Scaling, Fixed Poles, and Electroproduction Sum Rules*

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Bjorken's scale invariance is imposed on the general form of the Deser-Gilbert-Sudarshan (DGS) representation for forward current-hadron scattering amplitudes to deduce features of the scaling limit, the Regge limit, and the asymptotic limit for fixed total mass of the produced hadrons. Two sum rules are derived, whose validity essentially asserts that the residue of the $J=0$ fixed pole in the virtual Compton amplitude is independent of the mass carried by the currents. One of these relations tests for the presence of operator Schwinger terms, and the other relates an integral over the total photoproduction cross section to an integral over a scale function observed in electroproduction. The significance of these ideas for the calculation of electromagnetic mass differences is also discussed.

I. INTRODUCTION

WITH the assumption of asymptotic scaling, as suggested by Bjorken[1] for electroproduction and neutrino cross sections, and as partially corroborated by the data,[2] we derive sum rules which involve electron and photon total cross sections. Although it will not be easy to test these relations experimentally, a check of their validity would illuminate interesting theoretical features of current-hadron amplitudes; for example, (1) the existence of operator Schwinger terms; (2) the argument (to be developed in this paper) that fixed-pole residues have a polynomial dependence[3] on the squared momentum, q^2, carried by the current; (3) the possibility of expressing electromagnetic mass differences in terms of the electroproduction data, even in the presence of fixed poles and subtraction[4] terms in the Cottingham formula.[5]

The sum rules given here have been presented[6] previously, but with a very brief discussion of their theoretical origin. This paper provides a full discussion of the assumptions and reasoning which underlie their derivation.

The primary mathematical tool for our study is the DGS[7] representation, which has proven itself useful[8,9] for investigating asymptotic features of current-hadron forward amplitudes. The DGS representation is a version of the Jost-Lehmann-Dyson (JLD) representation of causal commutators, restricted to the forward direction, which incorporates the familiar requirements of causality. Although it has not been derived on as general grounds as the JLD representation, it has been shown by Nakanishi[7] to hold in every order of perturbation theory.

As a prototype of the two appropriately defined (in Sec. V) invariant functions which describe the forward Compton amplitude, we consider in Secs. II–IV an amplitude T, even under crossing symmetry,

$$T(q^2,-\nu)=T(q^2,\nu)\,, \tag{1.1}$$

where q is the momentum of current, p is the momentum of hadron, and $\nu=-q\cdot p=q_0p_0-\mathbf{q}\cdot\mathbf{p}$, and discuss general features of W,

$$W=-2\pi^{-1}\,\mathrm{Im}T\,, \tag{1.2}$$

which follow from the scaling hypothesis[1]

$$W\underset{B}{\rightarrow}F(\omega)\quad(0<\omega<1)\,. \tag{1.3}$$

Here $\omega\equiv q^2/2\nu$, and the limit denoted by B indicates that q^2 and ν are going to infinity with a fixed value of ω.

In Sec. II it is shown that (1.3) and the validity of the DGS representation given in (2.1) imply that

(1) W also scales in the region of timelike q^2, where $-1<\omega<0$, and the value of this limit, except for a sign, is given by the same F as in (1.3),

$$W\underset{B}{\rightarrow}F(\omega)=-F(-\omega)\quad(-1<\omega<0)\,. \tag{1.4}$$

This antisymmetry of F reflects the symmetry of T in Eq. (1.1), and F would be even in ω for an amplitude odd under crossing.

(2) In the "extreme" timelike region where $q^2<-2\nu$, or equivalently $\omega<-1$, W decreases "rapidly" in the B limit,

$$W\underset{B}{\rightarrow}0\quad(\omega<-1)\,. \tag{1.5}$$

(3) The most general form for T consistent with the scaling hypothesis (1.3) allows for an exotic behavior of W in the limit, which we shall call the S limit, where ν goes to infinity with a fixed hadronic mass, i.e., $2\nu-q^2$ fixed. There can be terms in W which grow

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[1] J. D. Bjorken, Phys. Rev. 179, 1547 (1969).
[2] E. D. Bloom *et al.*, Phys. Rev. Letters 23, 930 (1969); M. Briedenbach *et al.*, *ibid.* 23, 935 (1969).
[3] On quite different grounds, T. P. Cheng and Wu-ki Tung, Phys. Rev. Letters 24, 851 (1970), have argued that this is so.
[4] H. Harari, Phys. Rev. Letters 17, 1303 (1966).
[5] W. N. Cottingham, Ann. Phys. (N.Y.) 25, 424 (1963).
[6] J. M. Cornwall, D. Corrigan, and R. E. Norton, Phys. Rev. Letters 24, 1141 (1970).
[7] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. 115, 731 (1959); M. Ida, Progr. Theoret. Phys. (Kyoto) 23, 1151 (1960); N. Nakanishi, *ibid.* 26, 337 (1961); Suppl. 18, 70 (1961).
[8] J. M. Cornwall and R. E. Norton, Phys. Rev. 173, 1637 (1968).
[9] J. M. Cornwall and R. E. Norton, Phys. Rev. 177, 2584 (1969).

polynomially in ν,

$$W \underset{s}{\longrightarrow} \sum \nu^m G_m(2\nu - q^2) + \lim_{\omega\to 1} F(\omega)\,, \qquad (1.6)$$

where the $G_m(z)$ decrease faster than z^{-m} for large z and do not contribute to scaling. [If the G_m were $O(z^{-m})$, W would be singular at $\omega+1$ in the scale limit.] Of course, one normally expects the "form factors" at fixed hadronic mass to decrease with ν (or q^2), and thus it is natural to presume that the G_m and the limit $F(1)$ in (1.6) are zero. However, there are amusing theoretical implications if the G_m do not vanish. In particular, the relations in Ref. 1, which express certain equal-time commutators as integrals over the scale functions of electroproduction, would acquire additional contributions involving the G_m. Also, for similar reasons, the sum rules discussed in this paper would require modification from their forms given in Ref. 6.

Features of the Regge (R) limit ($\nu\to\infty$, q^2 fixed) of the electroproduction cross sections and their relationships with the asymptotic behavior in the B limit have been discussed by various authors.[10] In Sec. III we study these questions and show that if W has the asymptotic form in the R limit

$$W \underset{R}{\longrightarrow} \sum_{\alpha\geqslant 0} C_\alpha(q^2)\nu^\alpha\,, \qquad (1.7)$$

then it follows from scaling that

(4) as $\omega\to 0$, the scale function $F(\omega)$ in (1.3) has a Regge form similar to (1.7), such that we can define

$$\tilde{F}(\omega) = \theta(1-\omega)F(\omega) - \sum_{\alpha\geq 0} f_\alpha \omega^{-\alpha}\,, \qquad (1.8)$$

where $\tilde{F}(\omega)$ vanishes as $\omega\to 0$. Any of the f_α could, however, be zero.

(5) As $q^2\to\pm\infty$, the Regge residues in (1.7) become simply related to the f_γ in (1.8),

$$C_\alpha(q^2) \xrightarrow{q^2\to\pm\infty} \pm|2/q^2|^\alpha f_\alpha\,. \qquad (1.9)$$

(6) Although there may be no term in (1.7) for which $\alpha=\alpha(0)=0$ for spacelike q^2, a term of this kind could develop for timelike q^2. However, there would be no f_0 corresponding to such a pole, and the C_0 would vanish in the limit described by (1.9). This phenomenon is essentially a fixed pole[11,12] at $\alpha=0$ which occurs in both the real and imaginary parts of T for $q^2<0$, but only in ReT for $q^2\geq 0$. As discussed in Sec. IV, it is essential for the derivation of our sum rules that ImT is free of fixed poles at $\alpha=0$ for all values of q^2.

In Sec. IV we discuss a dispersion relation for $T(q^2,\nu)$ obtained by dispersing in q^2 for fixed ν. The validity of this relation follows trivially from the DGS representation (for the time-ordered product, not the retarded commutator). In particular, it involves an integral of W over both timelike and spacelike q^2. For spacelike q^2, W refers to lepton-hadron cross sections which can be measured experimentally. However, in the timelike region W either refers to cross sections which are unmeasurable in practice—for example, $e^+e^-p\to$ anything—or for $q^2<-\nu^2/M^2$ ($M^2=-p^2$) to processes for which $q^2=\mathbf{q}^2-q_0^2<-q_0^2$ and which therefore are unmeasurable in principle. Despite this situation, we deduce the following conclusions, *provided there is no term in* (1.7) *with* $\alpha=\alpha(0)=0$[13]:

(7) Any part of the dispersion integral contribution to ReT—that is, excluding the possible contribution of subtraction terms—which is constant in ν in the R limit (i.e., a $J=0$ fixed pole) can be expressed as an integral over spacelike q^2 only, where the integrand is determined by the asymptotic form of measurable cross sections.

(8) If the G_m in (1.6) are zero, the residue of this fixed pole at $J=0$ in ReT is independent of q^2; if the possible contributions of the G_m and of subtraction terms are allowed, the residue is a polynomial in q^2. The sum rule in (5.12) epitomizes points (7) and (8).

The assumption stated above that there is no term in the Regge limit of W which is constant in ν [i.e.,

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[10] H. D. I. Abarbanel, M. Goldberger, and S. B. Treiman, Phys. Rev. Letters **22**, 500 (1969); R. A. Brandt, *ibid.* **22**, 1149 (1969); H. Harari, *ibid.* **22**, 1078 (1969); R. A. Brandt, Phys. Rev. D **1**, 2808 (1970).

[11] See, e.g., J. B. Bronzan, I. S. Gerstein, B. W. Lee, and F. E. Low, Phys. Rev. **157**, 1448 (1967), and references therein.

[12] For brevity, throughout this paper we refer to asymptotic behavior of the type ν^{-m}, m an integer, as a fixed pole, even though it may merely signify a moving pole whose trajectory passes through an integer at $t=0$.

[13] The most general form of the DGS representation for the amplitude is

$$T = P(q^2,\nu) + \sum_{m=0}^{M}\sum_{n=0}^{N} \nu^m (q^2)^n \int_0^\infty d\sigma \int_{-1}^{1} d\beta \frac{h_{mn}(\sigma,\beta)}{q^2+2\beta\nu+\sigma-i\epsilon}$$

and differs from (2.1) to the extent that weight functions h_{mn} with $n>0$ are present. The results obtained in Secs. II–IV and stated in Sec. I are obtained from the restricted form (2.1). The question may thus arise as to the generality of our conclusions. However, note that since $q^2=(q^2+2\beta\nu+\sigma)-(2\beta\nu+\sigma)$, the general form given above can be rewritten as in (2.1) if the weight functions h_{mn} decrease sufficiently rapidly as $\sigma\to\infty$ for the resulting integrals over σ to converge. Thus, if weight functions h_{mn} with $n>0$ are required, they must essentially go for large σ as σ^{-P} with $P\leq 1$. However, it can be verified that the presence of weight functions with $n>0$ and an asymptotic behavior in σ characterized by $P<1$ would lead to violation of the basic scaling assumption (1.3); they thus need not be considered further. For $P=1$ it is possible to satisfy (1.3) and to violate (1.4). However, for this contradiction to our conclusions to obtain, it is necessary that in the Regge asymptotic limit ($\nu\to\infty$, q^2 fixed) W has a part constant in ν. As mentioned prior to conclusion (7) in Sec. I, and as discussed in Sec. IV, even in the absence of necessary subtractions in q^2 in the DGS representation for T [i.e., even when (2.1) is valid] an essential assumption for the derivation of our sum rules (5.12) and (5.14) and the conclusions listed as (7) and (8) is that W has no fixed pole at $\alpha=0$ [i.e., no term in (1.7) with $\alpha=0$]. Thus, except for the technical necessity of imposing the no-fixed-pole assumption earlier in the hierarchy of assumptions and conclusions presented in Sec. I, the results which would be obtained from the general DGS form given at the beinning of this footnote would agree with those obtained from (2.1), as derived in this paper. Henceforth, in this paper, only the form (2.1) will be considered.

no term in (1.7) with $\alpha=0$] is essential[13] to the conclusions listed as (7) and (8). It appears that this assumption is particularly powerful technically, and it would be fortunate if we could construct a convincing physical argument for its validity. However, except to say that this assumption appears plausible to us, and to mention that it is certainly related to (although not equivalent to) the often stated notion that there are no fixed poles in purely hadronic (or semihadronic, e.g., photoproduction[3]) processes, we must leave this question to the tastes of our readers.

Our second sum rule, Eq. (5.14), is valid if there are no operator Schwinger terms in the forward current-hadron amplitude. Any such terms which are present necessitate subtractions in the fixed-ν dispersion relations for the amplitude T_L (defined in Sec. V) and contribute quadratic divergences to electromagnetic mass differences.

In the Appendix we illustrate the general discussion of Secs. II–IV by appealing to two simple models. The first, a "parton" model,[14] satisfies all our assumptions and conclusions in an extreme form; the second, the Born approximation with form factors, satisfies neither.

II. GENERAL FORM OF W

In this section we develop the general form of W defined in (1.1) and (1.2) which follows from the assumptions of scaling and the validity of the DGS representation and deduce from it the conclusions listed under (1)–(3) in Sec. I.

The DGS representation[13] for $T(q^2,\nu)$ is[7–9]

$$T(q^2,\nu)=P(q^2,\nu)+\sum_{m=0}^{M}\nu^m\int_0^\infty d\sigma\int_{-1}^{1}d\beta\frac{h_m(\sigma,\beta)}{q^2+2\beta\nu+\sigma-i\epsilon},\quad(2.1)$$

where, because of (1.1),

$$h_m(\sigma,-\beta)=(-1)^m h_m(\sigma,\beta),\quad(2.2)$$

and where $P(q^2,\nu)$ is an arbitrary, real polynomial in q^2 and ν, which we retain for generality, but which will play a relatively minor role in the subsequent discussion. The imaginary part of T comes only from the $-i\epsilon$ in the denominator of (2.1). Thus, W defined in (1.2) is

$$W=-2\sum_{m=0}^{M}\nu^m\int_0^\infty d\sigma\int_{-1}^{1}d\beta\,h_m(\sigma,\beta)\delta(q^2+2\beta\nu+\sigma).\quad(2.3)$$

The lower limit of 0 in the σ integration in (2.1), (2.3), and subsequent formulas, is purely formal; for the forward Compton amplitude h vanishes for $\sigma<\sigma(\beta)$, where $\sigma(\beta)\geq 0$ for $-1\leq\beta\leq 1$. We use the δ function to integrate (2.3) over β; the limits of the resulting integral over σ are determined by the requirement that the δ function can actually vanish. Thus, (2.3) is evaluated as[15]

$$W(q^2,\nu)=-\sum_{m=0}^{M}\nu^{m-1}\int_{\bar\sigma}^{2\nu(1-\omega)}d\sigma\,h_m\left(\sigma,-\omega-\frac{\sigma}{2\nu}\right),\quad(2.4)$$

where the lower limit of the σ integration in (2.4) is

$$\bar\sigma=0\qquad(-1<\omega<1)\quad(2.5a)$$

$$=2\nu(-1-\omega)\quad(\omega<-1).\quad(2.5b)$$

We now consider the general form of W in the region $0<\omega<1$, as given by (2.4) and (2.5a), and impose the scaling hypothesis (1.3). If M in (2.4) is greater than unity, it is clear that (1.2) requires the integral over h_M in (2.4) to vanish in the B limit. That is,

$$\int_0^\infty d\sigma h_M(\sigma,-\omega)=0\quad(0<\omega<1)\quad(2.6)$$

and thus, if we define

$$h_M(\sigma,\beta)\equiv\partial_\sigma g_M(\sigma,\beta),\quad(2.7)$$

we can write

$$h_M\left(\sigma,-\omega-\frac{\sigma}{2\nu}\right)=\frac{d}{d\sigma}g_M\left(\sigma,-\omega-\frac{\sigma}{2\nu}\right)+(2\nu)^{-1}g_{M,\beta}\left(\sigma,-\omega-\frac{\sigma}{2\nu}\right),\quad(2.8)$$

where $g_{M,\beta}\equiv\partial g_M/\partial\beta$, and where

$$g_M(0,\beta)=g_M(\infty,\beta)=0.\quad(2.9)$$

By substituting (2.8) into (2.4), we obtain for $0<\omega<1$

$$W(q^2,\nu)=-\nu^{M-1}g_M(2\nu-q^2,-1)-\sum_{m=0}^{M-1}\nu^{m-1}\int_0^{2\nu(1-\omega)}d\sigma\,h_m'\left(\sigma,-\omega-\frac{\sigma}{2\nu}\right),\quad(2.10)$$

where

$$h_m'=h_m\qquad(0<m<M-2)\quad(2.11a)$$

$$=h_m+\tfrac{1}{2}g_{M,\beta}\quad(m=M-1).\quad(2.11b)$$

If $M-1>1$, it follows from (1.3) and (2.10) that the integral over h_{M-1}' in (2.10) also must vanish in the B limit. We could then proceed by analogy with the steps leading from (2.4) to (2.10), define a g_{M-1} in terms of

[14] See, e.g., J. D. Bjorken and E. A. Paschos, Phys. Rev. 185, 1975 (1969).

[15] It may be puzzling how the Born pole proportional to $\delta(\omega-1)$ is included in (2.4). To understand this feature is one of the reasons for considering the second example in the Appendix. The conclusion from this study is that the Born pole is contained in (2.4) by having an h_m singular at the point $\sigma=0$, $\beta=\pm1$, and it follows from this singular behavior that many of the steps leading to (2.13) are illegitimate when $2\nu\pm q^2=0$. However, all our efforts are directed toward understanding the asymptotic form of W, where the elastic form factors have dragged the Born pole into insignificance. Therefore, we will consider $2\nu\pm q^2$ to always differ from zero by a finite amount and henceforth ignore this point.

h_{M-1}', make the decomposition of h_{M-1}' analogous to (2.8), and obtain W for $0<\omega<1$ in the form

$$W=-\nu^{M-1}g_M(2\nu-q^2,\,-1)-\nu^{M-2}g_{M-1}(2\nu-q^2,\,-1)-\sum_{m=0}^{M-2}\nu^{m-1}\int_0^{2\nu(1-\omega)}d\sigma\, h_m''\left(\sigma,\,-\omega-\frac{\sigma}{2\nu}\right), \quad (2.12)$$

where

$$h_m''=h_m' \quad (0<m<M-3) \quad (2.12')$$

$$=h_m'+\tfrac{1}{2}g_{M-1,\beta} \quad (m=M-2). \quad (2.12'')$$

Clearly, we could continue in this manner and obtain W for $0<\omega<1$ in the canonical form $(N=M-1)$

$$W=\sum_{m=1}^{N}\nu^m G_m(2\nu-q^2)-\sum_{m=0}^{1}\nu^{m-1}\int_0^{2\nu(1-\omega)}d\sigma\,\bar{h}_m\left(\sigma,\,-\omega-\frac{\sigma}{2\nu}\right), \quad (2.13)$$

where $\bar{h}_0$ is the same as h_0 in (2.1), and $\bar{h}_1$ differs from h_1 to the extent that $M\geq 2$ in (2.1). However, from the way in which (2.13) is obtained, it is evident from (2.2) that

$$\bar{h}_m(\sigma,\,-\beta)=(-1)^m\bar{h}_m(\sigma,\beta), \quad (2.14)$$

and thus the form of W in (2.13) written for an arbitrary value of M $(1\leq M<\infty)$ in (2.1) differs only by the presence of the first term on the right-hand side of (2.13) from the form of W that is obtained from the choice $M=1$ in (2.1).

How can we understand the scale limit in (1.3) from the form of W in (2.13)? Since $2\nu-q^2=2\nu(1-\omega)$, the only way the G_m in (2.13) could contribute to $F(\omega)$ would be if $F(\omega)$ contained terms behaving like $(\omega-1)^{-m}$ which are singular as $\omega\to 1$. We will ignore this unrealistic possibility and require each $G_m(x)$ to decrease faster than $x^{-m-\epsilon}$ as $x\to\infty$. What about the term involving the integral over $\bar{h}_0=h_0$ in (2.13)? Because of the explicit factor of ν^{-1}, this term could contribute to the scale limit only if $h_0(\sigma,\beta)$ went to a constant for large σ; but then the contribution of h_0 to (2.1) would be logarithmically divergent. We are left, therefore, with $F(\omega)$ coming only from the integral over $\bar{h}_1$ in (2.13), and by comparing (1.3) and (2.13) we obtain for $0<\omega<1$

$$F(\omega)=\int_0^\infty d\sigma\,\bar{h}_1(\sigma,\omega), \quad (2.15)$$

where we have used (2.14) to effect a sign change.

Let us now ask ourselves what could prohibit the validity of all the steps leading from (2.1) to (2.13) and (2.15) for $-1<\omega<0$, provided they are valid for $0<\omega<1$. Because of (2.5a), (2.2), and (2.14), the answer is nothing, except possibly the occurrence of singularities in the $\bar{h}_m(\sigma,\beta)$ at $\beta=0$ which would contribute in the B limit of (2.13) for $-1<\omega<0$ but not for $0<\omega<1$. Of course, these singularities must not lead to divergences in (2.1) nor can they violate (2.14). Thus, one might try a term in $\bar{h}_1$ of the form $h_1(\sigma)\delta'(\beta)$. However, it is easy to check that a term of this kind would lead to an asymptotic behavior of $W\sim\nu^2$ in the R limit, which is sufficiently unrealistic not to require complicating our discussion by including its possible presence. Higher-order derivatives of $\delta(\beta)$ in $\bar{h}_1$ would, of course, give even more violent growth of W and can be ignored. What about a term like $h_0(\sigma)\delta(\beta)$ in $\bar{h}_0$? This possibility will be discussed again in Secs. III and IV, but at present let us simply observe that for such a term to contribute in the B limit of (2.13), it would be necessary for $h_0(\sigma)$ to approach a constant for large σ; but then (2.1) would diverge logarithmically. We conclude, therefore, that (2.13) and (2.15) are as valid in $-1<\omega<0$ as they are in $0<\omega<1$, as stated under (1) in Sec. I. The fact that $F(-\omega)=-F(\omega)$ follows immediately from (2.15) and (2.14).

The general form of W in the S limit (obtained by letting ν go to infinity with a fixed value of $2\nu-q^2$) is given in Eq. (1.6). This form clearly follows from (2.13) and the preceding discussion.

Let us consider the assertion listed under (2) in Sec. I. For $\omega<-1$ the lower limit on the integral in (2.4) is given by (2.5b). If we repeat the steps leading from (2.4) to (2.13), but take into account this crucial change in the lower limit of the σ integration, we obtain for $\omega<-1$

$$W=\sum_{m=1}^{N}\nu^m[G_m(2\nu-q^2)-(-1)^m G_m(-2\nu-q^2)]-\sum_{m=0}^{1}\nu^{m-1}\int_{2\nu(-1-\omega)}^{2\nu(1-\omega)}d\sigma\,\bar{h}_m\left(\sigma,\,-\omega-\frac{\sigma}{2\nu}\right). \quad (2.16)$$

The mathematical validity of the steps leading to this result follows from the validity of the steps leading to (2.13), which in turn is a consequence of the assumption of scaling in the region $0<\omega<1$. Since the G_m do not contribute to the B limit of (2.13), they do not contribute to the B limit of (2.16). But both the lower and upper limit of the σ integrations in (2.16) approach infinity in the B limit for $-1<\omega$. Thus, since $\bar{h}_0=h_0(\sigma,\beta)$ and $\sigma\bar{h}_1(\sigma,\beta)$ must be less than $O(\sigma^{-\epsilon})$ for large σ in order that (2.1) and (2.15) exist, it follows that Eq. (1.5) must be satisfied.

The general form of W given in (2.13) allows the possibility that the G_m and the limit $F(1)$ are not zero. If the G_m did not vanish, in contrast to our theoretical prejudices and to the indications of the electroproduction data, it would require modifications in the relations of Ref. 1 which express certain equal-time commutators as integrals over the scale functions of electroproduction. In effect, the relations of Ref. 1 are obtained by imposing the scaling hypothesis similar to (1.3) on the general commutator relations[9] of the form

$$C=\lim_{q^2\to\infty}(q^2)^{2n}\int\frac{d\nu}{\nu^{2n+1}}W(q^2,\nu) \tag{2.17}$$

and of interchanging the orders of performing the limit and the integration over ν. In the notation of (1.3), the commutator in (2.17) would then be replaced by

$$C'=2^{2n}\int_0^1 d\omega\,\omega^{2n-1}F(\omega). \tag{2.18}$$

However, it is easy to verify that if the G_m in (2.13) are not equal to zero, the limit in (2.17) cannot be taken inside the integral, and the commutator differs from the expression in (2.18) by integrals over the G_m.

III. REGGE LIMIT OF W

In this section we discuss the implications of requiring the general form of W given in (2.13) to satisfy the Regge asymptotic behavior illustrated in Eq. (1.7). In particular, we wish to deduce the assertions listed under (4)–(6) in Sec. I.

Let us begin by observing that since the G_m in (2.13) do not contribute to the B limit of W, they will also not contribute to the R limit. Next we note that the asymptotic form in (1.7) naturally arises from the integral over $\bar{h}_1$ in (2.13), if $\bar{h}_1$ has parts which go like $\beta^{-\alpha}$. That is, if we write $[\epsilon(\beta)=\beta/|\beta|]$

$$\bar{h}_1,(\sigma\beta)=\sum_{\alpha\geq 0}h_1{}^{\alpha}(\sigma)|\beta|^{-\alpha}\epsilon(\beta)+\tilde{h}_1(\sigma,\beta), \tag{3.1}$$

where $\tilde{h}_1(\sigma,\beta)$ vanishes as $\beta\to 0$, then the contribution of $\bar{h}_1$ to (2.13) leads to the asymptotic behavior in (1.7) with

$$C_\alpha(q^2)=2^\alpha\int_0^\infty d\sigma\,h_1{}^\alpha(\sigma)\frac{\epsilon(q^2+\sigma)}{|q^2+\sigma|^\alpha}. \tag{3.2}$$

Note that the Pomeranchon, with $\alpha=1$, can be accommodated in (3.1), even though $\bar{h}_1\sim\beta^{-1}$ at the origin. This singularity is harmless, because of the presence of $\epsilon(\beta)$, as one easily checks. Assuming for the moment that there are no other contributions to the C_α, and therefore that (3.2) is in fact correct, assertions (4) and (5) of Sec. I follow immediately from (2.15), (3.1), and (3.2). Clearly, if

$$\int_0^\infty d\sigma\,h_1{}^\alpha(\sigma)=0, \tag{3.3}$$

the corresponding f_α in (1.8) is zero.

Suppose there is a term in (1.7) with an $\alpha=0$ and with C_0 given by (3.2). Suppose also that

$$\int_0^\infty d\sigma\,h_1{}^0(\sigma)=0. \tag{3.4}$$

Under these circumstances, the $C_0(q^2)$ in (3.2) is equal to zero for $q^2>0$, whereas because of the $\epsilon(q^2+\sigma)$ in the integrand, for q^2 timelike C_0 becomes

$$C_0=-2\int_0^{-q^2}d\sigma\,h_1{}^0(\sigma)\quad(q^2<0). \tag{3.5}$$

An identical phenomenon occurs if $\bar{h}_0$ in (2.13) has a part of the form $h_0{}^0(\sigma)\delta(\beta)$, since then C_0 picks up an additional contribution proportional to $h_0{}^0(-q^2)$.

The possibility of a term in (1.7) with $\alpha=0$ which can arise either from an $h_1{}^0$ satisfying (3.3) or from a term in $\bar{h}_0$ proportional to $\delta(\beta)$ is the reason for assertion (6) in Sec. I. Except for this latter occurrence, Eq. (3.2) is correct.

IV. REGGE LIMIT OF ReT; FIXED POLES

A look at the R limit of the real part of the scattering amplitude T will allow us to study the possible presence of fixed poles of the conventional kind, i.e., those that occur in ReT but not in ImT. We are particularly interested in a pole of this kind at $J=0$, which corresponds to a term constant in ν in the R limit of ReT.

It is evident from (2.1) that T contains poles at integer values of J if the weight functions h_m have δ functions or derivatives of δ functions at $\beta=0$. Also, as discussed in Sec. III for a pole at $J=0$, the same phenomena arise if the h_m have parts proportional to $\epsilon(\beta)$ as $\beta\to 0$; and as far as the asymptotic behavior in the R limit is concerned, a term like $\delta(\beta)$ in h_m is equivalent to one like $\epsilon(\beta)$ in h_{m+1}, or to one like $\beta\epsilon(\beta)$ in h_{m+2}, etc. However, fixed poles which arise in this fashion are distinguished by the fact that they occur in ImT, as well as in ReT—at least for timelike q^2.

In this section we explicitly assume that there is no fixed pole in W at $J=0$, that is, that there is no term in (1.7) with $\alpha=0$, and show that

$$\mathrm{Re}T(q^2,\nu)\underset{R}{\longrightarrow}\tfrac{1}{2}\pi\sum_{\alpha>0}\nu^\alpha C_\alpha(q^2)\cot\tfrac{1}{2}\pi\alpha+\sum_{n=1}^{N'}t_n(q^2)\nu^{2n}+K(q^2), \tag{4.1}$$

where $C_\alpha(q^2)$ is given in (1.7) and (3.2), and where

$$K(q^2)=P(q^2,0)+\sum_{n=1}^{N}2^{-m}\int d\sigma(\sigma+q^2)^{m-1}G_m(\sigma)+\int_0^\infty\frac{d\omega}{\omega}\tilde{F}(\omega). \tag{4.2}$$

The t_m are polynomials in q^2 which can be expressed in terms of $P(q^2,\nu)$ and the G_m, where $P(q^2,\nu)$ is the arbitrary polynomial in (2.1). G_m is the function in (2.13) which governs any polynomial growth of W as ν is increased at fixed hadronic mass, and $\tilde{F}$ is the truncated scale function defined in (1.8). The relation (4.2) will be exploited in Sec. V to derive sum rules involving the electron and photon total cross sections; these sum rules actually test for the absence of P and the G_m, which amounts [by (4.2)] to saying that the fixed pole at $J=0$ has a residue independent of q^2. For the moment, however, let us simply note that assertions (7) and (8) in Sec. I follow from (4.2), since the G_m and $\tilde{F}$ are determined completely by the asymptotic form of W in the region of spacelike q^2, and since the first two terms on the right-hand side of (4.2) are polynomials in q^2.

From (2.1) it is evident that T satisfies a dispersion relation in q^2 at fixed ν with a discontinuity in q^2, $T(q^2+i\epsilon,\nu)=T(q^2-i\epsilon,\nu)$ equal to $-2\,\mathrm{Im}T$. Thus, from (2.1) and (1.2) we can write

$$\mathrm{Re}T(q^2,\nu)=P(q^2,\nu)+\tfrac{1}{2}P\int_{-\infty}^{2\nu}\frac{dq'^2}{q'^2-q^2}W(q'^2,\nu)\,, \quad (4.3)$$

where $P(q^2,\nu)$ is the same polynomial in q^2 and ν that occurs in (2.1).

We are interested in the limit of ReT as ν goes to infinity at fixed q^2, and we wish to make use of the forms for W given in (2.13) for $-2\nu<q^2<2\nu$ and in (2.16) for $q^2<-2\nu$. The decompositions of W in these expressions are invalid at the two points $q^2=\pm 2\nu$, as is suggested by the fact that some kind of singularity must occur for the expressions in (2.13) and (2.16) to accommodate the Born-pole terms proportional to $\delta(q^2\pm 2\nu)$. The second model studied in the Appendix is included primarily to lend understanding to this feature and, as shown there, some of the $h_m(\sigma,\beta)$ in (2.1) must have a singular behavior when $\sigma=0$ and $\beta=\pm 1$. These two points come into the expressions in (2.13) and (2.16) only when $q^2=\pm 2\nu$.

Despite the technical complication associated with the points $q^2=\pm 2\nu$, we wish to use the forms of W in (2.13) and (2.16) to discuss the R limit of the integral in (4.3). Suppose we eliminate from the range of integration in (4.3) the regions $2\nu-\Delta\leq q'^2\leq 2\nu$ and $-2\nu-\Delta\leq q'^2\leq 2\nu+\Delta$, where Δ is finite, but less than the gap in q^2 between the pole at 2ν and the beginning of the inelastic continuum (i.e., $\Delta<m_\pi M+\tfrac{1}{2}m_\pi^2$ for lepton scattering off a proton of mass M). Deletion of the first of these two intervals only eliminates the contribution of the Born pole at $q'^2=2\nu$ from (4.3); and this modification certainly does not affect the R limit of (4.3) because the elastic form factors which accompany the $\delta(q'^2-2\nu)$ in W decrease very rapidly in ν. Further, since $W(q^2,\nu)=0$ for $2\nu-\Delta<q^2<2\nu$, it is clear from (2.13) that $G_m(\sigma)=0$ for $0<\sigma<\Delta$, and therefore from (2.16) that elimination of the interval $-2\nu-\Delta\leq q'^2\leq -2\nu+\Delta$ from (4.3) only omits the contribution to (4.3) from the pole at $q'^2=-2\nu$, plus possibly an integral over this interval of a background part of W which is bounded as ν goes to infinity. Because of the denominator of the integrand in (4.3), this background contribution must vanish in the R limit, as must also the effect of the pole—provided, of course, that the elastic form factors decrease sufficiently rapidly for large timelike q^2. Thus, although we will not complicate the presentation by explicitly including Δ in all of the following equations, let us understand that we may consider the integral in (4.3) to be truncated in this manner when it is convenient for the discussion.

Consider the contribution to (4.3) which comes from the interval $-\infty<q'^2<-2\nu-\Delta$, and specifically consider the part of this contribution which comes from the second sum on the right-hand side of (2.16). By a change of variable to $\omega'=q'^2/2\nu$, this contribution to (4.3) in the R limit is of the form

$$\lim_{\nu\to\infty}\int_{-\infty}^{-1-(\Delta/2\nu)}\frac{d\omega'}{\omega'}\bar{W}(\omega',\nu)\,, \quad (4.4)$$

where by looking at the second term on the right-hand side of (2.16) and, if necessary, by considering the discussion of Sec. II, it is clear that

$$\text{(i)}\quad \bar{W}(\omega,\nu)\xrightarrow{\nu\to\infty}0\quad(-\infty<\omega<-1-\Delta/2\nu)$$

$$\text{(ii)}\quad \bar{W}[-1-\Delta/2\nu,\nu]\xrightarrow{\nu\to\infty}\leq\text{const}.$$

Thus, the limit in (4.4) is zero, and we can ignore this part of (4.3) as ν goes to infinity.

Consider next the contributions to (4.3) coming from the first terms on the right-hand sides of (2.13) and (2.16) which are applicable, respectively, to the intervals $-2\nu+\Delta\leq q'^2\leq 2\nu-\Delta$ and $q'^2<-2\nu-\Delta$. Substituting these parts of W into (4.3), one obtains

$$\tfrac{1}{2}\sum_{m=1}^{N}\nu^m\int_0^\infty d\sigma\, G_m(\sigma)\left[\frac{1}{2\nu-\sigma-q^2}+\frac{(-1)^m}{2\nu+\sigma+q^2}\right], \quad (4.5)$$

and expanding this expression in powers of ν^{-1} gives for the R limit of (4.3)

$$\mathrm{Re}T(q^2,\nu)\xrightarrow{R}P(q^2,\nu)+\sum_{m=1}^{N}\sum_{n=0}^{\frac{1}{2}[m-1]}\nu^{2n}2^{2n-m}\times\int_0^\infty d\sigma\, G_m(\sigma)(\sigma+q^2)^{m-1-2n}+\tfrac{1}{2}P\int_{-1}^{1}\frac{d\omega'}{\omega'-q^2/2\nu}W'(\omega',\nu)\,, \quad (4.6)$$

where $[m-1]$ is either $m-1$ or $m-2$, whichever is even; and W' refers to the part of W contained in the second sum on the right-hand side of (2.13)

$$W'(\omega,\nu)=-\sum_{m=0}^{1}\nu^{m-1}\int_0^{2\nu(1-\omega)}d\sigma\,\bar{h}_m\!\left(\sigma,-\omega-\frac{\sigma}{2\nu}\right). \quad (4.7)$$

It should be clear that $\bar{h}_0$ in (4.7) will give no contribution to (4.6) which survives in the R limit unless it contains a part proportional to $\delta(\beta)$, which we have excluded by assumption. Thus we need only consider the contribution of $\bar{h}_1$ in (4.7).

Suppose we decompose the $\bar{h}_1$ in (4.7) as in (3.1) before substituting (4.7) into (4.6). The $\tilde{h}_1$ in (3.1) vanishes as $\beta \to 0$ and for its contribution to (4.6) the limit $\nu \to \infty$ can be performed before the integration over ω'; thus from (1.8), (2.14), (2.15), (3.1), and (4.7)

$$P\int_{-1}^{1} \frac{d\omega}{\omega - q^2/2\nu} W'(\omega,\nu) \xrightarrow{R} P\int_{-1}^{1} \frac{d\omega}{\omega} \tilde{F}(\omega) + \sum_{\alpha>0} \int_0^\infty d\sigma\, h_1{}^\alpha(\sigma) P\int_{-1}^{1} \frac{d\omega}{\omega - q^2/2\nu} \frac{\epsilon(\omega+\sigma/2\nu)}{|\omega+q^2/2\nu|^\alpha}, \quad (4.8)$$

where we have again excluded the possibility of an $\alpha=0$ in the asymptotic form of W. The last integral on the right-hand side of (4.8) is

$$P\int_{-1}^{1} \frac{d\omega}{\omega - q^2/2\nu} \frac{\epsilon(\omega+\sigma/2\nu)}{|\omega+q^2/2\nu|^\alpha} = \frac{(2\nu)^\alpha}{|q^2+\sigma|^\alpha} P\int_{-\infty}^{\infty} \frac{dx}{x-1} \frac{\epsilon(x)}{|x|^\alpha} - 2\int_1^\infty \frac{d\omega}{\omega^{\alpha+1}} + O(1/\nu). \quad (4.9)$$

Because of the antisymmetry of $F(\omega)$ as given in (1.4) and elaborated in Sec. II, the first term on the right-hand side of (4.8) can be written as an integral between zero and one. Further, with the definition (1.8)—and remembering (2.15) and (3.1)—the second term on the right-hand side of (4.9) can be included in (4.8) simply by extending this integral from one to infinity. Thus, if we also perform explicitly the first integral on the right-hand side of (4.9) and make use of (3.2), Eq. (4.8) becomes

$$P\int_{-1}^{1} \frac{d\omega}{\omega - q^2/2\nu} W'(\omega,\nu) \xrightarrow{R} \pi \sum_{\alpha>0} \nu^\alpha C_\alpha(q^2) \times \cot\tfrac{1}{2}\pi\alpha + 2\int_0^\infty \frac{d\omega}{\omega} \tilde{F}(\omega). \quad (4.10)$$

If (4.10) is substituted into (4.6), it is evident that the relation in (4.1) is obtained with K given by (4.2) and with $t_n(q^2)$ containing contributions from both the first and second terms on the right-hand side of (4.6). As stated at the beginning of this section, assertions (7) and (8) in Sec. I follow from (4.1) and (4.2). Note also that the Regge residues for ReT contained in the first term on the right-hand side of (4.1) are expressed in terms of the $C_\alpha(q^2)$, the Regge residues in ImT as given by (1.2) and (1.7), for all values of q^2.

V. TWO SUM RULES

In this section we derive the two sum rules for electron and photon total cross sections that were given in Ref. 6. The precise forms of these relations, which are also given here in Eqs. (5.12) and (5.14), presume that the G_m's in (2.13) are zero for the two invariant Compton amplitudes (defined below).[16] As we have discussed, it is extremely unlikely that any of the G_m's are not zero; nevertheless, if they are present, they will eventually be measured in electroproduction, and our sum rules—although modified from (5.12) and (5.14)—would still relate experimental quantities.

The forward, virtual (spin-averaged) Compton amplitude $T_{\mu\nu}$ can be decomposed into two invariant amplitudes T_1 and T_2, defined as

$$T_{\mu\nu} = -\tfrac{1}{4}i\int d^4x\, e^{-iq\cdot x}\langle p|T^*[J_\mu(x)J_\nu(0)]|p\rangle \quad (5.1a)$$

$$= (\delta_{\mu\nu} - q_\mu q_\nu/q^2)T_1 + (p_\mu - q_\mu q\cdot p/q^2)(p_\nu - q_\nu q\cdot p/q^2)T_2, \quad (5.1b)$$

where the W_i given in terms of the ImT_i by (1.2) are related to the cross sections for the scattering of transverse and longitudinally polarized photons, σ_T and σ_L, by

$$W_1 = (4\pi^2\alpha)^{-1}(\nu - \tfrac{1}{2}q^2)\sigma_T, \quad (5.2a)$$

$$W_2 = (4\pi^2\alpha)^{-1}(\nu - \tfrac{1}{2}q^2)(\nu^2 + q^2M^2)^{-1}q^2(\sigma_T + \sigma_L). \quad (5.2b)$$

Here M is the hadron mass, and $\alpha = 1/137$. Since σ_L vanishes at $q^2=0$,

$$W_1 \xrightarrow[q^2\to 0]{} \frac{\nu^2}{q^2} W_2 \xrightarrow[q^2\to 0]{} (4\pi^2\alpha)^{-1}\nu\sigma_\gamma(\nu), \quad (5.3)$$

where $\sigma_\gamma(\nu)$ is the total cross section for real photon scattering from the hadron.

As suggested by Bjorken[1] and supported by the electroproduction data,[2] we will take the W_i to scale as

$$W_1 \xrightarrow[B]{} F_1(\omega), \quad (5.4a)$$

$$\nu W_2 \xrightarrow[B]{} F_2(\omega). \quad (5.4b)$$

Thus, if we construct two new amplitudes as

$$T_+ \equiv (2\nu^2/q^2)T_2, \quad (5.5a)$$

$$T_L \equiv (\nu^2/q^2 + M^2)T_2 - T_1, \quad (5.5b)$$

[16] Even when the B limit exists, it is common in perturbation theory to find other of the assumptions leading to (5.14) violated; i.e., either the G_m are unequal to zero, or W has a term constant in ν in the R limit. If only the first of these is violated, then the sum rule (5.14) exists in a modified form, as we discuss in the paper. It is not surprising, therefore, that the sum rule (5.14) is contradicted in perturbation theory, as noted in Ref. 19 for a model where $F_L(\omega)\sim\omega$.

and define, as in Sec. II,

$$W_+ = -2\pi^{-1}\,\mathrm{Im}T_+\,, \tag{5.6a}$$

$$W_L = -2\pi^{-1}\,\mathrm{Im}T_L\,, \tag{5.6b}$$

the W_+ and W_L scale in the B limit as

$$W_+ \underset{B}{\rightarrow} F_+ = F_2/\omega\,, \tag{5.7a}$$

$$W_L \underset{B}{\rightarrow} F_L = F_2/2\omega - F_1\,. \tag{5.7b}$$

Consider the amplitude T_+, defined in (5.5a), at $q^2=0$. From (5.3) and (5.6a) it follows that $\mathrm{Im}T_+(0,\nu) = -(4\pi\alpha)^{-1}\nu\sigma_\gamma(\nu)$. If we write for $T_+(0,\nu)$ a dispersion relation in ν, subtracted at $\nu=0$ with the help of the Thomson theorem, we obtain

$$T_+(0,\nu) = Q^2 - \frac{\nu^2}{2\pi^2\alpha}\int_{\nu_0}^{\infty}\frac{d\nu'\sigma_\gamma(\nu')}{\nu'^2-\nu^2}\,, \tag{5.8}$$

where Q is the charge of the target hadron in units of e ($Q=+1$ for the proton), and where ν_0 is the threshold for photoproduction (equal to $Mm_\pi + \frac{1}{2}m_\pi^2$).

We are interested in $T_+(0,\nu)$ as ν goes to infinity. If we assume that $\mathrm{Im}T_+(0,\nu)$ has the Regge asymptotic form typified by (1.7), we can define

$$\tilde{\sigma}_\gamma(\nu) \equiv \theta(\nu-\nu_0)\sigma_\gamma(\nu) - \sum_{\alpha>0} C_\alpha(0)\nu^{\alpha-1}\,, \tag{5.9}$$

where $\tilde{\sigma}_\gamma(\nu)$ vanishes faster than $\nu^{-1-\epsilon}$, provided there is no part of $\mathrm{Im}T$ in the high-energy limit characteristic of a Regge pole with $\alpha=\alpha(0)=0$. If we substitute (5.9) into (5.8) and let ν go to infinity, we obtain

$$\mathrm{Re}T_+(0,\nu) \underset{R}{\rightarrow} \sum_{\alpha>0} C_\alpha'\nu^\alpha + K_\gamma(0)\,, \tag{5.10}$$

with

$$K_\gamma(0) = Q^2 + \frac{1}{2\pi^2\alpha}\int_0^\infty d\nu\,\tilde{\sigma}_\gamma(\nu)\,. \tag{5.11}$$

Our objective is to relate $K_\gamma(0)$ to a truncated electroproduction scale function by using Eq. (4.2). We can do this, since T_+ is an example of the general amplitude T discussed in Secs. II–IV. First, however, note that $P(q^2,0)$, which occurs on the right-hand side of (4.2), is evidently zero for the amplitude T_+; this follows from the general DGS form (2.1), since T_+ defined in (5.5a) vanishes like ν^2 as $\nu\rightarrow 0$, except for the Born terms. What about the contribution to $K_\gamma(0)$ coming from the second term on the right-hand side of (4.2)? As we have discussed in connection with Eq. (2.13), if any of the G_m's did not vanish for the amplitude T_+, it would imply that $W_+ \sim (\nu^2/q^2)W_2$ would grow as ν^n ($n\geq 1$) as ν increased with a fixed mass for the produced hadronic system. Despite the fact that the present electroproduction data may not unequivocally rule out the possibility of small terms of this kind in the cross sections, we consider their presence to be sufficiently implausible to warrant our writing the sum rule as if they were absent. Thus, if for T_+ the first two terms on the right-hand side of (4.2) are zero, we obtain by comparing (4.2), (5.7a), and (5.11) (and setting $Q^2=1$)

$$1 + \frac{1}{2\pi^2\alpha}\int_0^\infty d\nu\,\tilde{\sigma}_\gamma(\nu) = \int_0^\infty \frac{d\omega}{\omega^2}\tilde{F}_2(\omega)\,, \tag{5.12}$$

where, by analogy with (1.8),

$$\tilde{F}_2 = \theta(1-\omega)F_2 - \sum_{\alpha>0} f_{2\alpha}\omega^{1-\alpha}\,. \tag{5.13}$$

If the G_m for T_+ are eventually found to be nonzero, the second term on the right-hand side of (4.2) can be measured and inserted as an additional, positive contribution to the right-hand side of (5.12).

The second sum rule is derived similarly, but for the amplitude T_L in (5.56). This amplitude, whose imaginary part is proportional to σ_L from (5.2), vanishes at $q^2=0$ to insure that $T_{\mu\nu}$ in (5.1) has no unwanted pole. Thus, the limit in (4.1) and the $K_L(0)$ in (4.2) must be zero for $T_L(0,\nu)$.

We shall also assume for T_L that the G_m in the second term on the right-hand side of (4.2) are zero, recognizing that the sum rule below can be modified appropriately in the unlikely event that this assumption is wrong.

Consider the $P_L(0,0)$ occurring first on the right-hand side of (4.2) for $T_L(0,\nu)$. We note from (5.1) that if either T_1 or T_2 has a part constant in q_0 in the limit $q_0\rightarrow\infty$, $\mathbf{q}$ fixed, then T_{0i} has a term behaving like q_0^{-1} in this limit. But according to Bjorken,[17] the equal-time commutator of j_0 and j_i would then contribute to the connected matrix element in (5.1a), implying that the Schwinger term was a q number (i.e., an operator). From the DGS representation for an amplitude T in (2.1), it is evident that the polynomial P must vanish if there is no part of T constant in q_0 in the above limit. Thus, if the Schwinger term is a c number, these polynomials must be zero for the amplitudes T_1, T_2, and $(\nu^2/q^2)T_2$ and hence also for T_L defined in (5.5b). The first two terms on the right-hand side of (4.2) are then zero for T_L, and from (5.7b) the expression in (4.2) becomes

$$\int_0^\infty \frac{d\omega}{\omega}\left(\frac{\tilde{F}_2}{2\omega} - \tilde{F}_1\right) = 0 \tag{5.14}$$

for a c-number Schwinger term. The $\tilde{F}_2$ is given in (5.13), and $\tilde{F}_1$ is related to F_1 as in (1.8).

Unfortunately, the sum rules in (5.12) and (5.14) cannot easily be tested with the existing electroproduction and photoproduction data. The left-hand side of (5.12) is known to some extent from the work of Damashek and Gilman,[18] who have determined it to

[17] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).
[18] M. Damashek and F. J. Gilman, Phys. Rev. D **1**, 1319 (1970).

lie somewhere in the range between 0.12 and 1.59, with a preferred value near unity—which amusingly is what it is in the absence of strong interactions. However, essentially nothing can be said about the numerical value of the right-hand side of (5.12). The integral is very sensitive to the choice of the $f_{2\alpha}$ in (5.13), whose determination depends upon accurate measurements of the scale function F_2 near $\omega=0$. At presently available energies, small values of ω tend to come associated with small q^2 ($q^2<1$ GeV^2), and it is doubtful that the scale limit has been reached. Further, in the region of small ω the errors introduced from the radiative corrections tend to become amplified. Note, however, that if σ_L/σ_T is shown experimentally to be zero in the scale limit, then the sum rule in (5.14) is satisfied.

The sum rule in (5.14) is the same as the relation in Eq. (19a) of Ref. 9, if the cross sections occurring in the latter are re-expressed in terms of the scale functions of (5.4). There is an essential difference in the derivation, however, since in Ref. 9 it was assumed that T_L had no fixed pole at $J=0$ for any q^2, whereas here we have simply imposed the necessary absence of a fixed pole in T_L at $q^2=0$. But from (4.2), with the vanishing of the P_L and the G_m, it is evident that any fixed pole residue is constant in q^2, so that its vanishing at $q^2=0$ implies its vanishing everywhere.

The sum rule in (5.14) has also been obtained by other authors[19] under the assumptions that the leading Regge trajectories (those with $\alpha>0$) do not contribute to W_L and that T_L satisfies an unsubtracted dispersion relation in ν (or, equivalently, that $\mathrm{Re}T_L$ has no fixed pole at $J=0$). If these leading trajectories are absent in T_L, the truncation of the F_L typified by (1.8) is unnecessary, and the integrand of (5.14) becomes $F_2/2\omega-F_1$, which is non-negative. The sum rule in (5.14) would then imply that $F_1=(2\omega)^{-1}F_2$ or, equivalently (for finite F_1), that σ_L/σ_T vanished in the B limit. The algebra-of-fields model, where $F_1=0$ [20] and the Schwinger term is a c number,[21] would then require that both F_1 and F_2 are zero, in clear contradiction to what is observed.

Sum rules similar to those which we have derived in this section have been given recently by Leutwyler and Stern.[22]

VI. CONCLUSIONS

In addition to the specific points listed in Sec. I and the two sum rules given in Eqs. (5.12) and (5.14), we have seen under what conditions a fixed-pole residue at $J=0$ in, for example, T_+ or T_L defined in (5.5) is independent of q^2. The essential condition from which this follows is that the fixed pole occurs only in the real part of the amplitude. The validity of this feature implies that the residue is a polynomial in q^2, as indicated specifically in Eq. (4.2).[3] The conclusion that the polynomial is a constant then follows to the extent that the first two q^2-dependent terms on the right-hand side of (4.2) can be shown to be zero. We have argued that these terms do not contribute for either T_+ or T_L, although—as we have discussed—the presence of any contributions from the G_m can be determined directly from the electroproduction data. Because of the feature listed under (6) in Sec. I, it is not possible to rule out the presence of a $J=0$ fixed pole in the imaginary part of the invariant Compton amplitudes simply by noting that there is no term with $\alpha=0$ in the Regge limit of the photoproduction or electroproduction cross sections. Experimental verification of this assumption must await a check on sum rules like (5.12) and (5.14).

It is amusing to note that the q^2 independence of the $J=0$ fixed pole residues in the invariant Compton amplitudes allow one to use a subtracted[4] Cottingham formula[5] and calculate the electromagnetic mass shifts only in terms of electroproduction data. For example, starting with

$$\delta M^2=\frac{\alpha}{\pi^2 M}\int_0^\infty \frac{dq^2}{q^2}\int_0^{M\surd(q^2)} d\nu\left(q^2-\frac{\nu^2}{M^2}\right)^{1/2} T_{\mu\mu}(q^2,i\nu)\,, \quad (6.1)$$

and noting from (5.1) and (5.5b) that

$$T_{\mu\mu}=-3T_L+2(M^2+\nu^2/q^2)T_2\,, \quad (6.2)$$

we can write a dispersion relation in ν for T_L and T_2, subtracting the one for T_L at $\nu=0$. To determine the subtraction constant, $T_L(q^2,0)$, we write

$$T_L(q^2,\nu)=T_L(q^2,0)-\nu^2\int\frac{d\nu' W_L(q^2,\nu')}{\nu'(\nu'^2-\nu^2)}\,, \quad (6.3)$$

and note that if the fixed pole residue is independent of q^2 its vanishing at $q^2=0$ implies that

$$T_L(q^2,0)=-\int_0^\infty \frac{d\nu'}{\nu'}\tilde{W}_L(q^2,\nu')\,, \quad (6.4)$$

where the truncated W_L,

$$\tilde{W}_L=W_L-\sum_{\alpha>0} C_L{}^\alpha(q^2)\nu^\alpha\,, \quad (6.5)$$

vanishes as $\nu\to\infty$. Note that the sum rule in (5.14) and the connection between the Regge residues and the

[19] R. Jackiw, R. Van Royen, and G. West, Phys. Rev. D 2, 2473 (1970).

[20] C. Callan and D. J. Gross, Phys. Rev. Letters 22, 156 (1969).

[21] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

[22] H. Leutwyler and J. Stern, Phys. Letters 31B, 458 (1970).

asymptotic terms in the $F(\omega)$ for $\omega \to 0$ given in (1.8) and (1.9) imply from (6.4) that $T_L(q^2,0)$ vanishes as $q^2 \to \infty$. This condition eliminates the quadratic divergence in the mass shift. The logarithmically divergent part comes from the coefficient of q^{-2} in the large q^2 limit of the right-hand side of (6.4) and from integrals over the scale functions F_L and F_2. Reliable estimates of these contributions must await very accurate measurements of the electroproduction cross sections. However, a finite answer is consistent with the positiveness of the cross sections σ_L and σ_T, whether or not W_L has the leading Regge trajectories.

APPENDIX

In this Appendix we give the DGS representation of Eq. (2.1) for two simple models of T and illustrate some of the features discussed more generally in the text.

Parton Model

The parton model[14] pictures the hadron as composed of elementary constituents, each of which carries a fraction β of the hadron momentum p. Further, the amplitude T is taken to be the sum of the amplitudes for the free scattering (i.e., the Born approximation) off the individual partons. Thus, if the momentum distribution in β weighted with the square of the current-parton charge is given by a function $f(\beta)$, the partonlike model of T would be of the form, for example,

$$T=\int_0^1 \frac{d\beta(-q\cdot\beta p)f(\beta)}{(q+\beta p)^2-(\beta p)^2}+(\nu\to-\nu)\,, \qquad \text{(A1)}$$

which has the DGS representation in (2.1) with $h_1=\beta\delta(\sigma)f(\beta)$ and $h_m=0$ for $m\neq 1$. This model has the feature that it scales for all values of q^2—that is, it satisfies (1.3)–(1.5) as equalities. However, it is difficult to incorporate the Regge behavior (1.7), since the $C_\alpha(q^2)$ as given by (3.2) would diverge at $q^2=0$ for $\alpha>0$.

Born Approximation with Form Factors

Here we take T to be

$$T=\nu F(q^2)[(q^2-2\nu)^{-1}-(q^2-2\nu)^{-1}]\,, \qquad \text{(A2)}$$

where the form factor (squared) F satisfies

$$F=\int\frac{dm^2\rho(m^2)}{q^2+m^2}. \qquad \text{(A3)}$$

By substituting (A3) into (A2) and using the Feynman trick to write the product of the two denominators as an integral, (A2) can be rewritten as

$$T=\nu\int dm^2\rho(m^2)\int_{-1}^{1}d\beta\frac{\epsilon(\beta)}{[q^2+2\beta\nu+m^2(1-|\beta|)]^2}. \qquad \text{(A4)}$$

If we introduce an integration over σ and integrate by parts, we can manipulate (A4) into the DGS form in (2.1) with

$$\begin{aligned} h_1(\sigma,\beta)&=\rho'(\sigma/1-|\beta|)(1-|\beta|)^{-2}\epsilon(\beta)\,,\\ h_m&=0 \quad (m\neq 1)\,, \end{aligned} \qquad \text{(A5)}$$

where the prime indicates a derivative. As discussed in Sec. IV, the h_1 in (A5) is well behaved except when $\sigma=0$ and $\beta=\pm 1$. The singular behavior at this point is necessary for the general form of W given in (2.13) to accommodate the Born poles at $q^2=\pm 2\nu$. Further, the fact that h_1 in (A5) behaves like $\epsilon(\beta)$ for small β implies that the expansion in (1.7) has a term with $\alpha=0$. This kind of occurrence, which arises here from the contribution to W from ImF in (A3), is explicitly assumed not to exist in the derivations of Secs. IV and V.

Although it is contrary to the spirit which motivated our assumption as stated in the text, there is the possibility that the fixed pole at $J=0$ arises only from the Born term as illustrated here. In that event, the sum rule in Eq. (5.12) would be modified by having the 1 deleted from the left-hand side.