Coupling-Constant Analyticity in the Charged-Scalar Static Model

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The analytic properties of the two-point function for the charged-scalar static model in the bare coupling λ are investigated. The proper self-energy function appears to be an analytic function in the λ plane cut along the positive real λ axis, and the value of λ determined by mass renormalization lies well below the nearest branch point on the real λ line. The perturbation series will, therefore, converge at this point. The Padé approximants to the full propagator and to an important subsequence of graphs are studied. Although these approximants appear to converge in the cut λ plane, the convergence is not rapid.

I. INTRODUCTION

THE field theory of the charged-scalar-meson field interacting with a fixed two-component nucleon source (CSSM) is a very interesting model for several reasons. It is sufficiently complex that it has many of the properties desired for a realistic pion-nucleon interaction, but sufficiently simple that many of its properties may be studied rigorously. Most important, the model is fully renormalizable and therefore provides a "theoretical laboratory" for the study of the general properties of renormalizable theories.

Usually, numerical calculations based on field theory are done with perturbation expansions in Lagrangian coupling-constant parameters. This computational technique has several drawbacks. The first is merely a calculational problem: The increase in number and complexity of contributions in increasing orders is so great that normally first- or second-order calculations must suffice. In addition, such a series cannot be used to study bound states, for example, since by definition a pole in an amplitude lies outside of the radius of convergence of the series.

In fact, it seems to be the case for some interactions that the radius of convergence of the perturbation series is zero. The $\lambda\phi^4$ interaction for bosons is an example of a field theory with this difficulty.[1] The problem here may be seen in the analog of this system in ordinary quantum mechanics, the anharmonic oscillator. For λ (the strength of the anharmonic term) ≥ 0 there is a potential well having normal bound states; for $\lambda<0$, the potential turns over and there is no ground-state level. The complete change in characteristics of solutions at $\lambda=0$ is indicative of a branch point in the λ plane at that point. Even if the radius of convergence is finite for some model, extension of the power-series solution to large coupling constants may not be possible.

For the CSSM it has recently been shown that the perturbation series actually converges for small λ.[2] The result obtained in that paper, henceforth referred to as I, is that the series converges for

$$\lambda<\left[4\int_1^\infty \frac{\sigma(x)dx}{m_0+x}\right]^{-1}. \tag{1}$$

This was a rather weak result, however, and it is interesting to investigate the exact structure of the various n-point functions in the complex λ plane to determine the actual radius of convergence of the perturbation series for these quantities. Sufficient basis has been found to conjecture that the power-series expansion for the proper self-energy function converges at the value of λ necessary for renormalization of the nucleon mass.

The problem of the large number and complexity of Feynman diagrams has been circumvented by preparation of a computer program which constructs and computes the coefficients of λ^N in the n-point function automatically.[3] Using this program, it is possible to consider not only the perturbation series but also the Padé approximants which may be obtained from the series.

The use of Padé approximants to represent amplitudes in quantum field theory has recently become quite popular.[4] The Padé approximant $[n,m]$ is defined as the ratio of two polynomials (numerator of degree m and denominator of degree n) whose power-series expansion agrees with the known coefficients up to and including order $n+m$.

Unfortunately, there is a dearth of rigorous mathematical results on the conditions necessary and sufficient for convergence of a sequence or a subsequence of the approximants to the desired function. The work of Baker contains perhaps the best available summary of known results applicable to physics.[5] If the power series of interest is a series of Stieltjes (to be defined below) and if the power series has a finite radius of convergence, then the sequence of approximants $[N, N+j]$, $j\geq -1$, converge to the function defined by the series in the complex plane with a cut on the real axis. Since it is quite difficult in practice to ascertain whether a power

[1] A. Jaffe, Commun. Math. Phys. 1, 127 (1965).

[2] P. B. James and G. R. North, Phys. Rev. D 2, 695 (1970).

[3] P. B. James and G. R. North, J. Comp. Phys. (to be published).

[4] D. Bessis and M. Pusterla, Nuovo Cimento **54A**, 243 (1968); J. Basdevant, D. Bessis, and J. Zinn Justin, *ibid.* **60A**, 185 (1969).

[5] G. Baker, Advan. Theoret. Phys. 1, 1 (1965).

series is a series of Stieltjes, this result is not of overwhelming practical significance. In addition, there are obvious subtle difficulties with any series which has zero radius of convergence, e.g., $\lambda\phi^4$ theory.

However, the sequence of $[N,N]$ approximants has been seen to converge for the anharmonic oscillator.[6] This has encouraged several authors to use very-low-order approximants as approximations to exact field-theoretic situations. It would be desirable to study the behavior of the approximants in the case of a reasonable field theory. In this paper, the methods of the Padé approximants are applied to the two-point function for CSSM. The four-point function will be analyzed in a subsequent paper.

In addition to studying the Padé-approximant results for various features of the full perturbation series for the two-point function, the paper considers a subsequence of graphs which can be shown to be a series of Stieltjes. The convergence properties of the approximants can be studied in detail in this case and the results compared to the series for the full propagator. It is likely, as indicated below, that the analytic structure of this subsequence closely resembles the full two-point function and, therefore, this is probably a better laboratory for studying the application of Padé approximants to field theory than many of the trial functions used elsewhere.

Section II considers the CSSM; in particular, an argument for the convergence of the perturbation series at the value of $\lambda=g_0^2$ necessary for renormalization is given, and the interesting properties of the two-point function which may be studied by use of Padé approximants are defined. In Sec. III, the subsequence mentioned above is considered. The numerical results obtained from the full propagator function are presented in Sec. IV, and the conclusions reached are summarized in Sec. V. The Feynman rules of the CSSM are given in I.

II. CSSM

In this section some of the more interesting properties of the two-point function in CSSM are presented. The most interesting feature is the mass renormalization. There are two parameters in the theory, the bare mass m_0 and the bare coupling constant, $\lambda=g_0^2$. However, when the propagator is required to have a pole at $E=0$, the position of the renormalized nucleon pole, a relation between λ and m_0 results.

It is more convenient to consider the inverse propagator $S^{-1}(E)$, which is related to the proper self-energy function $\Sigma(E-m_0,\lambda)$ by

$$S^{-1}(E)=E-m_0-\Sigma(E-m_0,\lambda)\,. \qquad (2)$$

Evidently, if $S^{-1}(E=0)=0$, the resulting condition on the proper self-energy function is

$$m_0=-\Sigma(-m_0,\lambda)\,. \qquad (3)$$

This condition leads to a relationship $\lambda(m_0)$ between the bare coupling and the bare mass.

The function $-\Sigma(-m_0,\lambda)$ is a positive definite, decreasing function of $+m_0$ for $\lambda>0$ within its radius of convergence. (This was shown in I.) It is of particular interest to consider the possibility that the perturbation series for $\Sigma(-m_0,\lambda(m_0))$ actually converges. The following argument indicates that this is the case although, as will be pointed out, there is a gap in the proof. The proper self-energy part satisfies an integral equation (Fig. 1)

$$\Sigma(E-m_0,\lambda)=\int_1^\infty \frac{\sigma(x)dx\Gamma_x(E-m_0,\lambda)}{E-m_0-x-\Sigma(E-m_0-x,\lambda)}\,, \qquad (4)$$

where $\sigma(x)$ is the cutoff function, including phase-space factors, and Γ_x is the proper three-point vertex function for external meson momentum x. If it is assumed that Γ_x contributes no closer singularities, the location of the first branch point will occur when the singularity of the integrand denominator pinches the endpoint of the integration (at $E=0$):

$$m_0+1=-\Sigma(-m_0-1,\hat{\lambda})\,. \qquad (5)$$

If this is the closest singularity to the origin, the power series for $\Sigma(-m_0,\lambda)$ will converge for $\lambda<\hat{\lambda}(m_0)$. Note that $\hat{\lambda}(m_0)=\lambda(m_0+1)$ by comparison of (5) and (3). Within the radius of convergence of the series for Σ, λ is an increasing function of m_0; this is due to the fact that the decrease of $-\Sigma$ in (3) with m_0 must be compensated by an increase in λ. Therefore, $\hat{\lambda}(m_0)=\lambda(m_0+1)>\lambda(m_0)$.

Therefore, except for the possibility that Γ_x provides a more nearby singularity than this one, the perturbation series will converge at the point necessary for renormalization. The convergence described above is supported by the numerical calculations described later and by the analytic structure of the Padé approximants to $S^{-1}(E=0)$.

The conclusion suggested then is that the proper self-energy part with $E=0$ is analytic in the λ plane except for a cut running from $\hat{\lambda}$ to ∞, and finally that $\hat{\lambda}>\lambda(m_0)$.

There is a possibility of a dynamical pole in the vertex function for some value of E, $0<E<1$. This pole would also appear as a pole in $S^{-1}(E)$ because of the relationship between Σ and Γ_x expressed in Eq. (4). [This pole could not occur for $\lambda<\lambda(m_0)$, however, at

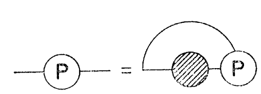

FIG. 1. Diagrammatic representation of Eq. (4). Open circles containing P's represent proper functions; shaded circle represents the full propagator including disconnected terms.

[6] C. Bender and T. T. Wu, Phys. Rev. Letters **21**, 406 (1968); J. J. Loeffel, A. Martin, B. Simon, and A. S. Wightman, Phys. Letters **164**, 656 (1969); B. Simon, Ann. Phys. (N. Y.) **58**, 76 (1970).

$E=0$. Since $-\Sigma$ is positive definite, if a pole were its first singularity, $-\Sigma$ would run through all positive values, including m_0, as λ was varied.] Such a pole would be detected when λ is fixed at $\lambda(m_0)$ and m_0 is varied in such a way that the $E-m_0$ variation in Σ mimics an increasing E from $E=0$ to $E=1$.

A dynamical argument also suggests no interference in the integrand of (4) by a pole in Γ_x. Such a pole would be of the "bound-state" type discussed by Goebel and Sakita.[7] Though of no phenomenological interest, such poles can occur. However, usually such poles approach $E=0$ only as $\lambda\to\infty$ in static models. This last suggests no nearby contributions from Γ_x.

The wave-function renormalization constant Z_2 is calculated as below by use of Padé approximants. The renormalization zero discussed above will appear as a pole in the $[n,n]$ Padé approximant for $S(E)$. That is, if the pole is factored out explicitly,

$$[n,n]_{[S(E)]}=\frac{f(E,\lambda)}{\lambda-\alpha(E-m_0)}. \qquad (6)$$

The renormalization constant is the residue of the pole at $E=0$ in the energy plane. Expanding $\alpha(E-m_0)$ about $E=0$,

$$\alpha(E-m_0)=\lambda(m_0)+\left.\frac{\partial\lambda}{\partial E}\right|_{E=0}E+\cdots. \qquad (7)$$

Therefore, the residue of the pole at $E=0$ is given by

$$\mathrm{Res}[n,n]=-\frac{f(0,\lambda(m_0))}{\partial\alpha/\partial E|_{E=0}}. \qquad (8)$$

The quantity $\partial\alpha/\partial E|_{E=0}$ may be easily computed by noting that the functions involved are actually functions only of $E-m_0$, so that

$$\left.\frac{\partial\alpha}{\partial E}\right|_{E=0}=-\left.\frac{\partial\alpha(m_0)}{\partial m_0}\right|_{m_0}. \qquad (9)$$

Since the functions involved are smooth functions of m_0, the derivative and, therefore, the wave-function renormalization, may easily be computed.

Strong-coupling theory makes specific predictions concerning properties of the two-point function.[8] In the normalization used here the relation $\lambda(m_0)$ takes the form, for large λ,

$$m_0=(25/24)\pi\lambda. \qquad (10)$$

Since $\lambda(m_0)$ is likely to be inside the radius of convergence of the power series, both the power series and the Padé approximants were studied for large λ to compare with (10). In addition, the wave-function renormalization should go to zero in the large-λ limit as[9]

$$Z_2^{sc}=\tfrac{1}{2}e^{-0.335\lambda}. \qquad (11)$$

The normalization used in (10) and (11) is such that the simple bubble graph (first term of Fig. 2) is given by

$$\lambda\int_1^\infty\frac{kd\omega}{(1+a^2k^2)(E-m_0-\omega)},$$

where a is the cutoff, taken here to be 5 in units of pion mass = 1.

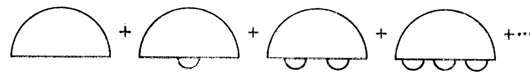

FIG. 2. First few terms of the sequence of turtle graphs. These graphs seem to play a particularly important role in determining the structure of the self-energy function in the λ plane.

III. SUBSEQUENCE

As a model for the proper self-energy part, we consider the subsequence of "turtle" graphs shown in Fig. 2. This subsequence seems to have analytic structure in λ similar to the full self-energy part and is by far the largest graph numerically in each order of perturbation theory studied. Table I compares the contributions from turtle graphs to the exact coefficients of λ^N for $m_0=5$. It is interesting to note that this subsequence was omitted in the crossing-symmetric formalism of Freeman, North, and Rubin,[10] no doubt contributing to their error in the relation $m_0(\lambda)$ in the strong-coupling regime.

The nth-order term for the turtle sequence may be written explicitly. The coefficient of λ^n in $-\Sigma(E=0)/\lambda$ (a positive function) is given by

$$a_n=\int_1^\infty\frac{\sigma(x)dx}{(m_0+x)^{n+1}}\left[\int_1^\infty\frac{\sigma(y)dy}{m_0+x+y}\right]^n. \qquad (12)$$

This series may be summed exactly, giving

$$-\frac{\Sigma(E=0,\lambda)}{\lambda}=\int_1^\infty\frac{\sigma(x)dx}{m_0+x-\lambda f(x)}, \qquad (13)$$

TABLE I. Numerically computed coefficients of the full perturbation series for $-\Sigma(E=0)$, c_n, are compared with the turtle graphs for the same order N. In addition, the number of graphs in each order is given.

N	n	c_n	a_n
1	1	3.768	3.768
2	1	0.6946	0.6946
3	3	0.3051	0.2073
4	13	0.1701	0.0747
5	71	0.1055	0.0300
6	461	0.0708	0.0129

[7] C. Goebel and B. Sakita, Phys. Rev. Letters **11**, 293 (1963).
[8] A. Pais and R. Serber, Phys. Rev. **105**, 1636 (1957). This paper contains many references to earlier work.
[9] T. D. Lee, Phys. Rev **95**, 1329 (1954). Note that λ(bare) $=4\lambda$(renormalized) in the strong-coupling limit.
[10] D. F. Freeman, G. R. North, and M. H. Rubin, Phys. Rev. **188**, 2426 (1969).

where

$$f(x)=\int_1^{\infty}\frac{\sigma(y)dy}{m_0+x+y}. \tag{14}$$

Actually, this is just (4) with Γ_ω truncated at its lowest-order term, 1, and Σ approximated by only the simple bubble graph in the integral.

The power series for $-\Sigma(E=0)/\lambda$, $\sum a_n\lambda^n$, is a series of Stieltjes. Define a function $u(x)=f(x)/(m_0+x)$. It is easily seen that du/dx is strictly ≤ 0 and $u(x)\geq 0$ in the region of integration. The coefficient a_n may be written

$$a_n=-\int_{u=0}^{u_0}\left[\frac{\sigma(u)}{m_0+x(u)}\frac{dx}{du}\right]du\, u^n, \tag{15}$$

where $u_0=f(1)/(m_0+1)$. The quantity in square brackets in (15) is a negative function in the region of integration. Therefore, the function

$$\hat{\phi}(u)=\int_0^u \frac{-\sigma(x)}{m_0+x}\frac{dx}{du}du \tag{16}$$

is a positive nondecreasing function of u for $u\geq 0$. The power-series coefficient may be expressed

$$a_n=\int_0^{u_0}u^n d\hat{\phi}(u), \tag{17}$$

where $\hat{\phi}(u)$ is a positive nondecreasing function on $[0,u_0]$. Define a function $\phi(u)$ by

$$\begin{aligned}\phi(u)&=\hat{\phi}(u), \quad 0\leq u\leq u_0\\ &=\hat{\phi}(u_0), \quad u_0<u\leq\infty.\end{aligned} \tag{18}$$

$\phi(u)$ is also a positive nondecreasing function of u and

$$a_n=\int_0^{\infty}u^n d\phi(u). \tag{19}$$

A power series whose coefficients may be expressed in the form of Eq. (19) is a series of Stieltjes.

Therefore, the turtle graphs form a series of Stieltjes, and the $[n, n+j]$ approximants will converge to the function everywhere in the complex λ plane cut on the real axis. The branch point in this case is determined by

$$m_0+1-\hat{\lambda}f(1)=0, \tag{20}$$

so that

$$\hat{\lambda}=(m_0+1)/f(1)=1/u_0. \tag{21}$$

Since $\lambda(m_0)$ for the subsequence is determined by

$$m_0=\lambda f(0)+\sum_{n=2}^{\infty}\lambda^n a_{n-1}, \quad a_{n-1}>0 \tag{22}$$

it is clear that $\lambda(m_0)<\hat{\lambda}$ determined from the turtle graphs alone.

The Padé approximants for $S^{-1}(E=0)=-m_0-\Sigma(E=0)$ have been calculated for this subsequence up to and including $[7,7]$. For $n>7$, the roundoff error from the inversion of the nearly singular matrices required makes computation difficult. The first property of interest is the location of poles and zeros of the various approximants $[n,n]$. The $[n,n]$ approximant to $S^{-1}(E=0)$ is equal to $-m_0+\lambda[n, n-1]$ approximant to $\Sigma(E=0)/\lambda$. Since Σ/λ is the function represented by a Stieltjes series, the poles and zeros of the approximants should interlace, as discussed by Baker. The poles of the $[n,n]$ approximant to S^{-1} will be those of the $[n, n-1]$ approximant to Σ/λ; the zeros are slightly displaced and, in addition, a new zero appears to the *left* of the first pole. This zero is just the $\lambda(m_0)$ computed from the subsequence. The positions of the poles and zeros in the λ plane are shown in Fig. 3. It will be noted that these singularities move to the left as the order n is increased and appear to be approaching $\hat{\lambda}$ as they attempt to represent the λ plane cut on the real axis.

It is of particular interest to consider the convergence of the sequence of $[n,n]$ approximants for values of $|\lambda|>\hat{\lambda}$. Two values of $|\lambda|$ were chosen, $|\lambda|=3$ and $|\lambda|=10$ (with $m_0=5$, $\hat{\lambda}=1.59$); the phase was chosen to approach the real axis along the appropriate semicircles from $\lambda=\pi$ to $\lambda=\frac{1}{64}\pi$. Since the exact value of the subsequence is known (13), the convergence may be examined in some detail. Tables II and III list the values for $S^{-1}(E=0)$ calculated from various Padé approximants for $m_0=5$. The corresponding power-series partial sums, of course, diverge rapidly.

It will be seen that, while the sequence of $[n,n]$ Padé approximants *does* converge, the convergence is not as rapid as might have been hoped. This result seems to shed some doubt on the usefulness of low-order approximants as calculational approximations for field theories outside of the radius of convergence of the power series, especially above a branch cut.[11] It would

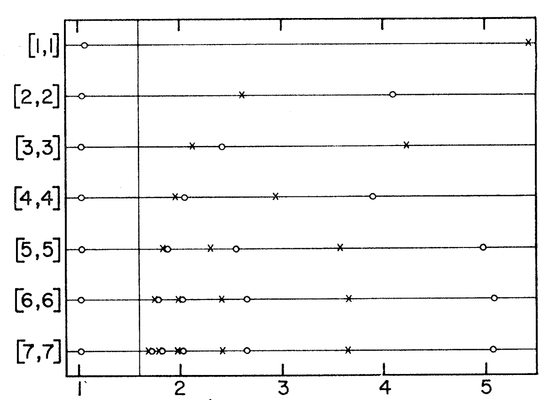

FIG. 3. Poles and zeros of the $[n,n]$ Padé approximants to $S^{-1}(E=0)$ for the turtle sequence. Here o's represent zeros and ×'s represent poles; the vertical line is the value $\lambda(m_0)$ for the bare mass used here, $m_0=5$. A few poles and zeros for $\lambda\gg 5$ were truncated for convenience.

[11] It is not difficult to show that the physical region is above the branch cut. This follows from letting E have a small positive-imaginary part to continue past the branch point in the λ plane.

TABLE II. Convergence of the $[n,n]$ Padé approximants to $S^{-1}(E=0)$ for the turtle subsequence examined for $m_0=5$ and $\lambda=3e^{i\phi}$, $\phi=\pi$, $\frac{1}{4}\pi$, $\frac{1}{16}\pi$, and $\frac{1}{64}\pi$. The exact value for each phase, computed from (13), is given for comparison.

	π	$\frac{1}{4}\pi$	$\frac{1}{16}\pi$	$\frac{1}{64}\pi$
[1,1]	−12.2812	−1.6663+15.2593*i*	16.8642+ 9.9624*i*	20.0350+ 2.7532*i*
[2,2]	−12.8031	−2.7005+11.2077*i*	−5.8011+14.1940*i*	−19.5941+ 8.6486*i*
[3,3]	−12.8382	−2.7585+12.1402*i*	9.5457+11.2913*i*	13.7009+ 3.6030*i*
[4,4]	−12.8408	−2.6966+11.9152*i*	1.9710+18.4599*i*	−12.2087+55.8327*i*
[5,5]	−12.8410	−2.7292+11.9513*i*	7.7925+14.6213*i*	17.0268+ 6.6958*i*
[6,6]	−12.8410	−2.7274+11.9519*i*	7.2265+14.0974*i*	14.0146+ 6.0193*i*
[7,7]	−12.8410	−2.7274+11.9519*i*	7.2197+14.0969*i*	13.9826+ 6.0189*i*
Exact	−12.8410	−2.7274+11.9519*i*	7.2197+14.0972*i*	13.9827+ 6.0191*i*

TABLE III. Convergence of the $[n,n]$ Padé approximants to $S^{-1}(E=0)$ for the turtle subsequence examined for $m_0=5$ and $\lambda=10e^{i\phi}$, $\phi=\pi$, $\frac{1}{4}\pi$, $\frac{1}{16}\pi$, and $\frac{1}{64}\pi$. The exact value for each phase, computed from (13), is given for comparison.

	π	$\frac{1}{4}\pi$	$\frac{1}{16}\pi$	$\frac{1}{64}\pi$
[1,1]	−18.259	−28.9180+14.9012*i*	−46.6162+ 9.4263*i*	−49.5350+ 2.5913*i*
[2,2]	−21.5215	−12.4126+35.2237*i*	53.0998+60.3266*i*	95.5694+23.3508*i*
[3,3]	−22.2890	−6.3371+22.8819*i*	6.7420+10.0182*i*	8.1375+ 2.6212*i*
[4,4]	−22.4891	−12.5962+20.5773*i*	−24.1757+14.3822*i*	−29.1310+ 4.2990*i*
[5,5]	−22.5265	−13.9373+22.9425*i*	−38.210 +47.0451*i*	−94.6997+33.8412*i*
[6,6]	−22.5272	−13.9087+23.0470*i*	−37.3063+49.8257*i*	−101.972 +40.5013*i*
[7,7]	−22.5272	−13.9086+23.0472*i*	−37.3033+49.8317*i*	−101.990 +40.5191*i*
Exact	−22.5272	−13.9086+23.0472*i*	−37.3033+49.8319*i*	−101.990 +40.5193*i*

seem unlikely that this behavior would improve as more classes of graphs are added.

Since the subsequence of turtle graphs is known to have only one singularity at a well-determined value of $\hat{\lambda}$, it is possible to employ an alternative scheme for computing the function outside of the region of convergence for the power series. A conformal mapping may be used to map the complex plane cut from $\hat{\lambda}$ to ∞ onto the interior of the unit disk; the real line from $[\hat{\lambda}, \infty]$ is mapped onto the boundary of the disk. It is then possible to compute the power-series coefficients for the function $f(w)$, w being the transformed variable. This series will converge for all $|w|<1$, that is, for all values of the original variable λ not on the real axis with $|\lambda|>\hat{\lambda}$.

The convergence of the resulting series was examined at the same points in the complex λ plane at which the Padé series was computed. The series did indeed converge, although the convergence was not quite as rapid as that of the Padé series. Since the effort required to compute the coefficients of the series for $f(w)$ is considerably greater than that for computation of the Padé approximants, it would seem that the mapping method had little to offer in this particular problem.

IV. FULL PROPAGATOR

In this section we investigate the full propagator numerically through the sixth power of λ. This allows construction of the [3,3] Padé approximant as a tool for exploring the analytic structure of $\Sigma(E=0,\lambda)$. As an example, Table I gives numerical values of the coefficients as well as the number of graphs contributing in each order for $m_0=5$.

It may be asked how it is possible to identify a particular zero of the Padé approximant for $S^{-1}(E=0)$, or, for that matter, of the perturbation series with the desired $\lambda(m_0)$. There are several reasons that the zero nearest the origin represents this root. For the power series for the function $-\Sigma(-m_0,\lambda)$, all coefficients are positive. For any finite polynomial, there will be one solution of $-\Sigma(-m_0,\lambda)=m_0$ for $\lambda>0$. The other roots of this equation will be for λ negative or complex; these roots will be farther from the origin, since the various terms in the polynomial tend to interfere destructively. Therefore, there is only one candidate for $\lambda(m_0)$ from the power series; whether or not this may actually be computed from the power series depends on the convergence of the series at that point.

For the Padé approximants, there are two reasons for choosing the zero of the approximant to S^{-1} which occurs for smallest λ. This first zero of $[n,n]$ results from effectively solving the equation $m_0=\lambda[n,n-1]$, where $[n,n-1]$ is an approximant to $-\Sigma(-m_0,\lambda)/\lambda$. The first zero is the additional root generated by this equation and lies to the left of the first pole of the $[n,n-1]$ approximant to $-\Sigma(-m_0,\lambda)$. In addition, if one examines the zeros of $[n,n]$ to $S^{-1}(E=0)$ as a function of n, it is seen that the first zero is relatively stable; this is taken to be an indicator of the dynamical significance of that zero.

It was seen in Fig. 3 that for the turtle graphs the first pole of the $[n,n]$ Padé approximant approaches the

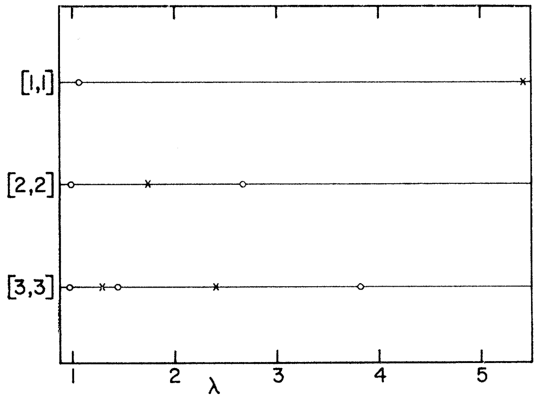

FIG. 4. Poles and zeros for the [1,1], [2,2], and [3,3] approximants to the full $S^{-1}(E=0)$ function for $m_0=5$. The structure of these quantities closely resembles that for the turtle sequence.

position of the actual branch point of the function. The poles and zeros of the approximants to the full propagator resemble those of the turtle graphs very closely, as may be seen in Fig. 4. This strongly suggests, therefore, that the zero representing $\lambda(m_0)$ lies within the radius of convergence of the power series. This is because $\lambda(m_0)$ always lies to the left of the first pole and, therefore, must lie to the left of the branch point represented by the location of the first pole of $[n,n]$ as $n\to\infty$. Again, this result does not prove that the series converges at $\lambda(m_0)$, but does strongly suggest such a conclusion.

The values $\lambda(m_0)$ computed in various approximations are shown in Tables IV and V for two values of m_0: $m_0=5$ and 36. It will be seen that the values obtained are consistent with convergence of both the

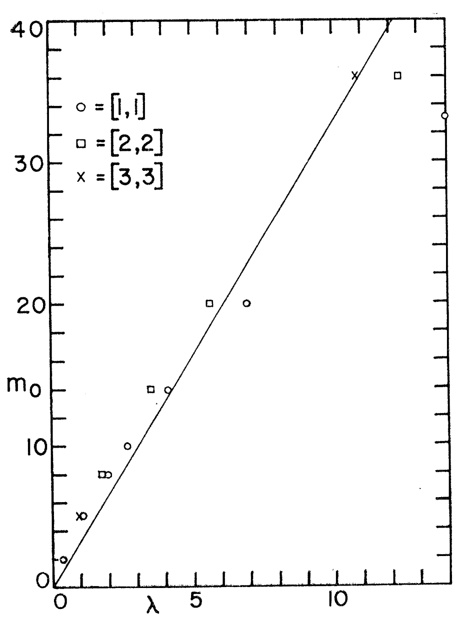

FIG. 5. Values $\lambda(m_0)$ computed from the [1,1], [2,2], and [3,3] Padé approximants for various m_0's compared to the strong-coupling result (solid line).

TABLE IV. Values $\lambda(m_0)$ computed from the truncated power series of order N and from the corresponding Padé approximant given for $m_0=5$. For odd N, the $[\frac{1}{2}(N-1), \frac{1}{2}(N+1)]$ approximant is used, because this corresponds to the $[\frac{1}{2}(N-1), \frac{1}{2}(N-1)]$ approximant to Σ/λ.

Power series (order)	$\lambda(m_0)$ (P.S.)	Padé approximant	$\lambda(m_0)$ (P.A.)
1	1.3268	⋯	
2	1.1027	[1,1]	1.0661
3	1.0378	[1,2]	0.9990
4	1.0091	[2,2]	0.9789
5	0.9940	[2,3]	0.9722
6	0.9850	[3,3]	0.9657

Padé sequence and the power series at the bare masses considered. Apparently the convergence improves with decreasing bare mass. The most interesting feature of the $\lambda(m_0)$ is the striking agreement obtained with the strong-coupling limit, as indicated in Fig. 5. Because of normalization factors, absorbed into the λ used in this work, $\lambda\simeq 1$ should be a "large" coupling. It will be seen in Fig. 5 that the $\lambda(m_0)$ computed from the various Padé approximants very nicely follows the strong-coupling curve until a point is reached where the results begin to deviate. This is presumably due to the less rapid convergence of the sequence to $\lambda(m_0)$ at large m_0. This is substantiated by the fact that the higher-order approximants agree with strong coupling to larger values of λ.

No evidence was found for a dynamical pole[7] in the inverse propagator function. Such a pole must, by definition, lie outside the region of convergence of the power series; the Padé approximants must be used, therefore, to search for this pole. Since the Padé approximants handle cuts by interlaced poles and zeros which move together as the order of approximants is increased, it is difficult to distinguish between this sort of singularity and an actual dynamical pole. Presumably, as in the case of the zero discussed above, a dynamical pole should remain stable as the order of approximant is varied. None of the poles which appeared in the Padé approximants to $S^{-1}(E=0)$ seemed to satisfy this

TABLE V. Values $\lambda(m_0)$ computed from the truncated power series of order N and from the corresponding Padé approximant given for $m_0=36$. For odd N, the $[\frac{1}{2}(N-1), \frac{1}{2}(N+1)]$ approximant is used, because this corresponds to the $[\frac{1}{2}(N-1), \frac{1}{2}(N-1)]$ approximant to Σ/λ.

Power series (order)	$\lambda(m_0)$ (P.S.)	Padé approximant	$\lambda(m_0)$ (P.A.)
1	24.523	⋯	
2	18.129	[1,1]	16.602
3	15.864	[1,2]	13.692
4	14.644	[2,2]	12.340
5	13.859	[2,3]	11.470
6	13.301	[3,3]	10.894

criterion. If one fixes $\lambda(m_0)$ by the position of the renormalized nucleon pole and varies the energy (or $E-m_0$), the pole of the Padé approximants which appears at the lowest value of λ corresponds to a pole at roughly $E=2.8$ for $m_0=5$, which is well above threshold. It therefore appears that no dynamical pole occurs in the Padé approximants for $S^{-1}(E)$, at least through [3,3] and $m_0<36$.

The results for the wave-function renormalization constant Z_2 were not in good agreement with the strong-coupling result of (11). For example, with $m_0=5$, $Z_2^{sc}=0.36$. The values of Z_2 computed from the [1,1], [2,2], and [3,3] approximants are 0.69, 0.55, and 0.47, respectively. The results are even farther from the strong-coupling result as λ is increased. Optimistically, the values of Z_2 computed in $[n,n]$ do seem to be converging in the direction of Z_2^{sc}, as they also do at the larger values of λ.

Apparently, the problem with Z_2 is that this quantity involves the derivatives of functions evaluated at $\lambda(m_0)$. It is well known that differentiation of a power series interferes destructively with the convergence of the series. This explanation is supported by a study of the convergence of the series

$$\frac{d}{dE}\Sigma(E-m_0, \lambda(m_0))\,|_{E=0}\,,$$

which may be obtained numerically from quantities which have been computed. This series converged much less rapidly than that for $\Sigma(E-m_0, \lambda(m_0))|_{E=0}$. In terms of the convergence proof given in I, differentiation of the perturbation series for Σ has two effects: An extra factor is added in each denominator, and there are $2N-1$ times more terms in the Nth order. The series for the derivative will still converge, but the factors of $2N-1$ will have the effect of slowing the rate of convergence.

It is true that Z_2 computed in each order of Padé approximant goes to zero as $\lambda\rightarrow\infty$. Also, $Z_2\rightarrow 0$ much more strongly as the order of approximant is increased.

V. CONCLUSIONS

The fact that the bare perturbation series for the CSSM appears to converge at the value of the bare coupling constant necessary for renormalization is probably the least expected but most important result of this work. Although there is no ironclad proof of this property, unfortunately, the result is suggested by the analyticity argument of Sec. II, by the locations of the poles and zeros of the Padé approximants to $\Sigma(E=0)$, and by the behavior of $\lambda(m_0)$ actually computed from the power series directly for the first few terms. In addition, this is a property of the important subsequence of turtle graphs.

Questions which arise in this context which will require further work are as follows.

(1) Can the result be proved rigorously?
(2) Does the result extend to other field theories?
(3) What is the behavior of $\lambda(m_0)$ in the point-source local-field-theory limit?

The last question is probably the most difficult one to answer because of the uncertainties involved in the point-source limit.

The behavior of $\lambda(m_0)$ computed directly from the perturbation series and from the Padé approximants appears to be in excellent agreement with the results of strong-coupling theory as $\lambda(m_0)$ becomes large. This may be contrasted with the work of Freeman, North, and Rubin,[10] in which contributions from all orders were included but certain classes of graphs were omitted. In particular, the turtle graphs, which appear to be particularly important in determining analytic structure in the λ plane, were omitted there. The behavior of Z_2, while not in such excellent agreement with strong-coupling theory, appears to not be inconsistent with that limit if one considers the slower convergence to be expected in this case.

The values of $\lambda(m_0)$ obtained from the Padé approximants do converge more rapidly than the values computed from the power series. However, the improvement is not great enough to allow the use of the [1,1] or [2,2] approximants for large values of m_0. The Padé approximants for the turtle graphs actually converged rather slowly, particularly near the positive real λ axis. Since such values of λ are the ones of interest in this model, this property detracts from the attractiveness of such approximants for CSSM. In particular, if $\lambda(m_0)$ is held fixed and E is varied, $E>1$, this is exactly the situation which is encountered. So use of the approximants to compute scattering amplitudes above threshold, for example, is questionable. Of course, this result depends on the fact that the λ-plane singularities in this model appear to be confined to the positive real axis. It is an interesting, but difficult, problem to study to λ-plane structure for other field theories to determine if this is a general rule.

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