Systematics of Elastic Scattering*

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(Received 19 October 1970)

We point out a simplification of the crossing matrix for elastic scattering. New amplitudes, with particularly simple properties under parity and time reversal, are defined. These are useful in theories where the quantity of interest is the amount of direct-channel helicity flip or sum. For example, in the accompanying paper, we employ this formalism to study suppression of direct-channel helicity flip for Pomeranchukon exchange, as well as other patterns of direct-channel amplitude suppression induced by exchange of particular quantum numbers in the crossed channel.

I. INTRODUCTION

PARTICLE physics abounds with methods for describing scattering amplitudes involving particles with spin: Lorentz-scalar invariant amplitudes, helicity amplitudes, transversity amplitudes, M functions, etc. Frequently, one type of amplitude is much more convenient for a given purpose than the others. For this reason, we are not embarrassed to define yet another set of amplitudes, for elastic scattering, which are particularly useful in the study of one type of theory.

We begin by noting that the crossing matrix between s- and t-channel helicity amplitudes simplifies greatly when the masses in the final state of the s channel are the same as those in the initial state (elastic scattering): Only two independent crossing angles are present in this case. We can then use a property of the rotation functions $d_{\lambda\mu}{}^J$ to reduce the crossing matrix to two $d_{\lambda\mu}{}^J$ functions (instead of the original four). By taking suitable linear combinations of the s- and t-channel helicity amplitudes, we obtain new amplitudes which cross according to the simplified matrix. Essentially, the matrix crosses each t-channel vertex separately.

Symmetry properties of these new amplitudes under operations such as parity and time reversal are discussed in Sec. III. In Sec. IV, we go on to study the consequences (for both s- and t-channel amplitudes) of exchange of a particular set of quantum numbers in the t channel.

Further simplifications result if the initial particles are identical (as in pp scattering). These are presented in Sec. V.

This work was motivated by a hypothesis of Gilman, Pumplin, Schwimmer, and Stodolsky[1] that true diffraction scattering (Pomeranchukon exchange) suppresses direct-channel helicity-flip amplitudes by at least one power of s. Our s-channel amplitudes can be labeled by helicity flip, and thus provide a natural framework for theoretical study of this hypothesis. In the following paper,[2] we study various patterns of s-channel amplitude suppression induced by exchange of particular t-channel quantum numbers.

* Work supported in part by the National Science Foundation under Grant No. NSF GP 19433 and NSF GP 13671.

[1] F. G. Gilman, J. Pumplin, A. Schwimmer, and L. Stodolsky, Phys. Letters **31B**, 387 (1970).

[2] Lorella M. Jones and D. G. Ravenhall, following paper, Phys. Rev. D **3**, 696 (1971).

II. DERIVATION OF SIMPLIFIED CROSSING MATRIX

We begin with the Trueman-Wick crossing matrix for helicity amplitudes[3]:

$$
\begin{aligned}
f_{cd;ab}{}^s &= \sum_{c'A'D'b'} d_{A'a}{}^{J_a}(\chi_a) d_{b'b}{}^{J_b}(\chi_b) \\
&\qquad \times d_{c'c}{}^{J_c}(\chi_c) d_{D'd}{}^{J_d}(\chi_d) f_{c'A';D'b'}{}^t , \\
\cos\chi_a &= \frac{-(s+m_a{}^2-m_b{}^2)(t+m_a{}^2-m_c{}^2)-2m_a{}^2\Delta^2}{s_{ab}\tau_{ac}} , \\
\cos\chi_b &= \frac{(s+m_b{}^2-m_a{}^2)(t+m_b{}^2-m_d{}^2)-2m_b{}^2\Delta^2}{s_{ab}\tau_{bd}} , \\
\cos\chi_c &= \frac{(s+m_c{}^2-m_d{}^2)(t+m_c{}^2-m_a{}^2)-2m_c{}^2\Delta^2}{s_{cd}\tau_{ac}} , \qquad (2.1)\\
\cos\chi_d &= \frac{-(s+m_d{}^2-m_c{}^2)(t+m_d{}^2-m_b{}^2)-2m_d{}^2\Delta^2}{s_{cd}\tau_{bd}} , \\
\Delta^2 &= m_c{}^2-m_a{}^2+m_b{}^2-m_d{}^2 , \\
s_{ij}{}^2 &= [s-(m_i+m_j)^2][s-(m_i-m_j)^2] , \\
\tau_{ij}{}^2 &= [t-(m_i+m_j)^2][t-(m_i-m_j)^2] .
\end{aligned}
$$

For elastic scattering, $m_b=m_d$, $m_a=m_c$, and, therefore, $\chi_d=\pi-\chi_b$, $\chi_a=\pi-\chi_c$. Also, $J_b=J_d$, $J_a=J_c$.[4] Hence the matrix reduces to

$$
\begin{aligned}
f_{cd;ab}{}^s &= \sum_{c'A'D'b'} d_{A'a}{}^{J_a}(\pi-\chi_a) d_{b'b}{}^{J_b}(\chi_b) \\
&\qquad \times d_{c'c}{}^{J_a}(\chi_c) d_{D'd}{}^{J_b}(\pi-\chi_b) f_{c'A';D'b'}{}^t . \qquad (2.2)
\end{aligned}
$$

Properties of the d operators, namely,

$$
\begin{aligned}
d_{\lambda\mu}{}^j(\theta) &= (-1)^{j+\lambda} d_{\lambda,-\mu}{}^j(\pi-\theta) , \\
&= (-1)^{\lambda-\mu} d_{\mu\lambda}{}^j(\theta) , \\
&= (-1)^{\lambda-\mu} d_{-\lambda,-\mu}{}^j(\theta) ,
\end{aligned}
$$

allow us to make the arguments of the functions in Eq. (2.2) all either χ_b or χ_c in a number of ways. For

[3] T. L. Trueman and G.-C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

[4] Much of the simplification of our method remains even when the particle *spins* are not equal in pairs. See the note at the end of Sec. II.

example,

$$f_{cd;ab}{}^{s}=\sum_{c'A'D'b'}(-1)^{J_a+J_b+A'+D'}d_{A',-a}{}^{J_a}(\chi_c)d_{b',b}{}^{J_b}(\chi_b)\times d_{c',c}{}^{J_a}(\chi_c)d_{D',-d}{}^{J_b}(\chi_b)f_{c'A';D'b'}{}^{t}. \quad (2.3a)$$

Two rotation functions with the same angle can be combined according to the formula

$$d_{M_1K_1}{}^{J_1}(\theta)d_{M_2K_2}{}^{J_2}(\theta)=\sum_{JMK}d_{MK}{}^{J}(\theta)\times(J_1J_2M_1M_2|JM)(J_1J_2K_1K_2|JK). \quad (2.4)$$

Thus Eq. (2.3a) can be rewritten more compactly in a form involving only two d functions:

$$f_{cd;ab}{}^{s}=\sum_{c'A'D'b'}(-1)^{J_a+J_b+A'+D'}\times\sum_{JMKJ'M'K'}d_{MK}{}^{J}(\chi_c)d_{M'K'}{}^{J'}(\chi_b)\times(J_aJ_aA'c'|JM)(J_aJ_a-ac|JK)\times(J_bJ_bD'b'|J'M')(J_bJ_b-db|J'K')f_{c'A';D'b'}{}^{t}. \quad (2.5)$$

The Clebsch-Gordan coefficient identity,

$$\sum_{M_1M_2}(J_1J_2M_1M_2|JM)(J_1J_2M_1M_2|J'M')=\delta_{JJ'}\delta_{MM'}, \quad (2.6)$$

allows us to write

$$\sum_{abcd}(J_bJ_b-db|J'''K''')(J_aJ_a-ac|J''K'')f_{cd;ab}{}^{s}=\sum_{MM'}d_{MK''}{}^{J''}(\chi_c)d_{M'K'''}{}^{J'''}(\chi_b)\times\sum_{A'c'D'b'}(-1)^{J_a+J_b+A'+D'}(J_aJ_aA'c'|J''M)\times(J_bJ_bD'b'|J'''M')f_{c'A';D'b'}{}^{t}. \quad (2.7)$$

If we now define new amplitudes $f_{JJ'MM'}{}^{s(--)}$ and $f_{JJ'KK'}{}^{t(++)}$, which are combinations of helicity amplitudes,

$$f_{JJ'MM'}{}^{s(--)}=\sum_{abcd}(J_aJ_a-ac|JM)\times(J_bJ_b-db|J'M')f_{cd;ab}{}^{s}, \quad (2.8a)$$

$$f_{JJ'KK'}{}^{t(++)}=\sum_{c'A'D'b'}(-1)^{J_a+J_b+A'+D'}(J_aJ_aA'c'|JK)\times(J_bJ_bD'b'|JK')f_{c'A';D'b'}{}^{t}, \quad (2.8b)$$

Eq. (2.7) becomes the simplified crossing relation

$$f_{JJ'MM'}{}^{s(--)}=\sum d_{KM}{}^{J}(\chi_c)d_{K'M'}{}^{J'}(\chi_b)f_{JJ'KK'}{}^{t(++)}. \quad (2.9a)$$

The superscripts $(--)$ and $(++)$ are related to the manner in which the argument replacement is performed in going from Eq. (2.2) to Eq. (2.3a). There are three other ways in which this may be done, namely,

$$f_{cd;ab}{}^{s}=\sum_{c'A'D'b'}(-1)^{J_a+J_b-a+D'}d_{-A'a}{}^{J_a}(\chi_c)d_{D'-d}{}^{J_b}(\chi_b)\times d_{b'b}{}^{J_b}(\chi_b)d_{c'c}{}^{J_a}(\chi_c)f_{c'A';D'b'}{}^{t} \quad (2.3b)$$

$$=\sum_{c'A'D'b'}(-1)^{J_a+J_b+A'-d}d_{A'-a}{}^{J_a}(\chi_c)d_{-D'd}{}^{J_b}(\chi_b)\times d_{b'b}{}^{J_b}(\chi_b)d_{c'c}{}^{J_a}(\chi_c)f_{c'A';D'b'}{}^{t} \quad (2.3c)$$

$$=\sum_{c'A'D'b'}(-1)^{J_a+J_b-a-d}d_{-A'a}{}^{J_a}(\chi_c)d_{-D'd}{}^{J_b}(\chi_b)\times d_{b'b}{}^{J_b}(\chi_b)d_{c'c}{}^{J_a}(\chi_c)f_{c'A';D'b'}{}^{t}. \quad (2.3d)$$

Similar manipulations allow us to define amplitudes identical to (2.8) and (2.9) except for signs on the helicity indices and for sign factors. The four possible s- and t-channel amplitudes may be expressed succinctly in the form

$$f_{JJ'KK'}{}^{s(\sigma_1,\sigma_2)}=\sum_{abcd\bar{a}\bar{d}}(J_aJ_a-\bar{a}c|JM)(J_bJ_b-\bar{d}b|J'M')\times C_{\bar{a}a}{}^{J_a}(\sigma_1,s)C_{\bar{d}d}{}^{J_b}(\sigma_2,s)f_{cd;ab}{}^{s} \quad (2.8c)$$

and

$$f_{JJ'KK'}{}^{t(\sigma_1,\sigma_2)}=\sum_{A'b'c'D'\bar{A}'\bar{D}'}(J_aJ_a-\bar{A}'c'|JK)(J_bJ_b-\bar{D}'b'|J'K')\times C_{\bar{A}'A'}{}^{J_a}(\sigma_1,t)C_{\bar{D}'D'}{}^{J_b}(\sigma_2,t)f_{c'A';D'b'}{}^{t}, \quad (2.8d)$$

where σ_1, σ_2 are $\pm$ independently. [The superscripts in Eqs. (2.8) and (2.9) follow this notation.] The metric tensors $C_{xy}{}^{J}$ in these equations have the following values:

$$\begin{aligned}C_{xy}{}^{J}(-,s)&=C_{xy}{}^{J}(-,t)=\delta_{xy},\\ C_{xy}{}^{J}(+,s)&=(-1)^{J-y}\delta_{x,-y},\\ C_{xy}{}^{J}(+,t)&=(-1)^{J+y}\delta_{x,-y}.\end{aligned} \quad (2.10)$$

These amplitudes all satisfy the one crossing relation

$$f_{JJ'MM'}{}^{s(\sigma_1,\sigma_2)}=\sum d_{KM}{}^{J}(\chi_c)d_{K'M'}{}^{J'}(\chi_b)f_{JJ'KK'}{}^{t(-\sigma_1,-\sigma_2)}. \quad (2.9b)$$

We see that amplitudes $f_{JJ'MM'}{}^{s(\sigma_1,\sigma_2)}$ labeled by $\sigma_1=-$ contain only amplitudes with helicity difference at the a-c vertex equal to M, whereas those with $\sigma_1=+$ contain only amplitudes with helicity sum equal to M (and similarly for M' and the b-d vertex). In this notation, therefore, the amplitudes useful for study of direct-channel helicity flip are the $f_{JJ'MM'}{}^{s(-,-)}$, Eq. (2.8a), which we obtained first.

Note that the character σ of a vertex changes sign under crossing in Eq. (2.9b). This can be understood by the following argument: Eq. (2.9b) can be interpreted as the crossing relation for one spin-J "particle" and one spin-J' "particle." Hence we are first combining the individual particle spins into a total spin (at each vertex) and then performing the crossing operation. Clebsch-Gordan coefficients $(J_1J_2M_1M_2|JM)$ combine two representations J_1 and J_2 of the rotation group to make representation J in a particular way. In the crossed channel, however, we have the antiparticle represented by the conjugate representation of spin J_1. In order to use the same coupling by Clebsch-Gordan coefficients, we must first convert the conjugate representation to a standard one. This involves the metric tensor $C_{xy}{}^J(+$, appropriate channel$)$,[5] and multiplication by $C_{xy}{}^J(+)$ changes the sign of the character σ. (The argument is somewhat more transparent if one uses transversity amplitudes, because in that case all the spin projections are taken on the same axis.)

The t-channel couplings have fairly simple physical interpretations. If the character σ of, say, the c-a vertex is minus, we are coupling together spins J_a and J_c to form total spin J in the standard way. If the character σ is plus, we are using the "contravariant" coupling.[5] It is more difficult to make a physical interpretation of the s-channel couplings because the helicities of (for example) c and a do not represent projections of spin along a single direction. However, if one chooses the normal to the scattering plane as the direction of spin quantization, by using transversity amplitudes, it becomes clear that the $\sigma=-$ amplitudes in the s channel again correspond to standard coupling of spins at a (t-channel) vertex, whereas the $\sigma=+$ amplitudes correspond to "contravariant" coupling.

Simplification of the matrix by the trick we use does not require $J_b=J_d$, $J_a=J_c$. We concentrate on this case because it applies to elastic scattering, discussed in detail in the following paper. Formulas for the more general case $m_b=m_d$, $m_a=m_c$, with all spins different, are summarized in the Appendix. These may be useful in treatment of more general reactions.

III. SYMMETRY PROPERTIES OF $f_{JJ'MM'}{}^{s(\sigma_1,\sigma_2)}$ AND $f_{JJ'MM'}{}^{t(\sigma_1,\sigma_2)}$

The properties of helicity amplitudes which follow from parity, time-reversal, and charge-conjugation invariance[6] are

$$f_{-c-d;-a-b}{}^{s,t}=(-1)^{a-b-c+d}f_{cd;ab}{}^{s,t} \quad \text{(parity)},$$
$$f_{ab;cd}{}^{s}=(-1)^{a-b-c+d}f_{cd;ab}{}^{s} \quad \text{(time reversal)},$$
$$f_{dc;ba}{}^{t}=(-1)^{a-b-c+d}f_{cd;ab}{}^{t} \quad \text{(charge conjugation)}.$$

These can be combined with symmetry properties of Clebsch-Gordan coefficients

$$(J_1J_2M_1M_2|JM) =(-1)^{J_1+J_2-J}(J_1J_2-M_1-M_2|J-M), \quad (3.1\text{a})$$
$$(J_1J_2M_1M_2|JM) =(-1)^{J_1+J_2-J}(J_2J_1M_2M_1|JM), \quad (3.1\text{b})$$

to yield relationships between our amplitudes.

From the behavior of helicity amplitudes under *parity*, we can deduce

$$f_{JJ'MM'}{}^{s(\sigma_1,\sigma_2)} =(-1)^{2J_a+2J_b-J-J'-M-M'}f_{JJ'-M-M'}{}^{s(\sigma_1,\sigma_2)} \quad (3.2\text{a})$$

and

$$f_{JJ'MM'}{}^{t(\sigma_1,\sigma_2)} =(-1)^{2J_a+2J_b-J-J'-M-M'}f_{JJ'-M-M'}{}^{t(\sigma_1,\sigma_2)}. \quad (3.2\text{b})$$

Symmetry of elastic scattering (the direct channel) under *time reversal* gives

$$f_{JJ'MM'}{}^{s(+,+)}=(-1)^{2J_a+2J_b-J-J'}f_{JJ'MM'}{}^{s(+,+)}, \quad (3.3\text{a})$$
$$f_{JJ'MM'}{}^{s(-,-)}=(-1)^{-M-M'}f_{JJ'-M-M'}{}^{s(-,-)}, \quad (3.3\text{b})$$
$$f_{JJ'MM'}{}^{s(+,-)}=(-1)^{2J_a-J}(-1)^{-M'}f_{JJ'M-M'}{}^{s(+,-)}, \quad (3.3\text{c})$$
$$f_{JJ'MM'}{}^{s(-,+)}=(-1)^{-M}(-1)^{2J_b-J'}f_{JJ'-MM'}{}^{s(-,+)}. \quad (3.3\text{d})$$

The corresponding symmetry in the t channel is *charge conjugation*. This gives

$$f_{JJ'MM'}{}^{t(+,+)}=(-1)^{2J_a+2J_b-J'-J}f_{JJ'MM'}{}^{t(+,+)}, \quad (3.4\text{a})$$
$$f_{JJ'MM'}{}^{t(-,-)}=(-1)^{-M-M'}f_{JJ'-M-M'}{}^{t(-,-)}, \quad (3.4\text{b})$$
$$f_{JJ'MM'}{}^{t(+,-)}=(-1)^{2J_a-J}(-1)^{-M'}f_{JJ'M-M'}{}^{t(+,-)}, \quad (3.4\text{c})$$
$$f_{JJ'MM'}{}^{t(-,+)}=(-1)^{-M}(-1)^{2J_b-J'}f_{JJ'-MM'}{}^{t(-,+)}. \quad (3.4\text{d})$$

We shall be concerned primarily with strong interactions, where all three symmetries are present. For this case, the above answers can be checked by use of the CPT theorem: Lorentz invariance (the crossing relations) plus two of the three symmetries, P, C, and T, guarantee the third symmetry.

By combining the results of C and P symmetry, we see that only those $f_{JJ'MM'}{}^{t(-,-)}$ are populated which have $2J_a-J+2J_b-J'$ an even integer. This can easily be derived by physical arguments. If $\bar{a}$ and c are in orbital state L_{out} in the t channel, and in total spin state J, then the parity of the $\bar{a}c$ state is $P=(-1)^{2J_a+L_{\text{out}}}$, and the charge conjugation is $C=(-1)^{J+L_{\text{out}}}$. Similar formulas (with L_{in} replacing L_{out}) apply to the $\bar{d}b$ state. Hence conservation of parity gives

$$(-1)^{2J_a+L_{\text{out}}}=(-1)^{2J_b+L_{\text{in}}}$$

and invariance under C gives

$$(-1)^{J+L_{\text{out}}}=(-1)^{J'+L_{\text{in}}}.$$

Combining these equations yields $(-1)^{2J_a-J}=(-1)^{2J_b-J'}$, as expected.

Charge-conjugation invariance alone gives the condition $(-1)^{2J_a-J+2J_b-J'}=+1$ for $f_{JJ'MM'}{}^{t(+,+)}$ and $f_{JJ'MM'}{}^{s(-,-)}$. As we do not often use the $\sigma=+$ coupling in nonrelativistic arguments, the corresponding "physical" arguments are less transparent and we will omit them.

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[5] E. P. Wigner, *Group Theory* (Academic, New York, 1959), p. 292.

[6] M. Jacob and G.-C. Wick, Ann Phys. (N. Y.) **7**, 404 (1959).

IV. CONSEQUENCES OF EXCHANGE OF DEFINITE SETS OF QUANTUM NUMBERS

For exchanges of a single Reggeon or particle of spin S, we can write $f_{c'A';D'b'}{}^t = R_{c'A';D'b'}{}^{(t)} d_{D'-b';c'-A'}{}^S(\theta_t)$. The functions $R_{c'A';D'b'}$ have symmetries which depend on the quantum numbers exchanged (see Table I). These can be summarized by introducing the notation $R_{c'A';D'b'} = \eta_p R_{c'A';-D'-b'}$, $R_{c'A';D'b'} = \eta_c R_{c'A';b'D'}$, where $\eta_p = \pm 1$ if $P = \pm(-1)^S$ and $\eta_c = \pm 1$ if $C = \pm(-1)^S$. From these we see that for exchange with $CP = \eta$, $R_{c'A';D'b'} = \eta R_{c'A';-b'-D'}$.

Once we know that only exchanges with a particular set of quantum numbers are present, these relationships lead to additional simple properties of the amplitudes $f_{JJ'MM'}{}^{t(\sigma_1,\sigma_2)}$ and $f_{JJ'MM'}{}^{s(\sigma_1,\sigma_2)}$. There are two types of new relationships: those which hold exactly and those which are true only to highest power in $\cos\theta_t$, and thus are useful for large $\cos\theta_t$. We treat these separately. In each case we list both the constraint on $f_{JJ'MM'}{}^{t(\sigma_1,\sigma_2)}$ and the constraint on $f_{JJ'MM'}{}^{s(-\sigma_1,-\sigma_2)}$ that crossing produces. Sometimes the s-channel results are particularly interesting when expressed in terms of helicity amplitudes. These may be obtained by inverting Eq. (2.8c):

$$\sum_{ad} C_{\bar{a}a}{}^{J_a}(\sigma_1,s) C_{\bar{d}d}{}^{J_b}(\sigma_2,s) f_{cd;ab}{}^s = \sum_{JJ'MM'} (J_a J_a - \bar{a}c | JM) \times (J_b J_b - \bar{d}b | J'M') f_{JJ'MM'}{}^{s(\sigma_1,\sigma_2)}. \quad (4.1)$$

Because of the properties of the tensor $C_{xy}{}^J$, Eq. (2.10), this expression, despite its appearance, involves only one helicity amplitude.

A. Exact Relations

(i) $f_{JJ'MM'}{}^{t(+,+)}$:

$$f_{JJ'MM'}{}^{t(+,+)} = \eta(-1)^M f_{JJ'-MM'}{}^{t(+,+)} = \eta(-1)^{M'} f_{JJ'M-M'}{}^{t(+,+)}, \quad (4.2a)$$

$$f_{JJ'MM'}{}^{s(-,-)} = \eta(-1)^M f_{JJ'-MM'}{}^{s(-,-)} = \eta(-1)^{M'} f_{JJ'M-M'}{}^{s(-,-)}, \quad (4.2b)$$

$$f_{cd;ab}{}^s = \eta(-1)^{c-a} f_{ad;cb}{}^s = \eta(-1)^{d-b} f_{cb;ad}{}^s. \quad (4.3)$$

For $\eta = -1$, as in pion exchange, Eq. (4.3) shows that only direct-channel helicity-*flip* amplitudes may be populated. This places fairly strong constraints on "dual" models of $CP = -1$ exchanges.

(ii) $f_{JJ'MM'}{}^{t(-,-)}$:

$$f_{JJ'MM'}{}^{t(-,-)} = \eta(-1)^{2J_a-J} f_{JJ'MM'}{}^{t(-,-)} = \eta(-1)^{2J_b-J'} f_{JJ'MM'}{}^{t(-,-)}. \quad (4.4)$$

Thus we see that for $CP = +$ exchanges, the only nonzero $f_{JJ'MM'}{}^{t(-,-)}$ and $f_{JJ'MM'}{}^{s(+,+)}$ occur for $2J_a - J$ and $2J_b - J'$ even; whereas for $CP = -$ exchanges, the other J, J' values are populated. This result may also be derived by physical arguments: For a particle-antiparticle pair of spin J_a constituents, in orbital angular momentum state L, and total spin J, we have $C = (-1)^{L+J}$, $P = (-1)^{2J_a+L}$. Hence $CP = (-1)^{2J_a-J}$. The constraints on the helicity amplitudes are, of course, the same as Eq. (4.3).

(iii) $f_{JJ'MM'}{}^{t(+,-)}$:

$$f_{JJ'MM'}{}^{t(+,-)} = \eta(-1)^M f_{JJ'-MM'}{}^{t(+,-)} = \eta(-1)^{2J_b-J'} f_{JJ'MM'}{}^{t(+,-)}, \quad (4.5a)$$

$$f_{JJ'MM'}{}^{s(-,+)} = \eta(-1)^M f_{JJ'-MM'}{}^{s(-,+)} = \eta(-1)^{2J_b-J'} f_{JJ'MM'}{}^{s(-,+)}. \quad (4.5b)$$

Only those states such that $CP = (-1)^{2J_b-J'}$ are populated, as predicted by the physical argument in (ii). Note that this argument can be applied only to the $\sigma = -$ vertex.

(iv) $f_{JJ'MM'}{}^{t(-,+)}$:

$$f_{JJ'MM'}{}^{t(-,+)} = \eta(-1)^{2J_a-J} f_{JJ'MM'}{}^{t(-,+)} = \eta(-1)^{M'} f_{JJ'M-M'}{}^{t(-,+)}, \quad (4.6a)$$

$$f_{JJ'MM'}{}^{s(+,-)} = \eta(-1)^{2J_a-J} f_{JJ'MM'}{}^{s(+,-)} = \eta(-1)^{M'} f_{JJ'M-M'}{}^{s(+,-)}. \quad (4.6b)$$

B. Relations Good to Highest Order in $\cos\theta_t$

There are relatively few exact relations because, for given values of σ_1 and σ_2, the $f_{JJ'MM'}{}^{t(\sigma_1,\sigma_2)}$ for different M tend to involve different $d_{\lambda\mu}{}^\alpha$ functions. However, at very large values of $\cos\theta_t$, $d_{\lambda\mu}{}^\alpha(\theta_t) \approx (-1)^{-\lambda} d_{-\lambda\mu}{}^\alpha(\theta_t)$. This allows us to obtain more relations. Many of these relations will be used in the accompanying paper about suppression of energy dependence in s-channel amplitudes.

(i) $f_{JJ'MM'}{}^{t(+,+)}$: By applying C,

$$f_{JJ'MM'}{}^{t(+,+)} = [1 + \eta_c(-1)^{2J_a-J}] O(s^\alpha) + (\text{lower-order terms}), \quad (4.7a)$$

$$f_{JJ'MM'}{}^{s(-,-)} = [1 + \eta_c(-1)^{2J_a-J}] O(s^\alpha) + (\text{lower-order terms}), \quad (4.7b)$$

and similar relationships with $J_a \to J_b$, $J \to J'$. Thus, the only $f_{JJ'MM'}{}^{s(-,-)}$ populated to order s^α are those with J, J' such that $\eta_c = (-1)^{2J_b-J'}$, $\eta_c = (-1)^{2J_a-J}$. The others are of order $s^{\alpha-1}$.

TABLE I. Symmetries of t-channel partial-wave amplitudes for exchange with spin S.

Quantum number of exchange	Symmetry
$P = (-1)^S$	$R_{c'A';D'b'} = R_{c'A';-D'-b'}$
$P = -(-1)^S$	$R_{c'A';D'b'} = -R_{c'A;-D'-b'}$
$C = (-1)^S$	$R_{c'A';D'b'} = R_{c'A';b'D'}$
$C = -(-1)^S$	$R_{c'A';D'b'} = -R_{c'A';b'D'}$

By applying P, or alternatively combining (4.7) with exact relation (4.2a), we get to order s^α

$$f_{JJ'MM'}{}^{t(+,+)} \sim (-1)^{2J_a-J}\eta_P(-1)^M f_{JJ'-MM'}{}^{t(+,+)}, \quad (4.8a)$$

$$f_{JJ'MM'}{}^{s(-,-)} \sim (-1)^{2J_a-J}\eta_P(-1)^M f_{JJ'-MM'}{}^{s(-,-)}, \quad (4.8b)$$

and relations with $J_a \to J_b$, $J\to J'$.

(ii) $f_{JJ'MM'}{}^{t(-,-)}$: By applying P and C, we get, respectively,

$$f_{JJ'MM'}{}^{t(-,-)} \sim (-1)^M(-1)^{2J_a-J}\eta_P f_{JJ'-MM'}{}^{t(-,-)} = (-1)^M\eta_C f_{JJ'-MM'}{}^{t(-,-)}, \quad (4.9a)$$

$$f_{JJ'MM'}{}^{s(+,+)} \sim (-1)^M(-1)^{2J_a-J}\eta_P f_{JJ'-MM'}{}^{s(+,+)} = (-1)^M\eta_C f_{JJ'-MM'}{}^{s(+,+)}, \quad (4.9b)$$

and relations with $J_a \to J_b$, $J\to J'$.

(iii) $f_{JJ'MM'}{}^{t(+,-)}$: By applying C at the upper vertex,

$$f_{JJ'MM'}{}^{t(+,-)} = [1+(-1)^{2J_a-J}\eta_C]O(s^\alpha) + (\text{lower-order terms}), \quad (4.10)$$

and by applying C at the lower vertex,

$$f_{JJ'MM'}{}^{t(+,-)} \sim \eta_C(-1)^{+M'} f_{JJ'M-M'}{}^{t(+,-)}. \quad (4.11)$$

(iv) $f_{JJ'MM'}{}^{t(-,+)}$: By applying C at the upper vertex,

$$f_{JJ'MM'}{}^{t(-,+)} \sim \eta_C(-1)^{+M} f_{JJ'-MM'}{}^{t(-,+)}, \quad (4.12)$$

and at the lower vertex,

$$f_{JJ'MM'}{}^{t(-,+)} = [1+\eta_C(-1)^{2J_b-J'}]O(s^\alpha) + (\text{lower-order terms}). \quad (4.13)$$

At this point we can see a pattern developing: $\sigma=+$ vertices are present to orders s^α only if $\eta_C=(-1)^{2J_i-J}$ for J_i the constituents' spin and J the resultant spin. This is in contrast to the *exact* relation for $\sigma=-$ vertices: $\eta=(-1)^{2J_i-J}$ or the contribution is zero.

Likewise, for $\sigma_2=-$ vertices, to order s^α, the amplitudes are related by

$$f_{JJ'MM'}{}^{t(\sigma_1,-)} \sim \eta_C(-1)^{M'} f_{JJ'M-M'}{}^{t(\sigma_1,-)}.$$

The corresponding exact relation for $\sigma_2=+$ amplitudes is

$$f_{JJ'MM'}{}^{t(\sigma_1,+)} = \eta(-1)^{M'} f_{JJ'M-M'}{}^{t(\sigma,+)}.$$

V. FURTHER SIMPLIFICATIONS IF INCIDENT PARTICLES ARE IDENTICAL

There are two essential simplifications if the two s-channel incident particles happen to be identical, as in the case of pp scattering. First, we can invoke the Pauli principle in the s channel and time-reversal invariance in the t channel. Second, the two crossing angles χ_c and χ_b are now identical. Hence the two remaining $d_{\lambda\mu}{}^J$ functions can be combined to form one, and new amplitudes defined which cross according to this rotation matrix.

(i) Translating the behavior of helicity amplitudes into our language, we find that the Pauli principle in the direct channel gives

$$f_{JJ'MM'}{}^{s(+,+)} = (-1)^{-J-J'} f_{J'JM'M}{}^{s(+,+)}, \quad (5.1a)$$

$$f_{JJ'MM'}{}^{s(-,-)} = (-1)^{-M-M'} f_{J'J-M'-M}{}^{s(-,-)}, \quad (5.1b)$$

$$f_{JJ'MM'}{}^{s(+,-)} = (-1)^{-M'}(-1)^{2J_a-J} f_{J'J-M'M}{}^{s(-,+)}, \quad (5.1c)$$

$$f_{JJ'MM'}{}^{s(-,+)} = (-1)^{-M}(-1)^{2J_a-J'} f_{J'JM'-M}{}^{s(+,-)}. \quad (5.1d)$$

When these properties are combined in various ways with the other symmetry relations in Sec. III, we obtain as the common result

$$f_{JJ'MM'}{}^{s(\sigma_1,\sigma_2)} = f_{J'JM'M}{}^{s(\sigma_2,\sigma_1)}. \quad (5.2)$$

The corresponding t-channel symmetry is time-reversal invariance. In a similar fashion, in combination with the C relations of Sec. III, it yields

$$f_{JJ'MM'}{}^{t(\sigma_1,\sigma_2)} = f_{J'JM'M}{}^{t(\sigma_2,\sigma_1)}. \quad (5.3)$$

(ii) The combination of the two d functions remaining in the crossing matrix can be achieved in two ways, as we expect from the methods of Sec. II. We denote the two possible combinations by yet another superscript σ_3, $\sigma_3=\pm$. The crossing relation becomes

$$f_{JM}{}^{s(\sigma_1\sigma_2\sigma_3)}(J'',J''') = \sum d_{KM}{}^J(\chi) f_{JK}{}^{t(-\sigma_1-\sigma_2\sigma_3)}(J',J''),$$

where, for both s- and t-channel amplitudes,

$$f_{JM}{}^{(\sigma_1\sigma_2\sigma_3)}(J'',J''') = \sum_{M'M''M'''} (J''J'''M''M'''|JM) \times C_{M''M'}{}^{J''}(-\sigma_3, s) f_{J''J'''M'M'''}{}^{(\sigma_1\sigma_2)}.$$

Here C^J is the tensor defined in (2.10).

The implications of various symmetries for the combined amplitudes are summarized in Table II. Note that when all symmetries are taken into account, the amplitudes $f_{JM}{}^{s(\sigma,\sigma)}(J'',J''')$ are nonzero only for $J''+J'''$ even. If $J''=J'''$, then J must be even in addition. Use of the combined amplitudes with these rules allows a quick counting of independent amplitudes in processes like $\rho\rho$ scattering.

VI. SUMMARY AND CONCLUSIONS

A simplification in the crossing matrix allowed for elastic scattering (more accurately, for scattering where the *masses* are equal in pairs) is exploited to construct new amplitudes with simpler crossing properties. These amplitudes prove very convenient for discussing s-channel consequences of t-channel Reggeon exchanges, and some simple selection rules appear. An application of this formalism may be found in the following paper.[2]

Our construction uses properties and Clebsch-Gordan coefficients of only the three-dimensional rotation group. The possibility that our results for elastic scattering are a simple special case of more complicated Lorentz-group properties of the general case has not

TABLE II. Symmetries of combined amplitudes when incident particles are identical. (1) Parity (in both s and t channels): $f_{JM}^{(\sigma_1,\sigma_2,\sigma_3)}(J'',J''')=(-1)^{-J-M}f_{J-M}^{(\sigma_1,\sigma_2,\sigma_3)}(J'',J''')$. (2) Other symmetries. Amplitudes on a given line of the table are equal. For s-channel amplitudes, column B equals column A by the Pauli principle, and column C equals column A by time-reversal invariance. For t-channel amplitudes, column B equals column A by time-reversal invariance, and column C equals column A by charge-conjugation invariance.

A	B	C
$f_{JM}^{(+,+,+)}(J''.J''')$	$(-1)^{-J}f_{JM}^{(+,+,+)}(J'',J''')$	$(-1)^{-J''-J'''}f_{JM}^{(+,+,+)}(J'',J''')$
$f_{JM}^{(+,+,-)}(J'',J''')$	$(-1)^{-M}f_{J-M}^{(+,+,-)}(J'',J''')$	$(-1)^{-J''-J'''}f_{JM}^{(+,+,-)}(J'',J''')$
$f_{JM}^{(-,-,+)}(J'',J''')$	$(-1)^{-M}f_{J-M}^{(-,-,+)}(J'',J''')$	$(-1)^{J''+J'''-J-M}f_{J-M}^{(-,-,+)}(J'',J''')$
$f_{JM}^{(-,-,-)}(J'',J''')$	$(-1)^{-J}f_{J-M}^{(-,-,-)}(J'',J''')$	$(-1)^{J''+J'''-J-M}f_{J-M}^{(-,-,-)}(J'',J''')$
$f_{JM}^{(+,-,+)}(J'',J''')$	$(-1)^{2J_a-J}f_{JM}^{(-,+,-)}(J'',J''')$	$(-1)^{2J_a+J''+J'''-J-M}f_{J-M}^{(+,-,-)}(J'',J''')$
$f_{JM}^{(+,-,-)}(J'',J''')$	$(-1)^{2J_a-M}f_{J-M}^{(-,+,+)}(J'',J''')$	$(-1)^{2J_a+J''+J'''-J-M}f_{J-M}^{(+,-,+)}(J'',J''')$
$f_{JM}^{(-,+,+)}(J'',J''')$	$(-1)^{2J_a-M}f_{J-M}^{(+,-,-)}(J'',J''')$	$(-1)^{2J_a+J''+J'''}f_{JM}^{(-,+,-)}(J'',J''')$
$f_{JM}^{s(-,+,-)}(J'',J''')$	$(-1)^{2J_a-J}f_{JM}^{(+,-,+)}(J'',J''')$	$(-1)^{2J_a+J''+J'''}f_{JM}^{(-,+,+)}(J'',J''')$

escaped us.[7] The exploration of more complicated reactions, either by finding this property or by more elementary expansions about the elastic scattering case, is a task we hope to pursue.

APPENDIX

In the combination of rotation functions performed in the main part of this paper, the requirement that $J_a=J_c$, $J_b=J_d$ is not necessary, although it simplifies the formalism and the phase factors. We give here expressions obtained for general spins.

Combined amplitudes are

$$f_{JJ'MM'}^{s(\sigma_1\sigma_2)}=\sum_{abcd\bar{a}\bar{d}}(J_aJ_c-\bar{a}c|JM)(J_dJ_b-\bar{d}b|J'M')\times C_{\bar{a}a}^{J_a}(\sigma_1,s)C_{\bar{d}d}^{J_d}(\sigma_2,s)f_{cd;ab}^{s},\quad(2.8\text{c}')$$

$$f_{JJ'KK'}^{t(\sigma_1\sigma_2)}=\sum_{A'b'c'D'\bar{A}'\bar{D}'}(J_aJ_c-\bar{A}'c'|JK)(J_dJ_b-\bar{D}'b'|J'K')\times C_{\bar{A}'A'}^{J_a}(\sigma_1,t)C_{\bar{D}'D'}^{J_d}(\sigma_2,t)f_{c'A';D'b'}^{t},\quad(2.8\text{d}')$$

and they also cross according to (2.9b).

Symmetry under *parity* is given by

$$f_{JJ'MM'}^{(\sigma_1\sigma_2)}=\eta_g(-1)^{\Sigma_iJ_i-J-J'+M+M'}f_{JJ'-M-M'}^{(\sigma_1\sigma_2)}\quad(3.2')$$

for both s- and t-channel amplitudes. Here $\sum_i J_i$ is the sum of the four particle spins, and η_g is the intrinsic parity factor:

$$\eta_g=(\eta_c\eta_d/\eta_a\eta_b)(-1)^{J_c+J_d-J_a-J_b}$$

for the s channel, with a corresponding one for the t channel. Time reversal in the s channel and charge conjugation in the t channel are no longer symmetries for the general spin case.

In considering the consequences of the t-channel exchange of a particle with parity factor η_P, and charge-conjugation η_C ($\eta_P\eta_C=\eta$), the case where the particles at one vertex are a particle-antiparticle pair still retains half of the symmetries of the elastic scattering case. If we choose c,A as this pair, those relationships in Sec. IV which arise by applying CP, or C or P at the upper vertex still apply [i.e., the first equalities in (4.2)–(4.6), and the approximate equalities cited in (4.7)–(4.10) and (4.12)]. The conclusions drawn at the end of Sec. IV, applied to the upper vertex (i.e., referring to σ_1) are still valid.

In the case where all four s-channel particles are distinguishable, only the parity-exchange symmetry remains. Specifically, to highest order in $\cos\theta_t$, the only t-channel symmetries are

$$f_{JJ'MM'}^{t(\sigma_1,\sigma_2)}\sim(-1)^{J_a+J_c-J}\eta_P(-1)^{M}f_{JJ'-MM'}^{t(\sigma_1,\sigma_2)},$$

with its s-channel consequence

$$f_{JJ'MM'}^{s(-\sigma_1,-\sigma_2)}\sim(-1)^{J_a+J_c-J}\eta_P(-1)^{M}f_{JJ'-MM'}^{s(-\sigma_1,-\sigma_2)},$$

and the obvious corresponding results for the lower vertex. The lack of other symmetries results in all $f_{JJ'MM'}^{t(\sigma_1,\sigma_2)}$ being populated to order s^{α}.

[7] The results involving direct-channel helicity-flip amplitudes are closely related to work by a number of authors [K. Bitar and G. L. Tindle, Phys. Rev. **175**, 1835 (1968); S. A. Klein, Phys. Rev. D **1**, 609 (1970); L. Durand III, P. M. Fishbane, S. A. Klein, and L. M. Simmons, Jr., Phys. Rev. Letters **23**, 201 (1969)]. The direct-channel helicity sum amplitudes do not appear to have been previously discussed.