Recoil Protons from Wide-Angle Bremsstrahlung

E. A. ALLTON*
Ecole Normale Supérieure, Laboratoire de l'Accélérateur Linéaire, Orsay, France
and
Daresbury Nuclear Physics Laboratory, Daresbury near Warrington, Great Britain

AND

E. F. ERICKSON†
Ecole Normale Supérieure, Laboratoire de l'Accélérateur Linéaire, Orsay, France
and
Institut de Recherches Nucléaires, Strasbourg, France

(Received 29 June 1970)

Inelastic electron-proton scattering may be studied experimentally by momentum analyzing and detecting the final proton, in which case it becomes necessary to consider the contribution made by wide-angle bremsstrahlung. In this paper we calculate exactly the Bethe-Heitler contribution to the cross section for this process, differential in proton momentum and solid angle, and integrated over the phase space available to the unobserved final electron and photon. The result may be applied to muon-proton bremsstrahlung as well. If the electron rest mass is neglected where possible, the final formula takes a simple form as the sum of two terms. Of these, the dominant term containing a logarithm is proportional to the elastic scattering cross section, while the minor nonlogarithmic term can usually be neglected. The resulting formula is similar to that for the concomitant process in which the incoming electron radiates a photon in a physical radiator upstream and then scatters elastically from a proton, which in turn recoils into the detector. The factoring of the elastic scattering cross section occurs also when the final electron rather than the recoil proton is observed, a result which depends upon the so-called peaking approximation. In the present calculation the factoring of the leading term is exact for relativistic electrons.

I. INTRODUCTION

In some work at Orsay on inelastic electron-proton scattering, it was desired to know the contribution of the process

$$e+p \rightarrow e'+p'+\gamma \tag{1}$$

to the momentum spectrum of recoil protons produced in a liquid-hydrogen target by an electron beam well defined in energy and angle. The requirement of appreciable momentum transfer to the proton in process (1) forces the electron or photon to scatter through a wide angle; i.e., one deals with wide-angle bremsstrahlung (WAB). The transition rate for this process was calculated by Berg and Lindner[1] (BL), who considered the "Bethe-Heitler" amplitude corresponding to Fig. 1, and also the "virtual Compton" amplitude of Fig. 2. However, the object of the BL calculation was the spectrum of secondary electrons.

A calculation involving the recoil proton spectrum was done by Schiff[2] to obtain radiative corrections for the experiment of Panofsky and Tautfest,[3] who detected protons from e-p scattering using photographic emulsions. Schiff began with the Bethe-Heitler[4] formula, which describes radiative electron scattering in a fixed Coulomb field. The integral over final states was done by a method now known as the peaking approximation, based on the observation that the integrand is large only when the momentum vector of the photon is nearly parallel to that of the incident or final electron. Schiff obtained a formula for the proton momentum distribution which contained the Mott[5]-cross-section

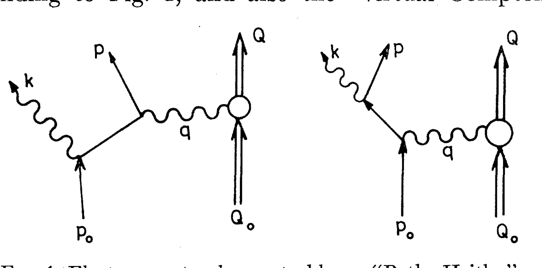

FIG. 1. Electron-proton bremsstrahlung: "Bethe-Heitler" diagrams. $q^\mu = Q^\mu - Q_0{}^\mu$.

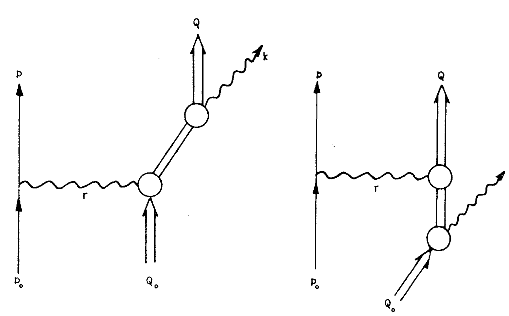

FIG. 2. Electron-proton bremsstrahlung: "virtual Compton-effect" diagrams. $r = p_0 - p$.

* Present address: 201 Residence des Eaux Vives, 91 Palaiseau, France.

† Present address: NASA Ames Research Center, Moffett Field, Calif. 94040.

[1] R. A. Berg and C. N. Lindner, Phys. Rev. **112**, 2072 (1958); Nucl. Phys. **26**, 262 (1961). See also, P. S. Isaev and I. S. Slatev, Nuovo Cimento **13**, 1 (1959); A. Costescu and T. Vescan, *ibid.* **48A**, 1041 (1967).

[2] L. I. Schiff, Phys. Rev. **87**, 750 (1952).

[3] W. K. H. Panofsky and G. Tautfest, Phys. Rev. **105**, 1356 (1957).

[4] H. Bethe and W. Heitler, Proc. Roy. Soc. (London) **146**, 83 (1934).

[5] N. F. Mott, Proc. Roy. Soc. (London) **A135**, 429 (1932).

elastic scattering as a factor. He then argued that the deficiencies of the no-recoil analysis could be made good by the simple substitution of the Rosenbluth[6] cross section for the Mott cross section, which was proved true in leading order of a recoil parameter by Drell.[7]

The BL analysis differs from that of Bethe-Heitler in representing the target current by an operator describing the properties of a physical proton including its finite mass, the anomalous magnetic moment, and the "structure" implied by the Dirac and Pauli form factors F_1 and F_2.

Meister and Yennie[8] (MY) treated radiative corrections to elastic electron-proton scattering using a current operator for the proton equivalent to that of BL. By differentiation of their formula applicable to proton detection, MY obtained an expression for the recoil momentum spectrum which should be valid near the elastic scattering peak (for soft-photon emission).

In this paper the required cross section, differential in final proton momentum and solid angle, is obtained by integrating the Bethe-Heitler terms of the BL matrix element over the phase space of the unobserved (e',γ) system. This integration can be done exactly in closed form because the momentum transfer to the proton vertex of Fig. 1 is fixed. Thus the result can be applied to muon bremsstrahlung, although for simplicity we always talk in terms of process (1).

In the text we consider only relativistic electrons. The formula for purely elastic scattering is presented as is that for the proton spectrum due to elastic scattering by electrons which have lost energy by bremsstrahlung in an upstream radiator. This second process is conveniently compared with WAB by defining an equivalent radiator thickness for WAB.

Appendix A contains outlined derivations of the elastic scattering and WAB cross sections, with similarities indicated. The soft-photon limit and ultra-relativistic limit are given. In Appendix B the differentiated radiative correction of Meister and Yennie[8] is compared with our result.

II. ELASTIC SCATTERING

Consider a proton of rest mass M, at rest in the laboratory until scattered elastically out of the target by an incoming energetic electron. The recoil proton emerges from the target at angle ϕ with respect to the electron beam, with momentum Q, total relativistic energy E, and kinetic energy $T=E-M$. The four-momentum transfer to the proton is $q^\mu=Q^\mu-Q_0{}^\mu$ (see Appendix A for notation). The invariant

$$\tau=-q_\mu q^\mu/4M^2=(E-M)/2M=T/2M=-q^2/4M^2 \qquad (2)$$

is a convenient dimensionless parameter.

[6] M. N. Rosenbluth, Phys. Rev. **79**, 615 (1950).
[7] S. D. Drell, Phys. Rev. **87**, 753 (1952).
[8] N. Meister and D. R. Yennie, Phys. Rev. **130**, 1210 (1963).

The incident electron energy necessary to eject the observed proton is

$$\epsilon_0{}^{\mathrm{el}}=MT/(Q\cos\phi-T) \qquad (3)$$

and the electron angle of scatter θ_e is determined by

$$\cot\tfrac{1}{2}\theta_e=\frac{Q\sin\phi}{Q\cos\phi-T}=\left(1+\frac{\epsilon_0{}^{\mathrm{el}}}{M}\right)\tan\phi\,, \qquad (4)$$

with the neglect of the electron's rest mass m relative to its energy.

The Rosenbluth cross section, differential in solid angle of the proton, is a function only of Q and ϕ:

$$\left(\frac{d\sigma}{d\Omega}\right)_R=\frac{C_{\mathrm{el}}G_{\mathrm{el}}}{\tau}\frac{Q(E+M)}{\epsilon_0{}^{\mathrm{el}}(\epsilon_0{}^{\mathrm{el}}+M)} =C_{\mathrm{el}}G_{\mathrm{el}}\left(\frac{2M}{\epsilon_0{}^{\mathrm{el}}}\right)^2(1+\tau)\sec\phi\,, \qquad (5)$$

where

$$G_{\mathrm{el}}=\frac{G_{Ep}{}^2+\tau G_{Mp}{}^2}{1+\tau}\cot^2(\tfrac{1}{2}\theta_e)+2\tau G_{Mp}{}^2=G_{\mathrm{el}}(Q,\phi)\,,$$
$$C_{\mathrm{el}}=(e^2/2M)^2=0.587\ 10^{-32}\ \mathrm{cm}^2\,, \qquad (6)$$

e being the elementary unit of charge in esu; G_{Ep} and G_{Mp} are, respectively, the τ-dependent electric and magnetic form factors of the proton.[9]

Equation (5) is valid when the final lepton is relativistic and

$$-q^2\gg 2m^2. \qquad (7)$$

III. RADIATION BEFORE SCATTERING

An electron of energy $\epsilon_0>\epsilon_0{}^{\mathrm{el}}$ may emit a photon of energy

$$k_f=\epsilon_0-\epsilon_0{}^{\mathrm{el}} \qquad (8)$$

into the forward direction while traversing the upstream physical radiator, subsequently scattering elastically on a proton.

The probability for emitting the photon with energy k_f in dk_f in a radiator of t radiation lengths is

$$t\Phi(\epsilon_0,k_f)dk_f/k_f\,, \qquad (9)$$

where $\Phi(\epsilon,k)\sim 1$ is a dimensionless function describing the shape of the bremsstrahlung energy spectrum[10]; $\Phi(\epsilon,0)\approx\frac{4}{3}$ in the soft-photon limit.

The effective cross section for detecting knock-on protons from this two-step process is given by

$$\frac{d^2\sigma}{d\Omega dQ}(\epsilon_0,Q,\phi)=\left|\frac{\partial k_f}{\partial Q}\right|\left(\frac{d\sigma}{d\Omega}\right)_R\frac{t\Phi(\epsilon_0,k_f)}{k_f}\,, \qquad (10)$$

[9] L. N. Hand, D. G. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 335 (1963).
[10] B. Rossi, *High Energy Particles* (Prentice-Hall, Englewood Cliffs, N. J., 1952).

in which the recoil factor

$$\left|\frac{\partial k_f}{\partial Q}\right| = \frac{\epsilon_0^{\text{el}}(\epsilon_0^{\text{el}}+M)}{EQ} \tag{11}$$

is obtained by differentiation from (8) and (3). For the purpose of computation it is convenient to combine the Rosenbluth cross section (5) with the recoil factor:

$$\left|\frac{\partial k_f}{\partial Q}\right|\left(\frac{d\sigma}{d\Omega}\right)_R = \frac{C_{\text{el}}G_{\text{el}}}{\tau}\frac{E+M}{E} = f(Q,\phi)\,, \tag{12}$$

a function of only Q and ϕ.

IV. WIDE-ANGLE BREMSSTRAHLUNG

The differential cross section for WAB is given by

$$\left[\frac{d^2\sigma}{d\Omega dQ}(\epsilon_0,Q,\phi)\right]_{\text{WAB}} = \frac{C_{\text{el}}}{\tau}\frac{E+M}{E}\frac{1}{k_f}$$

$$\times\left\{\frac{\alpha}{\pi}x(1-x)(G_{Ep}{}^2-2\tau G_{Mp}{}^2)+G_{\text{el}}t_{\text{eq}}\right\},$$

$$t_{\text{eq}} = \frac{\alpha}{\pi}\left[(1+x^2)\ln\left(\frac{-q^2}{Wmx}\right)+u^2-\frac{x+3x^2}{2}\right], \tag{13}$$

where $\alpha\approx 1/137$ is the fine-structure constant, x is an invariant "inelasticity parameter,"

$$x = -2M^2\tau/p_0{}^\mu q_\mu = \epsilon_0^{\text{el}}/\epsilon_0\,,\quad 0<x<1 \tag{14}$$

and t_{eq} is the equivalent radiator thickness. In the zero-momentum system of the final electron-photon system, W is the total energy:

$$W = [m^2-q^2(1-x)/x]^{1/2} \tag{15}$$

and the photon energy divided by W is

$$u = \tfrac{1}{2}(1-m^2/W^2)\,. \tag{16}$$

The validity of (13) is restricted only by (7). In questionable cases, such as muon-proton scattering, one can always use the exact formulas in Appendix A.

The dominant term within the curly brackets of (13) is clearly the second, which is proportional to the elastic factor G_{el}. For fixed τ, G_{el} has its minimum value when the proton comes forward ($\phi=0$). Even then the second term accounts for typically more than 95% of the cross section. This suggests rewriting (13) with the Rosenbluth cross section factored out:

$$\left[\frac{d^2\sigma}{d\Omega dQ}(\epsilon_0,Q,\phi)\right]_{\text{WAB}} = \left|\frac{\partial k_f}{\partial Q}\right|\left(\frac{d\sigma}{d\Omega}\right)_R\frac{t_{\text{eq}}}{k_f}(1+b)\,, \tag{17}$$

which is comparable in form to (10). The small correction factor b is the ratio of the first term to the second term in the curly bracket of (13):

$$b = \frac{\alpha}{\pi}\frac{x(1-x)(G_{Ep}{}^2-2\tau G_{Mp}{}^2)}{t_{\text{eq}}G_{\text{el}}}\,. \tag{18}$$

The explicit occurrence of k_f in (17) is rather curious, since the final-state integral was carried out exactly over all photon directions in the zero-momentum system. This integral is dominated by contributions in which the photon momentum vector lies close to the forward direction. Nevertheless, *our result has nothing to do with the peaking approximation.*

Consider the similarity between (17) for WAB and (10), which refers to the two-step process; their relative contributions to the observed proton spectrum are roughly in the ratio t_{eq}/t.

Near the infrared photon (elastic scattering) limit, $x\to 1$; from (13)–(16),

$$\lim_{x\to 1} t_{\text{eq}} = (2\alpha/\pi)[\ln(-q^2/m^2)-1] \tag{19}$$

for electron bremsstrahlung [see Eq. (A31) ff.]. The same expression holds in the infrared limit for electron detection.[11] However, the equivalent radiator for proton detection decreases faster with increasing photon energy than that for electron detection because of the behavior of the exact argument of the logarithm [cf. Eqs. (A27)–(A30)].

V. DISCUSSION

We see that for relativistic electrons the Rosenbluth cross section factors almost completely from the WAB cross section, as indicated by the approximate methods of Schiff and of Meister and Yennie.

In the study of electroproduction of hadrons by recoil proton detection, the formulas given here can be used to subtract the WAB contribution to the measured spectrum. Of course, the information so obtained is not in general equivalent to that found from electron detection. It is interesting to note that for fixed incident electron energy the missing mass increases linearly with decreasing momentum of a detected electron, whereas the missing mass for proton detection increases and then decreases again as the momentum of the detected proton decreases monotonically from its maximum value.

It is noteworthy that recoil proton detection has some advantages over scattered electron detection in the study of elastic e-p scattering: Radiative effects are reduced because photon emission along the direction of the final electron is suppressed. Calculation of the major contribution to the radiative tail is accomplished without approximation. Measurements at constant momentum transfer made with a magnetic spectrometer are made at constant spectrometer current, tending to reduce systematic errors.

[11] E. A. Allton, Phys. Rev. **135**, B570 (1964).

Corrections to our calculation will arise from radiative corrections and the neglected proton Compton effect. We intend to investigate the latter in a subsequent paper, although some idea of its importance may be found by examining the work of MY.[8]

ACKNOWLEDGMENTS

It is a pleasure to thank Professor A. Blanc-Lapierre of the Laboratoire de l'Accélérateur Linéaire, Professor A. W. Merrison of the Daresbury Nuclear Physics Laboratory, and Professor S. Gorodetsky and Dr. M. Croissiaux of the Institut de Recherches Nucléaires de l'Université de Strasbourg for their support of this work. One of us (E. F. E.) is grateful to Professor D. R. Yennie for an interesting discussion. We much appreciate the care and patience of Mme. G. Algardy and her staff at the LAL in preparing the manuscript.

APPENDIX A: THEORY

Introduction

The notation used here is that of Bethe, Schweber, and de Hoffmann,[12] with the spinor normalization of Feynman[13]: $\bar{u}u=2m$, where m is the rest mass of the particle in question. The target is a proton at rest in the laboratory for which the proton current operator for both Figs. 1 and 3 is that of Hand, Miller, and Wilson.[9] The contribution of the proton current to the square of the matrix element is

$$T^{\mu\nu}=4\mathfrak{F}_2(\tau)Q_0{}^{\mu}Q_0{}^{\nu}+q^2G_M{}^2(\tau)g^{\mu\nu}, \qquad \text{(A1)}$$

where $q^2=-4M^2\tau$ is the square of the four-momentum transfer, M is the proton mass, and

$$\mathfrak{F}_2(\tau)=[G_E{}^2(\tau)+\tau G_M{}^2(\tau)]/(1+\tau), \qquad \text{(A2)}$$

where $G_E(0)=1$ and $G_M(0)=2.793$.

Elastic Scattering

The lepton current contributes

$$S^{\mu\nu}=2(p_0,p)^{\mu\nu}+q^2g^{\mu\nu}, \qquad \text{(A3)}$$

where we have introduced the anticommutator symbol $(a,b)^{\mu\nu}=a^{\mu}b^{\nu}+b^{\mu}a^{\nu}$. The square of the matrix element is then

$$|\mathfrak{M}_{el}|^2=(1/q^4)S_{\mu\nu}T^{\mu\nu}=G_{el}/\tau, \qquad \text{(A4)}$$

where, with ϵ_0 (ϵ) the initial (final) lepton energy, and m the lepton mass,

$$G_{el}=\mathfrak{F}_2(\tau)\left[\frac{\epsilon\epsilon_0}{M^2\tau}-1\right]+2\tau G_M{}^2(\tau)\left[1-\frac{1}{2\tau}\left(\frac{m}{M}\right)^2\right]. \qquad \text{(A5)}$$

When the final electron is relativistic ($\epsilon\gg m$),

$$G_{el}=\mathfrak{F}_2(\tau)\cot^2(\tfrac{1}{2}\theta_e)+2\tau G_M{}^2(\tau) \qquad \text{(A6)}$$

is an accurate approximation; here θ_e is the laboratory scattering angle of the lepton.

The differential cross section is

$$d^2\sigma=(4\pi e^2)^2\frac{(2\pi)^4\delta^{(4)}(p_0+Q_0-p-Q)}{\beta_0(2\epsilon_0)2M}\times\frac{d^3p}{2\epsilon(2\pi)^3}\frac{d^3Q}{2E(2\pi)^3}|\mathfrak{M}_{el}|^2, \qquad \text{(A7)}$$

where $\beta_0=|\mathbf{p}_0|/\epsilon_0$ is the velocity of the incoming projectile relative to a target at rest. If the final proton is detected in solid angle $d\Omega$, the final-state integral gives

$$\left(\frac{d\sigma}{d\Omega}\right)_{el}=\frac{C_{el}G_{el}}{\tau}\frac{Q(E+M)}{p_0(\epsilon_0+M)}, \qquad \text{(A8)}$$

where $C_{el}=(e^2/2M)^2$. This formula is exact if G_{el} is taken from (A5); otherwise, one can use (A6) for G_{el} and set $p_0=\epsilon_0$ in (A8).

Electron Bremsstrahlung

The lepton current operator for Fig. 1 is given by

$$j_e{}^{\mu}=\boldsymbol{e}\frac{1}{\boldsymbol{p}+\boldsymbol{k}-m}\gamma^{\mu}+\gamma^{\mu}\frac{1}{\boldsymbol{p}_0-\boldsymbol{k}-m}\boldsymbol{e}, \qquad \text{(A9)}$$

where e^{μ} is the photon polarization four-vector, with $e^{\mu}e_{\mu}=-1$ and $\boldsymbol{e}=e^{\mu}\gamma_{\mu}$, etc. One easily finds from this the equivalent plane-wave operator:

$$j_e{}^{\mu}=\boldsymbol{ek}\gamma^{\mu}/2\lambda+\gamma^{\mu}\boldsymbol{ke}/2\lambda_0+\chi\gamma^{\mu}, \qquad \text{(A10)}$$

$$\chi=(e\cdot p)/\lambda-(e\cdot p_0)/\lambda_0, \qquad \text{(A11)}$$

$$\lambda=(k\cdot p),\quad \lambda_0=(k\cdot p_0). \qquad \text{(A12)}$$

The required tensor, averaged over initial lepton spins and summed over final lepton spins and photon

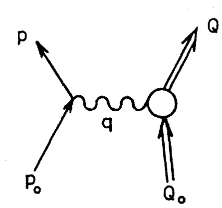

FIG. 3. Elastic electron-proton scattering: "Rosenbluth" diagram. $q^{\mu}=Q^{\mu}-Q_0{}^{\mu}$.

---

[12] S. S. Schweber, H. A. Bethe, and F. de Hoffmann, *Mesons and Fields* (Row Peterson, Evanston, Ill., 1955).

[13] R. P. Feynman, *Quantum Electrodynamics* (Benjamin, New York, 1962).

polarizations, is

$$S^{\mu\nu}=\sum_{\text{pol}}\tfrac{1}{2}\,\mathrm{Tr}[j_e^{\mu}(p_0+m)j_e^{\nu}(p+M)]$$

$$=4\Bigg\{\left(2\sigma+\frac{m^2}{M^2}\right)\left[\frac{(p_0,s)^{\mu\nu}+(p,s_0)^{\mu\nu}}{2M^2}-2\tau g^{\mu\nu}\right]\left(\frac{M^4}{\lambda\lambda_0}\right)-\frac{m^2k^\mu k^\nu}{\lambda\lambda_0}+\frac{(p,p)^{\mu\nu}+(p_0,s)^{\mu\nu}}{2\lambda}-\frac{(p_0,p_0)^{\mu\nu}+(p,s_0)^{\mu\nu}}{2\lambda_0}$$
$$-\tfrac{1}{2}g^{\mu\nu}\left(\frac{\lambda_0}{\lambda}+\frac{\lambda}{\lambda_0}\right)-\left[\frac{(p_0,s)^{\mu\nu}}{2M^2}-\tau g^{\mu\nu}\right]\left(\frac{mM}{\lambda}\right)^2-\left[\frac{(p,s_0)^{\mu\nu}}{2M^2}-\tau g^{\mu\nu}\right]\left(\frac{mM}{\lambda_0}\right)^2\Bigg\}, \quad \text{(A13)}$$

where $s_0=p-k$, $s=p+k$, and $\sigma=-(p_0-p)^2/4M^2$.

The square of the bremsstrahlung matrix element is found by contracting the two tensors:

$$|\mathfrak{M}_{\mathrm{WAB}}|^2=\frac{S^{\mu\nu}T_{\mu\nu}}{q^4}=\frac{1}{2M^2\tau^2}[\mathfrak{F}_2(\tau)Y_{EM}+2\tau G_M^2(\tau)Y_M], \quad \text{(A14)}$$

$$Y_M=\left(2\tau-\frac{m^2}{M^2}\right)\left[\left(2\sigma+\frac{m^2}{M^2}\right)\frac{2M^4}{\lambda\lambda_0}-\left(\frac{mM}{\lambda}\right)^2-\left(\frac{mM}{\lambda_0}\right)^2\right]+\frac{\lambda_0}{\lambda}+\frac{\lambda}{\lambda_0}, \quad \text{(A15)}$$

$$Y_{EM}=[A^2+A_0^2-2\sigma(1+\tau)]\frac{4M^4\tau}{\lambda\lambda_0}+[AA_0-\tau(1+\tau)]\frac{4M^2m^2}{\lambda\lambda_0}$$
$$-2[A_0^2-\tau(1+\tau)]\left(\frac{mM}{\lambda}\right)^2-2[A^2-\tau(1+\tau)]\left(\frac{mM}{\lambda_0}\right)^2-(1+\tau)\left(\frac{\lambda_0}{\lambda}+\frac{\lambda}{\lambda_0}\right), \quad \text{(A16)}$$

where $A=(P\cdot p)/2M^2$ and $A_0=(P\cdot p_0)/2M^2$. Equations (A14)–(A16) are equivalent to the result of Berg and Lindner.[1] The invariant λ_0 (λ) is small when the photon follows the incident (final) electron direction, so that photon emission along the electron trajectories is favored.

For proton detection, however, λ is fixed, so only forward photon emission is enhanced. Indeed, τ and A_0 are also determined, and the variables of integration will be λ_0, σ, and A. The identity $\sigma=\tau+(\lambda-\lambda_0)/2M^2$ eliminates σ. To manifest the infrared behavior, we want to factor out of both Y_M and Y_{EM} a quantity proportional to k^{-1}. Since $\lambda=(k\cdot p)=q^2/\tau-(p_0\cdot q)$, a convenient parameter is

$$x=q^2/2(p_0\cdot q),\quad 0<x<1. \quad \text{(A17)}$$

Then

$$\lambda=2M^2\tau(1-x)/x \quad \text{(A18)}$$

and so we factor out $(1-x)^{-1}$. Note that in the elastic scattering limit $x\rightarrow 1$. Another useful parameter is

$$z=xm^2/2M^2\tau. \quad \text{(A19)}$$

Clearly $z<1$ for $-q^2>2m^2$, or equivalently for $T>m^2/M$, where T is the kinetic energy of the final proton.

Finally we express Y_M and Y_{EM} as linear combinations of six quantities which will be integrands in the integration over final states, with coefficients given in terms of τ, A_0, x, and z. The integrands are

$$\begin{aligned}I_1'&=M^2/\lambda_0,\\ I_2'&=(mM/\lambda_0)^2,\\ I_3'&=\lambda_0/M^2,\\ I_4'&=M^2A^2/\lambda_0,\\ I_5'&=(mMA/\lambda_0)^2,\\ I_6'&=M^2A/\lambda_0.\end{aligned} \quad \text{(A20)}$$

Then from (A15) and (A16) one finds

$$\begin{aligned}(1-x)Y_M=&(2\tau/x)[1+x^2-2z(1-x+z)]I_1'\\ &-(2\tau/x)(1-x)(x-z)I_2'+(x/2\tau)I_3'\\ &-(x-z)(2-2x+z)/(1-x),\end{aligned} \quad \text{(A21)}$$

$$\begin{aligned}(1-x)Y_{EM}=&2\{xA_0^2-\tau(1+\tau)[(1+x^2)/x+2z]\}I_1'\\ &+2\tau(1+\tau)(1-x)I_2'-(x/2\tau)(1+\tau)\\ &\times I_3'+2x(1+\tau)+xz(1+\tau-A_0^2/\tau)/\\ &(1-x)+2xI_4'-2(1-x)I_5'\\ &+4zA_0I_6'.\end{aligned} \quad \text{(A22)}$$

The completely differential cross section, comparable to (A7), is

$$d\sigma=(4\pi e^2)^3\frac{(2\pi)^4\delta^{(4)}(p_0-q-k-p)}{\beta_0(2\epsilon_0)2M}$$
$$\times\frac{d^3p}{2\epsilon(2\pi)^3}\frac{d^3k}{2k(2\pi)^3}\frac{d^3Q}{2E(2\pi)^3}|\mathfrak{M}_{\mathrm{WAB}}|^2. \quad \text{(A23)}$$

When integrated over the phase space of the unobserved electron and photon, this becomes

$$\left(\frac{d^2\sigma}{d\Omega dQ}\right)_{\rm WAB}=\frac{\alpha}{\pi}\frac{C_{\rm el}}{\tau}\frac{E+M}{p_0E}\frac{1}{1-x}\times[\mathfrak{F}_2(\tau)H_{EM}+2\tau G_M{}^2(\tau)H_M],\qquad (A24)$$

$$H_{EM,M}=\frac{\tilde{k}}{W}\frac{1}{4\pi}\int d\tilde{\Omega}_k(1-x)Y_{EM,M}.$$

The tilde indicates that the integration is to be carried out in the zero-momentum frame of the photon–final-electron system, where

$$p^\mu+k^\mu=(W,0),$$
$$W^2=(\tilde{\epsilon}+\tilde{k})^2=2M^2\tau(2-2x+z)/x=-q^2(1-x)/x+m^2.\qquad (A25)$$

The six integrals contributing to (A24) have the form

$$I_i=\frac{\tilde{k}}{W}\frac{1}{4\pi}\int d\tilde{\Omega}_kI_i',\quad i=1,\ldots,6\qquad (A26)$$

where the I_i''s are given in (A20). Integrals I_1–I_3 involve only λ_0, and are straightforward. To do those involving A and λ_0 (integrals I_4–I_6), the coordinate system is oriented so that the spacelike components of $\tilde{p}_0$, $\tilde{Q}_0$, $\tilde{Q}$, and $\tilde{P}=\tilde{Q}_0+\tilde{Q}$ lie in the xz plane, with the z axis parallel to the spacelike part of $\tilde{p}_0$. Let the direction of photon emission (momentum three-vector of $\tilde{k}$) have angular coordinates ξ and ζ, so that $d\tilde{\Omega}_k=\sin\xi d\xi d\zeta$. In this system

$$\lambda_0=(2M^2\tau/x)u(1+z)(1-\tilde{\beta}_0\cos\xi)$$

and

$$A=A_0(1-u)+A_0\frac{(2x-1)u}{(1+2xz)^{1/2}}\cos\xi+\left[\frac{2\tau}{x}(1-x)u\left(\frac{x^2A_0{}^2}{\tau(1+2xz)}-1-\tau\right)\right]^{1/2}\sin\xi\cos\zeta,$$

where

$$\tilde{\beta}_0=\tilde{p}_0/\tilde{\epsilon}_0=(1+2xz)^{1/2}/(1+z),$$
$$u=\tilde{k}/W=(1-x)/(2-2x+z).$$

Defining

$$I_0=\frac{1}{(1+2xz)^{1/2}}\ln\left[\frac{-q^2}{Wmx}\left(\frac{1+z+(1+2xz)^{1/2}}{2}\right)\right],\qquad (A27)$$

the six integrals are

$$I_1=(x/2\tau)I_0,$$

$$I_2=x/2\tau(1-x),$$

$$I_3=\frac{2\tau}{x}(1+z)u^2,$$

$$I_4=\tfrac{1}{2}I_0\left[\frac{x^2A_0{}^2}{\tau}\frac{x(1+2z)^2-z(1-x)^2}{(1+2xz)^2}+\frac{z(1-x)^2}{1+2xz}(1+\tau)\right]-\frac{xA_0{}^2}{2\tau}u\left[\frac{x(2x-1)(1+2z)-x(1-x)(1+z)}{(1+2xz)^2}+\frac{(1-u)(2x-1)}{1+2xz}\right]-(1+\tau)\frac{(1-x)(1+z)u}{2(1+2xz)},$$

$$I_5=-I_0z\left[\frac{x^2A_0{}^2}{\tau}\frac{(2x-1)(1+2z)-(1-x)(1+z)}{(1+2xz)^2}+\frac{(1-x)(1+z)}{1+2xz}(1+\tau)\right]+\frac{xA_0{}^2}{2\tau}\frac{1}{(1-x)}\times\left[x^2\left(\frac{1+2z}{1+2xz}\right)^2+\frac{zu(1-x)[2(2x-1)^2-1-2xz]}{(1+2xz)^2}\right]+(1+\tau)z(1-x)/(1+2xz),\qquad (A28)$$

$$I_6=\frac{x^2A_0}{2\tau}\left(\frac{1+2z}{1+2xz}\right)I_0-\frac{xA_0}{2\tau}\left(\frac{2x-1}{1+2xz}\right)u.$$

Using these integrals one finds from (A21), (A22), and (A24)

$$H_M=I_0[1+x^2-2z(1-x+z)]+u^2(1+z)-2x+2z,\qquad (A29)$$

$$H_{EM}=I_0\left\{\frac{x^2A_0^2}{\tau}[1+x^2r^2-zt^2(2+x+2z)+2zr(1-t+2xt)]-(1+\tau)[1+x^2+2xz-zt(1-x)(2+x+2z)]\right\}$$
$$-\frac{x^2A_0^2}{\tau}\left\{xr^2+\frac{u}{1+2xz}[(2x-1)(1-u+xr)-xt(1+z)]\right\}-\frac{xA_0^2}{\tau}\left\{1-2u\right.$$
$$\left.+\frac{zu}{1+2xz}[2(2x-1)(1-t+2xt)-1+x]\right\}-(1+\tau)\{u^2(1+z)-2x+t[xu+z(xu+2-2x)]\},\quad \text{(A30)}$$
$$r=(1+2z)/(1+2xz),\quad t=(1-x)/(1+2xz),\quad u=(1-x)/(2-2x+z).$$

Note that only I_0 contains a logarithm. These expressions are to be used in (A24) to calculate the cross section, and constitute the main result of this appendix. We will consider two limiting cases.

In principle the scattering becomes elastic as $x\to 1$. Of course one cannot set $x=1$ because of the infrared divergence, but the cross section does simplify considerably if $u\equiv\tilde{k}/W\ll 1$, or, equivalently,

$$\frac{m^2}{4M^2\tau}\frac{x}{1-x}\gg 1. \quad \text{(A31)}$$

Then $u\simeq 0$, $r\simeq 1$, and $t\simeq 0$. For $-q^2>4m^2$, we can set $x\simeq 1$ in the numerators of all terms in (A29) and (A30) which become

$$H_M\approx 2(1-z)[(1+z)I_0-1],$$
$$H_{EM}\approx 2(\epsilon_0\epsilon/M^2\tau-1)[(1+z)I_0-1], \quad \text{(A32)}$$

where I_0 is to be evaluated from (A27) using $x=1$.

In the latter expression, we have used the elastic scattering kinematical relation $\epsilon_0=\epsilon+2M\tau$ with the exact formula

$$A_0=\epsilon_0/M-\tau/x \quad \text{(A33)}$$

to show that for $x\approx 1$,

$$A_0^2x^2/\tau-(1+\tau)\approx\epsilon_0\epsilon/M^2\tau-1.$$

Thus in the elastic scattering limit (A31)

$$\lim_{x\to 1}\left[(1-x)\left(\frac{d^2\sigma}{d\Omega dQ}\right)_{\text{WAB}}\right]$$
$$=\left(\frac{d\sigma}{d\Omega}\right)_{\text{el}}\frac{\epsilon_0+M}{QE}\frac{2\alpha}{\pi}[(1+z)I_0-1], \quad \text{(A34)}$$

where the exact elastic scattering cross section (A8)—using (A5) for G_{el}—appears as a factor. The factoring of the exact Rosenbluth cross section is a necessary result that could be proved directly from the matrix element. If, in addition to (A31), the condition $z\ll 1$ is valid, then $(1+z)I_0\simeq\ln(-q^2/m^2)$.

The other case is the relativistic limit. We consider the cross section under the assumptions

$$\epsilon_0\gg m,\quad -q^2\gg 2m^2; \quad \text{(A35)}$$

then $z\ll 1$, $r=1$, and $t=1-x$ are good approximations. Then (A29) and (A30) become

$$H_M=I_0(1+x^2)+u^2-2x,$$
$$H_{EM}=[A_0^2x^2/\tau-1-\tau][I_0(1+x^2)+u^2-\tfrac{1}{2}x-\tfrac{3}{2}x^2]+(1+\tau)x(1-x). \quad \text{(A36)}$$

However, (A35) assures that it is a good approximation to take

$$x^2A_0^2/\tau-1-\tau=\cot^2(\tfrac{1}{2}\theta_e). \quad \text{(A37)}$$

Then from (A24), (A36), and (A37), one finds

$$\left(\frac{d^2\sigma}{d\Omega dQ}\right)_{\text{WAB}}=\frac{\alpha}{\pi}\frac{C_{\text{el}}}{\tau}\frac{E+M}{p_0E}\frac{1}{1-x}\times\{G_{\text{el}}[I_0(1+x^2)+u^2-\tfrac{1}{2}x-\tfrac{3}{2}x^2]+[G_E^2(\tau)-2\tau G_M^2(\tau)]x(1-x)\}. \quad \text{(A38)}$$

Here G_{el} is to be calculated using (A6) and $I_0=\ln(-q^2/Wmx)$ according to (A25) and (A27). Equation (A38) is the result discussed in the text.

APPENDIX B: COMPARISON WITH FORMULA OF MEISTER AND YENNIE

Here we show that there exists a range of moderately soft photon energies within which a result of Meister and Yennie[8] agrees well with our formula (17) in the text.

Meister and Yennie obtain the differential cross section for WAB by differentiating their radiative correction with respect to Q. In fact they give two results, one being essentially their Eq. (2.26b), the other their Eq. (4.4). We examine the former, since the latter involves additional approximations which obscure the comparison with our result when the photon is very soft. Actually, their Eq. (2.26b) is not complete, and we are obliged to add to it the derivative of their Eq. (3.4). It is understood that we concern ourselves only with the Z_1^2 ("Bethe-Heitler") terms in these expressions.

The Z_1^2 terms inside the curly brackets of MY's Eq. (2.26b), in our notation, are

$$2[\ln(-q^2/m^2)-1-\tfrac{1}{2}\ln(1+2\lambda/m^2)], \quad \text{(B1)}$$

since the Γ_2 of MY is the same as our $\lambda=-q^2(1-x)/2x$. Noting that $W^2=2\lambda+m^2$, we combine the two logarithms to give

$$2[\ln(-q^2/Wm)-1]. \tag{B2}$$

To this we must add the derivative with respect to Q of the coefficient of α/π in MY's Eq. (3.4), holding constant the incident electron energy ϵ_0 and the proton angle ϕ:

$$\frac{\partial}{\partial Q}\left[\tfrac{1}{4}\ln\left(1+\frac{2\lambda}{m^2}\right)\right]=\frac{u}{2}\frac{1}{\lambda}\frac{\partial\lambda}{\partial Q}, \tag{B3}$$

where u is given by (16) in the text. Differentiation and some algebra leads to

$$\begin{aligned}\frac{1}{\lambda}\frac{\partial\lambda}{\partial Q}&=\frac{\partial\tau}{\tau\partial Q}-\frac{1}{x(1-x)}\frac{\partial x}{\partial Q}\\&=\frac{Q}{TE}+\frac{1}{xk_f}\left(\frac{\partial k_f}{\partial Q}\right),\end{aligned} \tag{B4}$$

where we have assumed $\epsilon_0 \gg m$, so that $k_f=\epsilon_0(1-x)$. Keeping only the dominant term ($\propto 1/k_f$) in (B4) can be justified by examining the relative magnitude of the two terms using (11) in the text for $\partial k_f/\partial Q$. Then MY's (3.4) contributes a term

$$\frac{u}{2x}\frac{1}{k_f}\left|\frac{\partial k_f}{\partial Q}\right|. \tag{B5}$$

The MY expression for the WAB cross section is then

$$\left(\frac{d^2\sigma}{dQd\Omega}\right)_{\mathrm{WAB}}^{\mathrm{MY}}=\left(\frac{d\sigma_0}{d\Omega}\right)_R\left|\frac{\partial k_f}{\partial Q}\right|\frac{t_{\mathrm{eq}}{}^{\mathrm{MY}}}{k_f}, \tag{B6}$$

where we have used $\Delta Q\approx k_f|\partial Q/\partial k_f|$, and the Rosenbluth cross section is to be evaluated at the proton momentum corresponding to the incident electron energy ϵ_0. The equivalent-radiator term in the trace is the sum of (B2) and (B5) multiplied by α/π:

$$t_{\mathrm{eq}}{}^{\mathrm{MY}}=\frac{2\alpha}{\pi}\left[\ln\left(\frac{-q^2}{Wm}\right)-1+\frac{u}{4x}\right]. \tag{B7}$$

In our formula (17) which corresponds to (B6), the Rosenbluth cross section is to be evaluated at the momentum Q of the detected proton. Our formula for the equivalent radiator is

$$t_{\mathrm{eq}}=\frac{2\alpha}{\pi}\left[\frac{1+x^2}{2}\ln\left(\frac{-q^2}{Wmx}\right)-\frac{x+3x^2}{4}+\frac{u^2}{2}\right]. \tag{13'}$$

The term-by-term correspondence between (B7) and (13′) is obvious. Both equations go to the correct infrared limit: $x=1$, $u=0$, $W=m$, and

$$t_{\mathrm{eq}}(k_f=0)=(2\alpha/\pi)[\ln(-q^2/m^2)-1].$$

There is a transition region of very soft photons ($\tilde{k}\sim m$) extending from the infrared limit up to the point where u approaches its asymptotic value $\frac{1}{2}$, while x has barely changed from unity. Beyond this transition region, $\frac{1}{2}u^2=\frac{1}{4}u=\frac{1}{8}$, and the theories are equivalent until x begins to differ appreciably from unity. In tracing out the momentum spectrum toward the lower momenta, the factor $(1+x^2)/2$ in (13′) decreases gradually from unity, so that $t_{\mathrm{eq}}{}^{MY}$ of (B7) increasingly overestimates our expression. This is not surprising since Meister and Yennie did not intend to apply their formula to the case of hard-photon emission.

PHYSICAL REVIEW D VOLUME 3, NUMBER 3 1 FEBRUARY 1971

Higher Symmetries and Nuclear Forces

SAYED S. EL-GHABATY, SURAJ N. GUPTA, AND WILLIAM H. WEIHOFEN
Department of Physics, Wayne State University, Detroit, Michigan 48202
(Received 8 September 1970)

The pion-nucleon couplings resulting from the $U(6,6)$ model and the chiral-dynamical model are applied to the investigation of the nucleon-nucleon scattering. It is shown (a) that in the $U(6,6)$ model the one-pion-exchange (OPE) contribution differs from the usual result only in short-range effects, while the two-pion-exchange (TPE) contribution is highly divergent; and (b) that the chiral-dynamical model leaves the usual OPE contribution unchanged, but leads to a significant modification of the usual TPE contribution.

I. INTRODUCTION

THE meson theory of nuclear forces has been investigated by many authors over several decades.[1] Although the one-pion-exchange (OPE) nucleon-nucleon interaction is easy to derive, the calculation of the two-pion-exchange (TPE) interaction is much more complicated. A precise evaluation of the TPE contribution has been carried out in our previous papers,[2–4] where the calculational techniques required for this purpose are also given. It has already been

[1] For a recent review, see G. Breit and R. D. Haracz, in *High Energy Physics*, edited by E. H. S. Burhop (Academic, New York, 1967), Vol. I, p. 21.

[2] S. N. Gupta, Phys. Rev. **117**, 1146 (1960); **122**, 1923 (1961).

[3] S. N. Gupta, R. D. Haracz, and J. Kaskas, Phys. Rev. **138**, B1500 (1965).

[4] S. S. El-Ghabaty, S. N. Gupta, and J. Kaskas, Phys. Rev. D **1**, 249 (1970).