multipole matrix elements, and we thus obtain the constants A, B_ρ, and C_ρ for each resonance. These results are listed in Table I. The *magnitudes* of the reduced multipole matrix elements have already been tabulated for our model, and may be found in Ref. 5 for a harmonic-

[1] For a discussion of the relevance of coincidence experiments to electroproduction, together with experimental references, see P. L. Pritchett and P. A. Zucker, Phys. Rev. D **1**, 175 (1970).
[2] P. L. Pritchett, J. D. Walecka, and P. A. Zucker, Phys. Rev. **184**, 1825 (1969).
[3] Previous references on coincidence cross sections are listed in Ref. 1.
[4] J. D. Bjorken and J. D. Walecka, Ann. Phys. (N. Y.) **38**, 35 (1966).
[5] N. S. Thornber, Phys. Rev. **169**, 1096 (1968).
[6] N. S. Thornber, Phys. Rev. **173**, 1414 (1968).
[7] P. L. Pritchett, N. S. Thornber, J. D. Walecka, and P. A. Zucker, Phys. Letters **27B**, 168 (1968).
[8] T. deForest and J. D. Walecka, Advan. Phys. **15**, 1 (1966).
[9] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957).
[10] J. D. Walecka, in *International School of Physics "Enrico Fermi," Italian Physical Society Course 38*, edited by T. E. O. Ericson (Academic, New York, 1967), p. 17.

TABLE I. Values of A, B_ρ, and C_ρ in the symmetric quark model.

State	I	(J^P)	A	B_+	C_+	B_-	C_-
940	$\frac{1}{2}$	$(\frac{1}{2}^+)$	1	···	···	···	-1
1520	$\frac{1}{2}$	$(\frac{3}{2}^-)$	1	$(\frac{3}{4})^{1/2}$	$(\frac{1}{4})^{1/2}$	$(\frac{1}{4})^{1/2}$	$-(\frac{3}{4})^{1/2}$
1535	$\frac{1}{2}$	$(\frac{1}{2}^-)$	1	···	···	1	···
1670	$\frac{1}{2}$	$(\frac{5}{2}^-)$	···	···	···	···	···
1688	$\frac{1}{2}$	$(\frac{5}{2}^+)$	1	$(\frac{2}{3})^{1/2}$	$(\frac{1}{3})^{1/2}$	$(\frac{1}{3})^{1/2}$	$-(\frac{2}{3})^{1/2}$
1700	$\frac{1}{2}$	$(\frac{1}{2}^-)$	···	···	···	···	···
2190	$\frac{1}{2}$	$(\frac{7}{2}^-)$	1	$(\frac{5}{8})^{1/2}$	$(\frac{3}{8})^{1/2}$	$(\frac{3}{8})^{1/2}$	$-(\frac{5}{8})^{1/2}$
1236	$\frac{3}{2}$	$(\frac{3}{2}^+)$	···	···	$-(\frac{3}{4})^{1/2}$	···	$+(\frac{1}{4})^{1/2}$
1650	$\frac{3}{2}$	$(\frac{1}{2}^-)$	1	···	···	1	···
1950	$\frac{3}{2}$	$(\frac{7}{2}^+)$	···	···	$(\frac{5}{8})^{1/2}$	···	$-(\frac{3}{8})^{1/2}$

oscillator quark potential and in Ref. 6 for a Coulomb quark potential. To obtain f_c, f_+, and f_-, we thus simply use Eq. (3) together with the values of A, B_ρ, and C_ρ listed in Table I and the tabulated magnitudes $|\langle J_f\|\hat{M}_J{}^{\mathrm{Coul}}\|\frac{1}{2}\rangle|$, $|\langle J_f\|\hat{T}_J{}^{\mathrm{el}}\|\frac{1}{2}\rangle|$, and $|\langle J_f\|\hat{T}_{J'}{}^{\mathrm{mag}}\|\frac{1}{2}\rangle|$ listed in Refs. 5 or 6.[11]

We note that since f_c, f_+, and f_- are amplitudes, only the relative phase between any two of them is of physical significance; the absolute phase of any one form factor is not observable. We have chosen the phases of the final-state wave functions such that f_c is real and positive in the physical region for each of our resonances.

We also observe that Table I contains a number of zero entries. By definition of f_+ [Eq. (1)], f_+ must vanish (in any model) whenever $J_f=\frac{1}{2}$, as has been pointed out previously.[4] This explains some of the zeros in Table I; the other zero entries arise in a quark-model calculation of this type, irrespective of which form $V(r)$ is chosen for the quark potential function.[6,12]

Some comments on the general shape of f_c, f_+, and f_- are in order. If the quark-model potential is that of a harmonic oscillator, the reduced multipole matrix elements of f_c, f_+, and f_- all have a Gaussian-type falloff in momentum transfer. This is as expected, since the Fourier transform of a Gaussian-type harmonic-oscillator wave function is just another Gaussian.[5] Thus, for a harmonic-oscillator potential, f_c, f_+, and f_- each have a common factor $\exp[-\frac{1}{3}(q^*b)^2]$, where

$$(q^*)^2=q^2+(1/4M^2)(q^2-M^2+m^2)^2$$

is the square of the three-momentum transfer as seen in the rest frame of the nucleon isobar, and b is a constant.

If the quark-model potential is proportional to $1/r$, then the reduced multipole matrix elements, and hence f_c, f_+, and f_-, fall off as inverse powers of $(q^*)^2$. For each nucleon resonance f_c (when not identically zero) contains no zeros in our model; instead there is a steady decline in absolute magnitude from an initial value towards zero. For resonances with a vanishing electric multipole matrix element but a nonvanishing magnetic multipole matrix element [940 (f_-), 1236, 1950], f_+ and f_- have no zeros in our model, and they also decay steadily from their initial values towards zero. For resonances with an electric but no magnetic multipole matrix elements [1535 (f_-), 1650 (f_-)], there can be a zero in f_- due to a possible zero in the electric multipole matrix element; this arises from competition between the convection and magnetic-moment parts of the nucleon current. In $f_-{}^{1535}$ the contributions add, and there is no zero; in $f_-{}^{1650}$ they subtract, and there is a zero. The position of this zero depends on the quark g factor g_q and lies at

$$q^*=0.48(g_q)^{-1/2}\ \mathrm{BeV}/c\ \text{(harmonic-oscillator model)},$$

$$q^*=0.53(g_q-1)^{-1/2}\ \mathrm{BeV}/c\ (1/r\text{ potential model for } g_q\neq 1;\text{ for } g_q=1 \text{ there is no zero}).$$

TABLE II. $f_+/f_c{}^{940}$ and $f_-/f_c{}^{940}$ in the harmonic-oscillator quark model (obtained from Table I and Ref. 5). $b=4(\mathrm{BeV}/c)^{-1}$; $g_q/2M_q=\mu_p\cong-\frac{3}{2}\mu_n$; M_q is undetermined.

State	I	(J^P)	$f_+/f_c{}^{940}$	$f_-/f_c{}^{940}$
1520	$\frac{1}{2}$	$(\frac{3}{2}^-)$	$\dfrac{1}{(\sqrt{6})M_qb}$	$\dfrac{1-g_q(q^*b)^2}{3\sqrt{2}M_qb}$
1688	$\frac{1}{2}$	$(\frac{5}{2}^+)$	$\dfrac{\sqrt{2}q^*b}{3(\sqrt{5})M_qb}$	$\dfrac{q^*b[1-\frac{1}{2}g_q(q^*b)^2]}{3(\sqrt{5})M_qb}$
2190	$\frac{1}{2}$	$(\frac{7}{2}^-)$	$\dfrac{(q^*b)^2}{4(\sqrt{21})M_qb}$	$\dfrac{(q^*b)^2[1-\frac{1}{3}g_q(q^*b)^2]}{4(\sqrt{35})M_qb}$

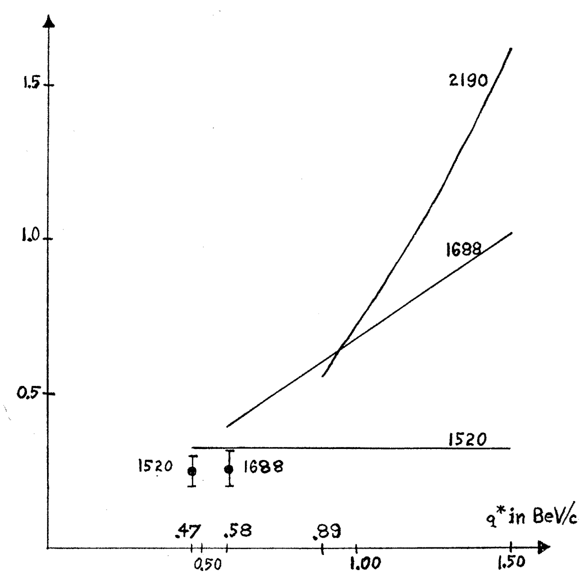

FIG. 1. $f_+/f_c{}^{940}$ in the symmetric quark model with a harmonic-oscillator potential and $M_q=\frac{1}{3}M_p$. The experimental points are taken from Ref. 1.

[11] The results for B_ρ in Table I correspond to choosing a positive phase in front of the convection part of the current in the reduced matrix element of $\hat{T}_J{}^{\mathrm{el}}$ in Refs. 5 and 6.

[12] R. G. Moorhouse, Phys. Rev. Letters 16, 772 (1966).

Thus all the form factors so far discussed have relatively simple behavior. These form factors have essentially already been plotted (in magnitude squared) in Refs. 5 and 6; the phases and normalization factors

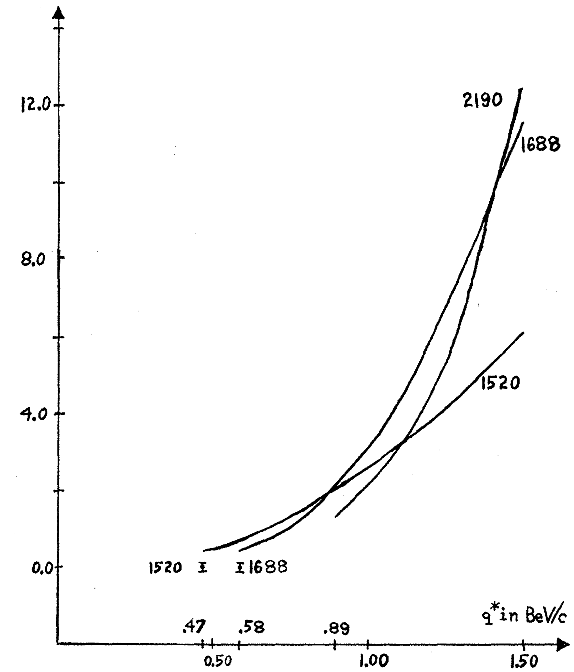

FIG. 2. Same as Fig. 1, except that the quantity plotted is now $-f_-/f_c^{940}$.

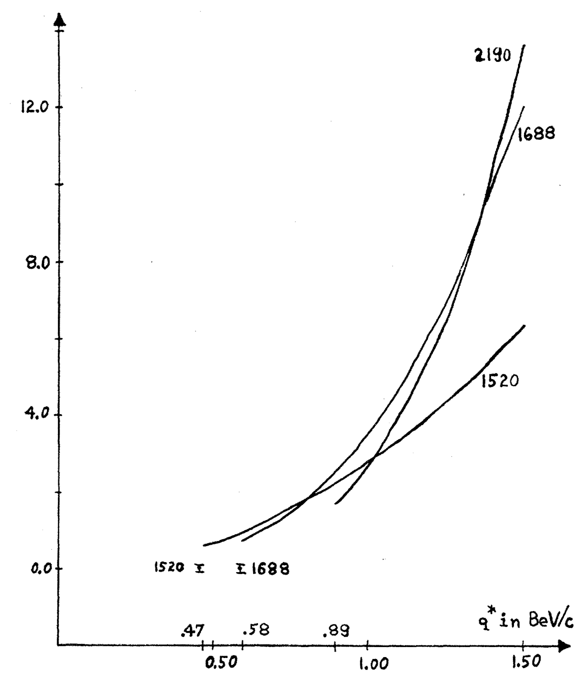

FIG. 3. Same as Fig. 2, but with $M_q=\infty$.

TABLE III. $f_\pm/f_c^{940}$ at threshold in the symmetric quark model with $V(r)\propto 1/r$.

State	f_+/f_c^{940} $M_q=\frac{1}{3}M_p$	f_+/f_c^{940} $M_q=\infty$	$-f_-/f_c^{940}$ $M_q=\frac{1}{3}M_p$	$-f_-/f_c^{940}$ $M_q=\infty$
1520	0.18	0.02	0.13	0.22
1688	0.03	0.008	0.03	0.04

TABLE IV. $f_\pm/f_c^{940}$ for the 1236 state in the symmetric quark model. h. o. stands for harmonic-oscillator potential; $1/r$ stands for a $1/r$ quark potential.

	$q^*\cong 0.26$	$q^*\cong 0.4$
f_+/f_c^{940}; h. o.	-0.45	-0.69
f_+/f_c^{940}; $1/r$	-0.45	-0.69
experiment (Ref. 1)	$\sim -0.50\pm0.06$	$\sim -0.72\pm0.1$
f_-/f_c^{940}; h. o.	0.26	0.40
f_-/f_c^{940}; $1/r$	0.26	0.40
experiment (Ref. 1)	$\sim 0.27\pm0.05$	$\sim 0.27\pm0.1$

can be obtained from Table I of the present note. The remaining transverse form factors in Table I, on the other hand, show potentially more complicated behavior. Table II shows the explicit behavior of $f_\pm/f_c^{940}$ for the 1520, 1688, and 2190 [13] states in the harmonic-oscillator quark model. f_- has one possible zero depending on the quark mass M_q; f_+ is seen to be fairly simple (no zeros). These form factors are plotted together with the experimental data in Figs. 1–3. We see that for f_+/f_c^{940} (Fig. 1) the agreement with the threshold experimental data is reasonably good; the theoretical curves can be diminished in magnitude if we increase M_q; in fact, they decrease to zero if we let $M_q\rightarrow\infty$. The agreement with experiment for f_-/f_c^{940} (Figs. 2 and 3), on the other hand, is not as good; the theoretical curves (for a harmonic-oscillator quark potential) are too large. We also note that in this model $|f_-|$ is expected to be much larger than $|f_+|$ for the 1520, 1688, and 2190 states, except at threshold. To check this prediction, additional experimental data for f_+ and f_- as functions of momentum transfer would be needed.

Table III shows the values of $f_\pm/f_c^{940}$ at threshold for the 1520 and 1688 states, using a $1/r$ potential in the symmetric quark model. $f_\pm/f_c^{940}$ (for the 1520 state) and f_-/f_c^{940} (for the 1688 state), each with $M_q=M_p/3$, are seen to be in reasonable agreement with the experimental data shown in Figs. 1–3, but f_+^{1688}/f_c^{940} is predicted too small. Here, again, more data would be useful.

Finally, there is experimental evidence on $f_\pm$ for the 1236 resonance.[1] This is compared with the theory in Table IV; the agreement is seen to be quite good.

[13] In Refs. 5 and 6 the reduced matrix element of $\hat{T}_J^{el}$ for the 2190 state is lacking a factor of $(\frac{1}{2})^{1/2}$.