PHYSICAL REVIEW D VOLUME 3, NUMBER 2 15 JANUARY 1971

Polarization Phenomena in Vacuum Nonlinear Electrodynamics

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(Received 6 July 1970)

We study the influence of the nonlinearities of vacuum quantum electrodynamics on the propagation of a low-frequency wave traversing an intense electromagnetic field. Cotton-Mouton and Kerr effects on the polarization, as well as the associated birefringence, appear in analogy with the corresponding phenomena in material media. The magnitude of these effects is very small. It is not, however, excluded that they might provide a direct test of photon-photon scattering.

I. INTRODUCTION

THE existence of sources of intense electromagnetic fields makes it desirable to reconsider a number of predictions of quantum electrodynamics in the low-energy range. In a previous article[1] we have studied the possibility with available techniques of testing the nonlinearities of quantum electrodynamics in the optical range through the mechanism of pair creation. It was found that the effect is exceedingly small. The suggestion was then to investigate dispersive, rather than absorptive, effects. Instead of addressing ourselves to the standard question of photon-photon scattering cross sections, we examine here the related question of the behavior of an optical wave in a constant electromagnetic field. In several cases this kind of propagation has already been studied,[2,3] but we concentrate here our attention on polarization phenomena. The reason for undertaking this study is to try to find a phenomenon which does not require the measurement of a differential effect like a scattering cross section, and hence is *a priori* more favorable.

There is a great wealth of effects in terms of optical phenomena, similar to those familiar in the study of light propagation through isotropic materials, in the presence of external fields. To be more precise, one finds in a magnetic or electric field the analogs of the Cotton-Mouton and Kerr effects on polarization as well as the associated birefringence.[4] Unfortunately, all these effects are very small.

In order to estimate their magnitude let us give a crucial figure for magnetic fields, for instance. The typical dimensionless parameter in the problem is

$$a=e\hbar\mathcal{H}/m^2c^3, \qquad (1)$$

where m is the electron mass and $\mathcal{H}$ the magnetic field. For $\mathcal{H}$ equal to 1 G, the parameter a is equal to 2.3×10^{-14}. Now, any effect of $\mathcal{H}$ on the incident wave is mediated by the vacuum-current fluctuations which are even under charge conjugation. As a result the polarization tensor is also even in $\mathcal{H}$. Furthermore, for frequencies ω small as compared to $mc^2/\hbar$, it is sufficient to limit oneself to the lowest order (quadratic) both in frequency and magnetic field. The polarization tensor is then proportional to $\alpha\omega^2a^2$ (α is the fine-structure constant) times numerical factors depending on the direction of propagation with respect to the applied field. Consequently, the difference in refraction index corresponding to a propagation perpendicular to the field (where the effect is maximum) is expected to be of order αa^2. The phase difference between the two components of a linearly polarized wave, traversing a distance L across the field, is then estimated to be of order $\varphi\sim(L/\lambda)\alpha a^2$, where λ is the wavelength. For a laboratory experiment with $\mathcal{H}=10^5$ G, i.e., $a=2.4\times10^{-9}$, $\lambda=0.1\,\mu$, and $L=10$ cm, the expected phase difference is of order $\varphi\sim10^{-14}$. Similarly, birefringence is characterized by the difference in indices of refraction αa^2. A 1% effect would require fields as high as 10^{13}–10^{14} G, which could only appear under very special conditions in astrophysics.

The quest for observable effects in laboratory experiments exhibiting these nonlinearities in the optical range remains therefore open.

As a matter of notation, we now use standard units with $\hbar=c=1$.

II. WAVE PROPAGATION IN NEARLY CONSTANT FIELD

The interaction between the incident wave and the applied field is mediated by the quantized electron-positron field. The integration over the variables of the charged particles is responsible for an extra term δS in the action for the electromagnetic wave, which reads

$$e^{i\delta S}=\frac{\langle 0|T\exp[-i\int d^4x\, j(x)\cdot A(x)]|0\rangle}{\langle 0|T\exp[-i\int d^4x\, j_0(x)\cdot A(x)]|0\rangle}, \qquad (2)$$

where $A_\mu(x)$ stands for the vector potential of the wave, and $j_\mu(x)$ stands for the current operator of the quantized Dirac field of the electron in the presence of the external field; $j_{0\mu}(x)$ is the same operator in the absence of field. The effect of the denominator in (2) is to cancel the remaining (divergent) contribution when the applied field vanishes. It thus provides a definite renormalization procedure, and is such that δS can be expanded in a power series in $A(x)$, the lowest order being

[1] E. Brezin and C. Itzykson, Phys. Rev. D **2**, 1191 (1970).
[2] J. Toll, thesis, Princeton University, 1952 (unpublished).
[3] J. J. Klein and B. P. Nigam, Phys. Rev. **135**, B1279 (1964).
[4] L. D. Landau and E. M. Lifchitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).

quadratic since $\langle 0|j(x)|0\rangle$ vanishes from Lorentz-invariance arguments. The quadratic term is responsible for the modification of the propagation properties. It can be written

$$\delta S=\tfrac{1}{2}\int\int d^4x d^4y\, A_\mu(x)\pi^{\mu\nu}(x-y)A_\nu(y)\,, \qquad (3)$$

where the polarization tensor is defined through

$$\begin{aligned}\pi^{\mu\nu}(x-y)&\\ &=i\langle 0|T[j^\mu(x)j^\nu(y)]|0\rangle-i\langle 0|T[j_0{}^\mu(x)j_0{}^\nu(y)]|0\rangle\\ &=ie^2\,\mathrm{tr}[G(xy)\gamma^\nu G(yx)\gamma^\mu]-(\text{value at zero field}). \qquad (4)\end{aligned}$$

Here G is the Green function of an electron in the presence of the field. It is easy to verify that $\pi^{\mu\nu}$ is invariant under gauge transformations (of the vector potential of the applied field) and space-time translations if this field is homogeneous. The problem is then in principle reduced to the calculation of the Green function in a constant (or nearly constant) external field. This is a case where an explicit solution for the Green function is known and the calculation could proceed using Eq. (4). Though this might appear to be an attractive mathematical exercise, it is not absolutely necessary for our present purposes.

Indeed, only the first nonvanishing term of $\tilde{\pi}^{\mu\nu}(k)$, the Fourier transform of $\pi^{\mu\nu}(x-y)$, in an expansion in powers of both k and the external field, is really needed. This is because of the smallness of the parameter a, and of the frequency, as compared to the electron mass. Under these conditions we can make use of a classical result due to Heisenberg and Euler which gives the first correction to the free Lagrangian of the Maxwell field in the presence of vacuum charge fluctuations to lowest order in frequency.[5]

Let f denote the total electromagnetic field $F+\mathfrak{F}$, where F corresponds to the optical field and $\mathfrak{F}$ is the (nearly) constant applied field. The additional Lagrangian of Euler and Heisenberg reads (with $f^{*\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f_{\rho\sigma}$ and $\epsilon^{0123}=+1$)

$$\delta\mathfrak{L}=(2\alpha^2/45m^4)[\tfrac{1}{4}(f^2)^2+\tfrac{7}{16}(f\cdot f^*)^2]\,, \qquad (5)$$

where $f^2=f_{\mu\nu}f^{\mu\nu}$ and $f\cdot f^*=f_{\mu\nu}f^{*\mu\nu}$. It is sufficient to keep in $\delta\mathfrak{L}$ only those terms that are quadratic both in F and $\mathfrak{F}$ (since we are not interested in the direct effect of the incident wave on itself, nor of the external field on itself). This procedure gives for the interaction

$$\begin{aligned}\delta\mathfrak{L}=(2\alpha^2/45m^4)[\tfrac{1}{2}F^2\mathfrak{F}^2+(F\cdot\mathfrak{F})^2&\\ +\tfrac{7}{8}(F\cdot F^*)(\mathfrak{F}\cdot\mathfrak{F}^*)+(7/4)(F\cdot\mathfrak{F}^*)^2]. \qquad (6)\end{aligned}$$

It is clear that the modified action computed from (6) will indeed be of the form (3) with $\tilde{\pi}^{\mu\nu}(k)$ of second order, both in frequency and applied field. It must then coincide to this order with the result from the general theory.

[5] W. Heisenberg and H. Euler, Z. Physik **38**, 714 (1936).

Let us set

$$\rho=(\alpha/45\pi)(e/m^2)^2. \qquad (7)$$

The modified Maxwell equations then read

$$\begin{aligned}(1-\rho\mathfrak{F}^2)\partial_\mu F^{\mu\nu}+2\rho\mathfrak{F}^{\nu\mu}\partial_\mu F\cdot\mathfrak{F}+\tfrac{7}{2}\rho\mathfrak{F}^{*\nu\mu}\partial_\mu F\cdot\mathfrak{F}^*=0\,,\\ \partial_\mu F^{*\mu\nu}=0\,.\end{aligned} \qquad (8)$$

Let us consider a plane-wave mode with four-dimensional wave vector k. Using the vanishing of $\partial_\mu F^{*\mu\nu}$, we set

$$F^{\mu\nu}(x)=(\epsilon^\mu k^\nu-\epsilon^\nu k^\mu)e^{ik\cdot x}\,, \qquad (9)$$

where $\epsilon^\mu(k)$ describes the polarization of the wave. Equation (8) takes the form

$$\begin{aligned}(1-\rho\mathfrak{F}^2)(k^2\epsilon^\mu-k\cdot\epsilon k^\mu)+4\rho k_\nu\mathfrak{F}^{\nu\mu}(\epsilon\mathfrak{F}\cdot k)&\\ +7\rho k_\nu\mathfrak{F}^{*\nu\mu}(\epsilon\mathfrak{F}^*\cdot k)=0. \qquad (10)\end{aligned}$$

In this formula $\epsilon\mathfrak{F}\cdot k$ stands for $\epsilon_\sigma\mathfrak{F}^{\sigma\tau}k_\tau$, and similarly for $\epsilon\mathfrak{F}^*\cdot k$. From this equation we easily see that $\epsilon(k)$ has to be of the form

$$\epsilon^\mu(k)=\xi_0 k^\mu+\xi_1 k_\nu\mathfrak{F}^{\nu\mu}+\xi_2 k_\nu\mathfrak{F}^{*\nu\mu} \qquad (11)$$

in the case of a general field $\mathfrak{F}$. The parameters ξ are to be determined from (10). It is clear that ξ_0 corresponds to an irrelevant gauge transformation. Inserting (11) into (10), we find with the help of some simple algebra that ξ_1 and ξ_2 are solutions of the homogeneous system

$$\begin{aligned}\xi_1[(1-\rho\mathfrak{F}^2)k^2+4\rho k\cdot\mathfrak{F}\cdot\mathfrak{F}\cdot k]-\xi_2\rho k^2(\mathfrak{F}\cdot\mathfrak{F}^*)=0\,,&\\ -\xi_1(7/4)\rho k^2(\mathfrak{F}\cdot\mathfrak{F}^*)&\\ +\xi_2[(1+\tfrac{5}{2}\rho\mathfrak{F}^2)k^2+7\rho k\cdot\mathfrak{F}\cdot\mathfrak{F}\cdot k]=0\,,\end{aligned} \qquad (12)$$

where obviously $k\cdot\mathfrak{F}\cdot\mathfrak{F}\cdot k$ means $k^\mu\mathfrak{F}_{\mu\nu}\mathfrak{F}^\nu{}_\sigma k^\sigma$.

III. PROPER MODES

To get the proper modes, we must then set the determinant of the system (12) equal to zero. In general, this procedure leads for the dispersion surface (frequency in terms of wave number) to a fourth-order cone in k space. In cases of practical interest (pure electric or pure magnetic field, for instance) $\mathfrak{F}\cdot\mathfrak{F}^*=-4\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{H}}$ vanishes, and the dispersion surface degenerates into two second-order cones corresponding to two privileged modes. We shall henceforth assume $\mathfrak{F}\cdot\mathfrak{F}^*=0$. Then the two proper modes correspond to the vanishing of ξ_2 and ξ_1, respectively. One has

(i) mode 1:

$$(1-\rho\mathfrak{F}^2)k^2+4\rho k\cdot\mathfrak{F}\cdot\mathfrak{F}\cdot k=0\,,\quad \epsilon_1{}^\mu(k)=\xi_0 k^\mu+\xi k_\nu\mathfrak{F}^{\nu\mu}\,;$$

(13)

(ii) mode 2:

$$(1+\tfrac{5}{2}\rho\mathfrak{F}^2)k^2+7\rho k\cdot\mathfrak{F}\cdot\mathfrak{F}\cdot k=0\,,\ \epsilon_2{}^\mu(k)=\xi_0 k^\mu+\xi k_\nu\mathfrak{F}^{*\nu\mu}.$$

Let us note the following simple identity:

$$\begin{aligned}k\cdot\mathfrak{F}\cdot\mathfrak{F}\cdot k=(k^0)^2\boldsymbol{\mathcal{E}}^2+(\mathbf{k}^2)\boldsymbol{\mathcal{H}}^2-2k^0(\mathbf{k},\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{H}})&\\ -(\mathbf{k}\cdot\boldsymbol{\mathcal{E}})^2-(\mathbf{k}\cdot\boldsymbol{\mathcal{H}})^2. \qquad (14)\end{aligned}$$

For definiteness let us consider the case of a pure magnetic field $\mathcal{H}$ and let θ be the angle between $\mathcal{H}$ and $\mathbf{k}$ (Fig. 1). Let us further introduce the index of refraction n through the definition $k=(\omega,\omega\mathbf{n})$, with $n=|\mathbf{n}|$. Then we can solve Eqs. (13) for n^2-1 and $\epsilon(k)$, to first order in ρ. We choose ξ_0 in such a way that $\epsilon_0=0$ and define $\boldsymbol{\epsilon}$ up to a scale factor. We find

(i) mode 1 or transverse mode:

$$n_1^2-1=4\rho\mathcal{H}^2\sin^2\theta\,,\quad \boldsymbol{\epsilon}_1=\mathbf{n}\times\boldsymbol{\mathcal{H}}\,; \tag{15}$$

(ii) mode 2 or parallel mode:

$$n_2^2-1=7\rho\mathcal{H}^2\sin^2\theta\,,\quad \boldsymbol{\epsilon}_2=\boldsymbol{\mathcal{H}}-(\boldsymbol{\mathcal{H}}\cdot\mathbf{n})\mathbf{n}\,.$$

The terms transverse and parallel refer to the situation of the polarization with respect to the plane $(\boldsymbol{\mathcal{H}},\mathbf{k})$. Note that n^2-1 is proportional to $\sin^2\theta$, i.e., is invariant in the replacement $\theta\to\pi\pm\theta$, reflecting the symmetry of the effect under the change of $\boldsymbol{\mathcal{H}}$ into $-\boldsymbol{\mathcal{H}}$. The effect vanishes at $\theta=0$, when the wave propagates along the field, and is maximal at $\theta=\frac{1}{2}\pi$, perpendicular to the field. We further observe that $\boldsymbol{\epsilon}_2$ fails to be orthogonal to $\mathbf{n}$ by an amount proportional to $(n_2^2-1)(\boldsymbol{\mathcal{H}}\cdot\mathbf{n})$.

In the case of a pure electric field it is readily verified that, to the same order in ρ, the roles of the two modes are interchanged: The transverse mode $\boldsymbol{\epsilon}=\mathbf{n}\times\boldsymbol{\mathcal{E}}$ has index $n^2-1=7\rho\mathcal{E}^2\sin^2\theta$, and the parallel mode $\boldsymbol{\epsilon}=\boldsymbol{\mathcal{E}}-(\boldsymbol{\mathcal{E}}\cdot\mathbf{n})\mathbf{n}$ has index $n^2-1=4\rho\mathcal{E}^2\sin^2\theta$, as one would expect.

A linearly polarized wave with frequency ω and wavelength $\lambda=2\pi/\omega$ propagating along a distance L perpendicular to a constant magnetic field $\boldsymbol{\mathcal{H}}$ would be transformed into an elliptically polarized wave according to $\boldsymbol{\epsilon}(t)=\alpha_1\boldsymbol{\epsilon}_1\cos(\omega t-n_1L)+\alpha_2\boldsymbol{\epsilon}_2\cos(\omega t-n_2L)$ if its initial polarization were $\boldsymbol{\epsilon}(t)=(\alpha_1\boldsymbol{\epsilon}_1+\alpha_2\boldsymbol{\epsilon}_2)\cos\omega t$. The elliptical locus of the final polarization vector is

$$\left(\frac{x}{\alpha_1}\right)^2+\left(\frac{y}{\alpha_2}\right)^2-\frac{2xy}{\alpha_1\alpha_2}\cos\Phi=\sin^2\Phi\,, \tag{16}$$

with the phase shift Φ given by

$$\Phi=2\pi(n_2-n_1)\frac{L}{\lambda}=\frac{\alpha}{15}\left(\frac{e\mathcal{H}}{m^2}\right)^2\frac{L}{\lambda}. \tag{17}$$

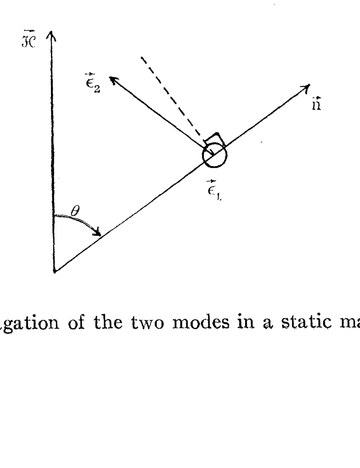

FIG. 1. Propagation of the two modes in a static magnetic field.

This is in agreement with the estimate given in the Introduction. We would have a similar effect in an electric field with $\mathcal{H}$ replaced by $\mathcal{E}$.

IV. ENERGY PROPAGATION

The most natural way to examine the propagation of energy in the wave is to compute the energy-momentum tensor density $T_{\mu\nu}$. The flux of energy is then given by $T^0{}_l$, where l runs from 1 to 3. This tensor can be derived from the action $S=S_{\text{free}}+\delta S$ and turns out to be equal to

$$\begin{aligned}T_{\mu\nu}=g_{\mu\nu}\tfrac{1}{4}[&(1-\rho\mathfrak{F}^2)F^2-2\rho(F\cdot\mathfrak{F})^2\\&-\tfrac{7}{2}(F\cdot\mathfrak{F}^*)^2-(7/4)F\cdot F^*\mathfrak{F}\cdot\mathfrak{F}^*]+(1-\rho\mathfrak{F}^2)F_{\mu\sigma}F^\sigma{}_\nu\\&-\rho[F^2\mathfrak{F}_{\mu\sigma}\mathfrak{F}^\sigma{}_\nu+2F\mathfrak{F}(F_{\mu\sigma}\mathfrak{F}^\sigma{}_\nu+\mathfrak{F}_{\mu\sigma}F^\sigma{}_\nu)]\\&-(7/4)\rho[F\cdot F^*\mathfrak{F}_{\mu\sigma}\mathfrak{F}^\sigma{}_\nu+\mathfrak{F}\cdot\mathfrak{F}^*F_{\mu\sigma}{}^*F^\sigma{}_\nu\\&\qquad+2F\cdot\mathfrak{F}^*(F_{\mu\sigma}{}^*\mathfrak{F}^\sigma{}_\nu+\mathfrak{F}_{\mu\sigma}{}^*F^\sigma{}_\nu)]\,.\end{aligned} \tag{18}$$

Inserting the expressions for F obtained in Sec. III, we could derive from (18) the flux of energy in the two modes. It is, however, possible to obtain the same result by computing the group velocity. We present this second method of calculation, which is much simpler. The group velocity is given by

$$\mathbf{v}=\boldsymbol{\nabla}_k\omega\,. \tag{19}$$

For the case of a pure magnetic field, say, we derive from (13) (to lowest order in ρ)

(i) transverse mode:

$$\mathbf{v}_1=(1-4\rho\mathcal{H}^2)\mathbf{n}+4\rho(\mathbf{n}\cdot\boldsymbol{\mathcal{H}})\boldsymbol{\mathcal{H}}\,; \tag{20}$$

(ii) parallel mode:

$$\mathbf{v}_2=(1-7\rho\mathcal{H}^2)\mathbf{n}+7\rho(\mathbf{n}\cdot\boldsymbol{\mathcal{H}})\boldsymbol{\mathcal{H}}\,.$$

Again, if $\mathcal{H}$ is replaced by $\mathcal{E}$, then v_1 and v_2 are interchanged. We see, of course, that the energy propagates in a direction different from its wave vector as occurs in a nonisotropic crystal (except for $\theta=p\frac{1}{2}\pi$, with p an integer).

If a wave enters a region where a field is present, it will split into the two modes giving rise to the phenomenon of birefringence.

V. CONCLUSION

We have presented a particular aspect of the scattering of light by light predicted by quantum electrodynamics.

The small magnitude of these nonlinear effects is such that the observation of the modes of propagation of a light wave in static fields, predicted by the theory, makes their direct observation in a laboratory experiment rather unlikely at present. However, high-intensity oscillatory fields with frequencies small compared with the wave frequency might serve as a quasistatic "external field" for the purpose of studying the effects discussed here. Thus, one could, for instance, use a laser beam to provide the "quasistatic" field and x rays as the incoming wave. With $\langle eH/m^2\rangle$ as high as 10^{-5} and λ of the order of 10^{-8} cm, a 10^{-6} effect would require an L of the order of 0.1 mm. These figures, while they might look a little futuristic, suggest that polarization phenomena might perhaps be a promising means of probing directly photon-photon scattering.

After the completion of this work, we were informed that a discussion of nonlinear electrodynamic phenomena has also been reported by a group in Princeton, with an eye towards astrophysical applications.[6]

ACKNOWLEDGMENTS

We have benefited from stimulating discussions with C. Cohen-Tanoudji and M. Froissart.

[6] S. L. Adler, J. N. Bahcall, C. G. Callan, and M. N. Rosenbluth, Phys. Rev. Letters **25**, 1061 (1970).

PHYSICAL REVIEW D VOLUME 3, NUMBER 2 15 JANUARY 1971

Covariant Classical Motion of Electron in a Laser Beam*

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The classical motion of an electron interacting with a plane-wave electromagnetic pulse is calculated in a fully covariant manner starting from the relativistic equation of motion neglecting radiation reaction.

THE motion of an electron in an intense plane-wave electromagnetic field has been given recently by Eberly and Sleeper[1] and by Sarachik and Schappert.[2] We wish to give in this note a direct calculation of the motion of the electron from the equation of motion instead of using the Hamilton-Jacobi equation. This method has the advantage that it gives the orbit directly in a covariant and gauge-invariant form. We introduce the four-vector potential for the plane-wave electromagnetic field in the Lorentz gauge,[3]

$$A_\mu(n)=a_\mu A(n)\,,\quad a_\mu=(\mathbf{a},ia_0)\,.$$

Relating this to the notation of Ref. 2, we have

$$\mathbf{a}A(n)=\mathbf{A}(\mathbf{x},t)=\mathbf{A}(n)P(n)\,.$$

n is defined by

$$n\equiv\omega t-\mathbf{k}\cdot\mathbf{x}=-k\cdot x\,,\quad k_\mu=(\mathbf{k},i\omega)\,,\quad k^2=0\,,$$

where k is the propagation vector for the laser beam. The Lorentz gauge requires the restriction $k\cdot a=0$. The electromagnetic field tensor is then

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu=-(k_\mu a_\nu-a_\mu k_\nu)A'(n)\,,$$

where the prime denotes a derivative with respect to the argument. The relativistic equation of motion neglecting radiation reaction is

$$\begin{aligned}dp_\mu/d\tau&=(e/m)F_{\mu\nu}p_\nu\\&=-(e/m)(k_\mu a\cdot p-a_\mu k\cdot p)A'(n)\,,\end{aligned}\qquad(1)$$

where τ is the proper time and $p_\mu=m dx_\mu/d\tau$ is the electron's momentum. This gives

$$d(k\cdot p)/d\tau=0\,,\qquad(2)$$

since $k\cdot a=k^2=0$. Thus $k\cdot p$ is a constant of the motion and can be evaluated from the initial conditions. Letting $p_\mu{}^0$ be the four-momentum of the electron for $\tau\to-\infty$, we have $k\cdot p=k\cdot p^0$. Now $k\cdot p=mk\cdot dx/d\tau=-mdn/d\tau$, or

$$dn/d\tau=-k\cdot p^0/m\,.\qquad(3)$$

In terms of n, Eq. (5) becomes

$$dp_\mu/dn=(e/k\cdot p^0)(k_\mu a\cdot p-a_\mu k\cdot p^0)A'(n)\,,\qquad(4)$$

giving

$$d(a\cdot p)/dn=-ema^2A'$$

or

$$a\cdot p=a\cdot p^0-ea^2A\,.$$

Substitution into (4) gives

$$\frac{dp_\mu}{dn}=-e\left(a_\mu-\frac{k_\mu a\cdot p^0}{k\cdot p^0}\right)\frac{dA}{dn}-\frac{e^2a^2}{2k\cdot p^0}\frac{dA^2}{dn}k_\mu\,.\qquad(5)$$

This can be immediately integrated to give the momentum

$$p_\mu=p_\mu{}^0-e\left(A_\mu-\frac{A_\nu p_\nu{}^0k_\mu}{k\cdot p^0}\right)-\frac{e^2A_\nu A_\nu k_\mu}{2k\cdot p^0}\,.\qquad(6)$$

Equation (6) gives the four-momentum of the electron in a covariant form valid for any reference frame, any initial momentum $p_\mu{}^0$, and is invariant under gauge transformations that remain in the Lorentz gauge $(A_\mu\to A_\mu+\lambda k_\mu)$. In the frame where $\mathbf{p}^0=0$ (lab frame), and specializing to the radiation gauge ($A_0=0$), (6)

* Work supported by the U. S. Atomic Energy Commission.
[1] J. H. Eberly and A. Sleeper, Phys. Rev. **176**, 1570 (1968).
[2] E. S. Sarachik and G. T. Schappert, Phys. Rev. D **1**, 2738 (1970).
[3] We take $c=1$, and our metric is chosen such that $a\cdot b=\mathbf{a}\cdot\mathbf{b}+a_4b_4(=\mathbf{a}\cdot\mathbf{b}-a_0b_0)$.