reduces to

$$\mathbf{p}=-e\mathbf{A}+(e^2\mathbf{A}^2/2m)(\mathbf{k}/\omega)\quad\text{(radiation gauge)},$$
$$E=p_0=m+(e^2\mathbf{A}^2/2m),$$

which agrees with Refs. 1 and 2.

We would like to point out that from Eq. (6) we obtain $p^2=-m^2$ (without any averaging), as we must since the original equation of motion (1) gives $p_\mu dp_\mu/d\tau=0$, a condition built into all classical equations of motion.[4] Eberly and Sleeper[1] consider the quantity

$$(P_\mu{}^{\text{can}})^2=(p_\mu+eA_\mu)^2=-m^2+2eA\cdot p^0-2e^2A\cdot A$$

and define an "interacting" mass by an average

$$-m^{*2}=\langle(P_\mu{}^{\text{can}})^2\rangle,$$

but since $(P_\mu{}^{\text{can}})^2$ is not a gauge-invariant quantity, it cannot have physical significance. To obtain the usual quantum-field-theory mass shift,[5] it would be better to consider the average of p_μ, since the usual momentum experiment (especially in the light of quantum mechanics) would automatically perform an average over many cycles of any optical frequency present. This procedure gives

$$\langle p_\mu\rangle=p_\mu{}^0-e^2\langle A\cdot A\rangle k_\mu/2k\cdot p^0$$

and

$$\langle p\rangle\cdot\langle p\rangle=-m^2-e^2\langle A\cdot A\rangle,$$

the mass shift found in Ref. 5.

[4] The last sentence in Sec. II A of Ref. 2 is a misprint (private communication). It should read $\langle E^2\rangle=\langle\mathbf{P}^2\rangle=m^2+\frac{1}{2}q^2m^2$.

[5] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).

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Note on the Scale Transformation*

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We derive a simple dimensional relation on the canonical energy-momentum tensor. This relation is used to express the deficiency of the scale invariance. It is shown that the scale transformation is not compatible with the Lorentz covariance in some cases which lie within the framework of the renormalizable field theory.

I. INTRODUCTION

The relevance of the notion of scale transformation to high-energy physics was noted sometime ago,[1,2] and its interest has revived in recent years.[3] However, as was noticed by many authors, the invariance of the theory under the scale transformation is badly broken due to the presence of the mass, and in some cases the coupling constant. One of the purposes of this paper is to examine how the presence of mass destroys the scale invariance. First, we derive a simple dimensional relation on the canonical energy-momentum tensor in Sec. II. The relation is then exploited in Sec. III to express the deficiency of the invariance in terms of the masses and the coupling constants. Sections IV and V are devoted to showing that even in renormalizable field theory the notion of scale transformation is not compatible with the Lorentz covariance. This is because in some field theories such as the Duffin-Kemmer and the Proca theory, the mass-zero limit does not exist and because of the presence of the mass the scale transformation becomes incompatible with Lorentz covariance. This incompatibility occurs unfortunately in unrenormalizable as well as in some renormalizable field theories.

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[1] J. M. Jauch, in Proceedings of the Theoretical Seminar, State University of Iowa, Iowa City, Iowa, 1956 (unpublished).
[2] Y. Takahashi and H. Umezawa, Nuovo Cimento **6**, 1324 (1958); **6**, 1382 (1958).
[3] S. P. de Alwis and P. J. O'Donnell, Phys. Rev. D **2**, 1023 (1970). Other references can be traced from here.

II. EULER EQUATION

We shall begin our discussion with the dimensional analysis of the canonical energy-momentum tensor. Consider a Lagrangian containing field variables, and their first-order derivatives:

$$L=L(\phi^{(\kappa)}(x),\partial_\mu\phi^{(\kappa)}(x),m^{(\kappa)},f_i),\qquad(2.1)$$

where we have written the mass $m^{(\kappa)}$ and the coupling constants f_i explicitly.[4]

The canonical energy-momentum tensor is

$$T_{\mu\nu}(x)=\sum_\kappa\frac{\partial L}{\partial\partial_\mu\phi^{(\kappa)}(x)}\partial_\nu\phi^{(\kappa)}(x)-\delta_{\mu\nu}L.\qquad(2.2)$$

[4] The field variable $\phi^{(\kappa)}(x)$ is in general of multicomponent form. However, we omit indices for the sake of simplicity.

By the aid of the Euler equation

$$4L=\sum_\kappa l^{(\kappa)}\frac{\partial L}{\partial\phi^{(\kappa)}}\phi^{(\kappa)}+\sum_\kappa(l^{(\kappa)}+1)\frac{\partial L}{\partial\partial_\mu\phi^{(\kappa)}}\partial_\mu\phi^{(\kappa)}+\sum_\kappa m^{(\kappa)}\frac{\partial L}{\partial m^{(\kappa)}}-\sum_i\eta_i f_i\frac{\partial L}{\partial f_i},\quad(2.3)$$

we immediately obtain

$$T_{\mu\mu}(x)=-\sum_\kappa l^{(\kappa)}\frac{\partial L}{\partial\phi^{(\kappa)}}\phi^{(\kappa)}+\frac{\partial L}{\partial\partial_\mu\phi^{(\kappa)}}\partial_\mu\phi^{(\kappa)}-\sum_\kappa m^{(\kappa)}\frac{\partial L}{\partial m^{(\kappa)}}+\sum_i\eta_i f_i\frac{\partial L}{\partial f_i},\quad(2.4)$$

where we have assigned the dimension as (in natural units)

$$[\phi^{(\kappa)}]=L^{-l^{(\kappa)}},\quad[m^{(\kappa)}]=L^{-1},\quad[f_i]=L^{\eta_i}.\quad(2.5)$$

The Euler-Lagrange equation enables us to rewrite (2.4) as

$$T_{\mu\mu}(x)=-\sum_\kappa l^{(\kappa)}\partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi^{(\kappa)}}\phi^{(\kappa)}\right)-\sum_\kappa m^{(\kappa)}\frac{\partial L}{\partial m^{(\kappa)}}+\sum_i\eta_i f_i\frac{\partial L}{\partial f_i},\quad(2.6)$$

$$\partial_\mu\left(T_{\mu\nu}x_\nu+\sum_\kappa l^{(\kappa)}\frac{\partial L}{\partial\partial_\mu\phi^{(\kappa)}}\phi^{(\kappa)}\right)=-\sum_\kappa m^{(\kappa)}\frac{\partial L}{\partial m^{(\kappa)}}+\sum_i\eta_i f_i\frac{\partial L}{\partial f_i}.\quad(2.6')$$

We further note that the 4-divergence term on the right-hand side can be eliminated by redefining the Lagrangian

$$L'\equiv L-\sum_\kappa\frac{l^{(\kappa)}}{3}\partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi^{(\kappa)}}\phi^{(\kappa)}\right).\quad(2.7)$$

Since the new Lagrangian contains second-order derivatives, the canonical energy-momentum tensor has to be redefined as

$$T_{\mu\nu}'(x)=\sum_\kappa\frac{\partial L'}{\partial\partial_\mu\phi^{(\kappa)}}\partial_\nu\phi^{(\kappa)}+\frac{\partial L'}{\partial\partial_\mu\partial_\lambda\phi^{(\kappa)}}\partial_\lambda\partial_\nu\phi^{(\kappa)}-\sum_\kappa\partial_\lambda\frac{\partial L'}{\partial\partial_\mu\partial_\lambda\phi^{(\kappa)}}\partial_\nu\phi^{(\kappa)}-\delta_{\mu\nu}L'.$$

The use of the explicit form (2.7) then yields

$$T_{\mu\mu}'(x)=-\sum_\kappa m^{(\kappa)}\frac{\partial L}{\partial m^{(\kappa)}}+\sum_i\eta_i f_i\frac{\partial L}{\partial f_i}.$$

Note that the right-hand side involves only the old Lagrangian. This relation was used previously in connection with the discussion of the self-stress.[5,5a]

III. GENERATOR OF SCALE TRANSFORMATION

Under the infinitesimal scale transformation

$$x_\mu\longrightarrow x_\mu'=(1+\epsilon)x_\mu,\quad(3.1)$$

the field is assumed to undergo the transformation

$$\phi^{(\kappa)}(x)\longrightarrow\phi^{(\kappa)\prime}(x')=(1-l^{(\kappa)}\epsilon)\phi^{(\kappa)}(x),\quad(3.2)$$

where $l^{(\kappa)}$ is defined by (2.5). The current associated with this transformation is[3]

$$S_\mu(x)=T_{\mu\nu}(x)x_\nu+\sum_\kappa l^{(\kappa)}\frac{\partial L}{\partial\phi_\mu{}^{(\kappa)}(x)}\phi^{(\kappa)}(x).\quad(3.3)$$

Taking the divergence, we obtain

$$\partial_\mu S_\mu(x)=T_{\mu\mu}(x)+\sum_\kappa l^{(\kappa)}\partial_\mu\left(\frac{\partial L}{\partial\phi_\mu{}^{(\kappa)}(x)}\phi^{(\kappa)}(x)\right),\quad(3.4)$$

on account of the conservation of the canonical energy-momentum tensor

$$\partial_\mu T_{\mu\nu}(x)=0.\quad(3.5)$$

If we make use of the dimensional relation (2.6), we arrive at

$$\partial_\mu S_\mu(x)=-\sum_\kappa m^{(\kappa)}\frac{\partial L}{\partial m^{(\kappa)}}+\sum_i\eta_i f_i\frac{\partial L}{\partial f_i},\quad(3.6)$$

which may be called the *scale deficiency*.

The generator of the scale transformation

$$D(\sigma)\equiv\int d\sigma_\lambda(x)S_\lambda(x)\quad(3.7)$$

is then shown to satisfy[6]

$$[D(\sigma),P_\mu]=iP_\mu-i\int d\sigma_\mu(x)\partial_\lambda\partial S_\lambda(x),\quad(3.8)$$

$$[D(\sigma),M_{\mu\nu}]=i\int d\sigma_\mu(x)x_\nu\partial_\lambda S_\lambda(x)-i\int d\sigma_\nu(x)x_\mu\partial_\lambda S_\lambda(x),\quad(3.9)$$

where P_μ and $M_{\mu\nu}$ are generators of the Poincaré group. If the scale deficiency vanishes, i.e., if the theory is scale invariant, we obtain

$$[D,P_\mu]=iP_\mu,\quad[D,M_{\mu\nu}]=0,\quad(3.10)$$

[5] J. Strathdee and Y. Takahashi, Nucl. Phys. **8**, 113 (1958).
[5a] *Note added in proof.* A similar argument on the dimension of a vector field is found in M. A. B. Beg *et al.*, Phys. Rev. Letters **25**, 1231 (1970).
[6] Y. Takahashi, Proc. Roy. Irish Acad. (to be published).

which agrees with the result obtained previously.[7] From Eq. (3.6), it seems clear that if there is no mass and no dimensional coupling constant, the scale deficiency vanishes. However, as is well known, most of the massive field equations we encounter in relativistic field theory do not allow a continuous transition to the massless case. We shall therefore examine how the generator $D(\sigma)$ induces the transformation on such fields.

IV. DUFFIN-KEMMER FIELD WITH SPIN ZERO

Since we defined the generator $D(\sigma)$ so that the field $\phi^{(\kappa)}(x)$ transforms according to (3.2), we would expect the relation

$$[\phi^{(\kappa)}(x), D(\sigma)] = i(x_\nu\partial_\nu + l^{(\kappa)})\phi^{(\kappa)}(x). \qquad (4.1)$$

However, this is not the case, as will be seen below.

Let us first take the Duffin-Kemmer equation of a spin-0 field

$$(\beta_\lambda\partial_\lambda + m)\psi(x) = 0, \qquad (4.2)$$

where the β_λ are 5×5 matrices satisfying[8]

$$\beta_\mu\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda\beta_\mu = \delta_{\mu\lambda}\beta_\nu + \delta_{\nu\lambda}\beta_\mu. \qquad (4.3)$$

In particular, the relation (4.3) yields

$$\beta_4{}^3 = \beta_4, \qquad (4.4)$$

$$\beta_4\beta_i\beta_4 = 0, \qquad (4.5)$$

$$\beta_i(1-\beta_4{}^2) = \beta_4{}^2\beta_i, \qquad (4.6)$$

$$(1-\beta_4{}^2)\beta_i = \beta_i\beta_4{}^2, \qquad (4.7)$$

with $i=1, 2, 3$. In this case, we put $l=\frac{3}{2}$. Namely, the field undergoes the transformation

$$\psi(x) \to \psi'(x') = (1-\tfrac{3}{2}\epsilon)\psi(x). \qquad (4.8)$$

The current is then given by

$$S_\mu(x) = -\bar{\psi}(x)\beta_\mu(x_\nu\partial_\nu + \tfrac{3}{2})\psi(x). \qquad (4.9)$$

We thus obtain

$$[\psi(x), S_\mu(x')] = -id(\partial)\Delta(x-x')\beta_\mu(x_\nu'\partial_\nu' + \tfrac{3}{2})\psi(x'), \qquad (4.10)$$

with

$$d(\partial) = -[m^{-1}(\Box - m^2) + \beta_\lambda\partial_\lambda - m^{-1}(\beta_\lambda\partial_\lambda)^2]. \qquad (4.11)$$

At equal time, we have,

$$d(\partial)\Delta(x-x') = -i(\beta_4 - m^{-1}\beta_i\beta_4\partial_i - m^{-1}\beta_4\beta_i\partial_i)\delta(\mathbf{x}-\mathbf{x}'), \qquad (4.12)$$

which gives

$$\begin{aligned}[\psi(x), S_4(x')]_{t=t'} &= -(\beta_4{}^2 - m^{-1}\beta_i\beta_4{}^2\partial_i) \\ &\quad \times\delta(\mathbf{x}-\mathbf{x}')(x_\nu'\partial_\nu' + \tfrac{3}{2})\psi(x') \\ &= -[\beta_4{}^2 - (1-\beta_4{}^2)m^{-1}\beta_i\partial_i]\delta(\mathbf{x}-\mathbf{x}')(x_\nu'\partial_\nu' + \tfrac{3}{2})\psi(x'),\end{aligned} \qquad (4.13)$$

where (4.5) and (4.7) have been used. However, we have

$$(1-\beta_4{}^2)\beta_i\partial_i(x_\nu\partial_\nu + \tfrac{3}{2})\psi(x) = -m(x_\nu\partial_\nu + \tfrac{5}{2})(1-\beta_4{}^2)\psi(x), \qquad (4.14)$$

which is a consequence of (4.2) and (4.3). Substituting (4.14) into (4.13) and integrating over $\mathbf{x}'$, we obtain

$$\begin{aligned}[\psi(x), D(t)] &= i(x_\nu\partial_\nu + \tfrac{3}{2})\beta_4{}^2\psi(x) \\ &\quad + i(x_\nu\partial_\nu + \tfrac{5}{2})(1-\beta_4{}^2)\psi(x),\end{aligned} \qquad (4.15)$$

with

$$D(t) = i\int_{t=t'} d^3x' S_4(x'). \qquad (4.16)$$

In other words, the components $\beta_4{}^2\psi(x)$ and $(1-\beta_4{}^2)\psi(x)$ transform with $l=\frac{3}{2}$ and $l=\frac{5}{2}$, respectively. This clearly contradicts our original assignment (4.8). The reason is the following. Because of the properties (4.4)–(4.7), we can split Eq. (4.3) into two:

$$[\beta_4\partial_4 - m^{-1}(\beta_i\partial_i)^2 + m]\beta_4{}^2\psi(x) = 0, \qquad (4.17)$$

$$(1-\beta_4{}^2)\psi(x) = -m^{-1}\beta_i\partial_i\beta_4{}^2\psi(x). \qquad (4.18)$$

The first equation determines $\beta_4{}^2\psi(x)$, and the $(1-\beta_4{}^2)\psi(x)$ component is determined from it by (4.18). As is seen in (4.18), if $\beta_4{}^2\psi(x)$ transforms with $l=\frac{3}{2}$, then $(1-\beta_4{}^2)\psi(x)$ must transform with $l=\frac{5}{2}$, since the scale transformation does not change the mass. As can easily be seen, the fourth component of the current $S_\mu(x)$ contains only the $\beta_4{}^2\psi(x)$ component. Therefore, it was not really necessary to assign the transformation (4.8) to start with. All we had to require was

$$\beta_4{}^2\psi(x) \to \beta_4{}^2\psi'(x') = (1-\tfrac{3}{2}\epsilon)\beta_4{}^2\psi(x), \qquad (4.19)$$

and the other component $(1-\beta_4{}^2)\psi(x)$ is automatically determined by (4.18).

The fact that the two components $\beta_4{}^2\psi(x)$ and $(1-\beta_4{}^2)\psi(x)$ transform differently causes a serious trouble in connection with the Lorentz covariance, since the separation is not a Lorentz-invariant concept. To see this explicitly, we note that the field $\psi(x)$ can be written as[9]

$$\psi(x) = \frac{1}{\sqrt{m}}\begin{pmatrix} -i\partial_4\phi(x) \\ \partial_1\phi(x) \\ \partial_2\phi(x) \\ \partial_3\phi(x) \\ -m\phi(x) \end{pmatrix}, \qquad (4.20)$$

$$\beta_4{}^2\psi(x) = \frac{1}{\sqrt{m}}\begin{pmatrix} -i\partial_4\phi(x) \\ 0 \\ 0 \\ 0 \\ -m\phi(x) \end{pmatrix}, \qquad (4.21)$$

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[7] G. Mack, Nucl. Phys. **B5**, 499 (1968).

[8] The explicit form of β_λ is given, for instance, in Refs. 9 and 11.

[9] Y. Takahashi, *An Introduction to Field Quantization* (Pergamon, New York, 1969).

$$(1-\beta_4{}^2)\psi(x)=\frac{1}{\sqrt{m}}\begin{bmatrix}0\\ \partial_1\phi(x)\\ \partial_2\phi(x)\\ \partial_3\phi(x)\\ 0\end{bmatrix}, \tag{4.22}$$

where $\psi(x)$ is a scalar field satisfying

$$(\Box-m^2)\phi(x)=0. \tag{4.23}$$

It is obvious that the two components are mixed by a Lorentz transformation. Thus, it is not possible to assign the transformation (4.19) in all Lorentz frames.

One can argue that the difficulty mentioned above is due to the Duffin-Kemmer formalism which can be replaced by the ordinary scalar field. Indeed, the difficulty does not arise in scalar and spin-$\frac{1}{2}$ spinor theory (if the interaction is simple enough). However, as is seen below, a similar difficulty arises for a neutral vector field.

V. MASSIVE NEUTRAL VECTOR FIELD

As another example of the difficulty of scale transformation, we take a massive neutral vector field which can interact with a spinor field within the framework of renormalizable theory.[10,11] The field equation is

$$[\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu-m^2\delta_{\mu\nu}]U_\nu(x)=0. \tag{5.1}$$

As before, we assign the transformation property with $l=1$,

$$U_\mu(x)\rightarrow U_\mu{}'(x')=(1-\epsilon)U_\mu(x), \tag{5.2}$$

and construct the current

$$S_\mu(x)=-\tfrac{1}{2}[\partial_\mu U_\sigma(x)-\partial_\sigma U_\mu(x)](x_\nu\partial_\nu+1)U_\sigma(x) +\tfrac{1}{2}U_\sigma(x)(x_\nu\partial_\nu+2)[\partial_\mu U_\sigma(x)-\partial_\sigma U_\mu(x)]. \tag{5.3}$$

The equal-time commutator of $U(x)$ and $S_4(x')$ can easily be calculated by the aid of the formula

$$[U_\lambda(x),U_\rho(x')]=i(\delta_{\lambda\rho}-m^{-2}\partial_\lambda\partial_\rho)\Delta(x-x'). \tag{5.4}$$

We then arrive at[12]

$$[U_\lambda(x),D(t)]=i(x_\nu\partial_\nu+1)\tfrac{1}{2}(\delta_{\lambda\mu}+g_{\lambda\mu})U_\mu(x) +i(x_\nu\partial_\nu+3)\tfrac{1}{2}(\delta_{\lambda\mu}-g_{\lambda\mu})U_\mu(x), \tag{5.5}$$

which means

$$\begin{aligned}[U_i(x),D(t)]&=i(x_\nu\partial_\nu+1)U_i(x),\\ [U_4(x),D(t)]&=i(x_\nu\partial_\nu+3)U_4(x).\end{aligned} \tag{5.6}$$

We again encounter the difficulty in which the space and the time components of $U_\mu(x)$ transform differently, with $l=1$ and $l=3$, respectively. The reason is the same as in the previous case, namely, the constraint equation

$$m^2U_4(x)=i\partial_i\pi_i(x) \tag{5.7}$$

determines $U_4(x)$ in terms of $\pi_i(x)$, the canonical momentum of $U_i(x)$.

VI. DISCUSSION

The above-mentioned difficulty arises in almost all theories of higher-spin fields. This is due to the following situation: In order to introduce spin, which is essentially a property of three-dimensional space, we have to supplement with extra components on the grounds of covariance in four-dimensional Minkowski space, and these extra components are connected with the original independent components through constraint equations. The constraint equations in general contain the finite mass and the space derivatives, as was seen in (4.18) and (5.6). The scale transformation only changes the latter but not the former. Thus the extra components transform differently from the original independent components, because of the space derivative. The above argument shows that as soon as the scale invariance is destroyed, it is always possible that the covariance will also be destroyed. In analyzing high-energy processes, is it then meaningful to talk about Lorentz covariance and broken scale invariance simultaneously?

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[10] It would be appropriate to point out here that the dimension of the coupling constant assigned in Sec. II does not give us any information on the renormalizability of the interaction. For the proper dimensional analysis in conjunction with the renormalizability, we refer to Ref. 11.

[11] H. Umezawa, *Quantum Field Theory* (North-Holland, Amsterdam, 1956); S. Sakata, H. Umezawa, and S. Kamefuchi, Progr. Theoret. Phys. (Kyoto) **7**, 377 (1952).

[12] $g_{\mu\nu}=1$ $(\mu=\nu=1,2,3)$; $g_{\mu\gamma}=-1$ $(\mu=\gamma=4)$; $g_{\mu\gamma}=0$ otherwise.