Direct- and Cross-Duality Amplitudes

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General considerations based on analyticity and asymptotic behavior lead us to construct two crossing-symmetric model amplitudes for equal-mass spinless scattering, interpolating imposed asymptotic forms at low and high energy. One, the direct-duality amplitude, is a generalized Veneziano model. In the other, Regge-cut exchange is dual to the direct-channel resonances while Regge-pole exchange is dual to the direct-channel cut.

I. INTRODUCTION

IN this paper we consider the construction of amplitudes possessing narrow resonances and cuts, and satisfying crossing symmetry and asymptotic behavior. We start from general assumptions on the asymptotic behavior of the scattering amplitude—for an equal-mass spinless process—at both low and high energies. At low energy, we consider contributions to the imaginary part of the amplitude from the elastic unitarity cut—a square-root branch point—and from the lowest-lying resonance (narrow and spinless). It is assumed that there are no bound-state poles. A high energy, contributions from the leading Regge pole and the leading Regge cut are considered. These asymptotic forms are then transformed into corresponding limiting behavior for the Laplace transform of the imaginary part of the amplitude. Model expressions for the Laplace transform interpolating these asymptotics then give model integral representations for the scattering amplitude.

We find that two simple constructions satisfying crossing and the imposed asymptotic behavior readily present themselves. In the first the amplitude consists of the sum of two terms; one interpolates the low-energy resonance behavior and the high-energy Regge-pole behavior, while the other connects the Regge-cut asymptotic form to that of the unitarity cut. This amplitude therefore possesses the usual property of duality[1] and introduces a definite term for the associated cuts.[2] We call this construction a "direct-duality model." In the second construction the asymptotic forms at low and high energy are linked crosswise: Regge pole to low-energy cut and Regge cut to low-energy resonance. We call the resulting amplitude a "cross-duality model." Figure 1 illustrates these connections.

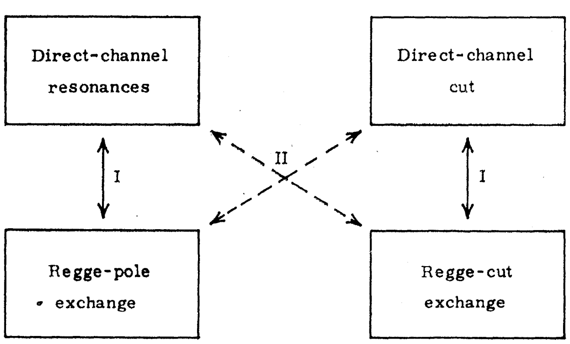

FIG. 1. Illustration of direct duality (I) and cross duality (II).

The complete scattering amplitude $M(s,t,u)$ is assumed to satisfy the unsubtracted Mandelstam representation

$$M(s,t,u)=A(s,t)+A(u,t)+A(s,u), \tag{1.1}$$

where $A(s,t)$ is the double-dispersive integral over the st spectral function. The direct-duality amplitude is given by

$$A(s,t)=C_1\int_0^1 y^{-\alpha(t)-1}(1-y)^{-\alpha(s)-1}dy+C_2\int_0^1 \frac{y^{-\alpha_c(t)-1}(1-y)^{-\alpha_c(s)-1}}{\{1-\ln[y(1-y)]\}^{3/2}}dy, \tag{1.2}$$

where C_1 and C_2 are arbitrary constants, $\alpha(t)$ is the leading Regge trajectory, and $\alpha_c(t)$ is the branch point of the leading Regge cut. Both $\alpha(t)$ and $\alpha_c(t)$ are real. The first term in (1.2) is, of course, the Veneziano amplitude.[3] The second term has square-root branch points in s and t, and behaves like a Regge cut (with a square-root branch point) asymptotically. It can be transformed into integral forms explicitly exhibiting these cuts [see Eqs. (3.10)–(3.13)]. Except for a slight difference in the integrand of the second term in (1.2), this amplitude has also been recently suggested by Matveev, Stoyanov, and Tavkhelidze.[4]

The cross-duality amplitude is given by

$$A(s,t)=D\int_0^1 \frac{y^{-\alpha_c(t)-1}(1-y)^{-\alpha(s)-1}}{(1-\ln y)^{3/2}}dy+(s\leftrightarrow t), \tag{1.3}$$

where D is an arbitrary constant. In the physical region

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[1] R. Dolen, D. Horn, and C. Schmid, Phys. Rev. **166**, 1768 (1968).

[2] These may be called "duality-preserving cuts" in the sense that Regge cuts are dual to unitarity cuts. An exotic channel will, however, still have these cuts in contrast to the duality-preserving cuts of V. Barger and R. J. N. Phillips, Phys. Letters **29B**, 676 (1969).

[3] G. Veneziano, Nuovo Cimento **57A**, 1395 (1968).

[4] V. A. Matveev, D. T. Stoyanov, and A. N. Tavkhelidze, Phys. Letters **32B**, 61 (1970).

of s-channel scattering ($s>4m^2$, $t<0$), the first term in (1.3) may be written as

$$\sum_{n=0}^{\infty} \frac{g_n(t)}{\alpha(s)-n}, \tag{1.4}$$

where the coefficients $g_n(t)$ are polynomials of order n in t. This term behaves like a Regge cut as $s\to\infty$. The second term in (1.3) has a cut for $s>4m^2$ and behaves like a Regge pole as $s\to\infty$. It is thus clear that (1.3) provides an analytically plausible description of the scattering amplitude and may be viewed as an s-channel or a t-channel dispersion relation. This model is therefore manifestly different from an interference model in which sums of s resonances and t resonances—convergent for all values of s and t—are added[5] or one in which the amplitude completely consists of resonances plus Regge poles.[6] It could, however, be similar to constructions called by Jengo "generalized interference models," whose validity as representations of the scattering amplitude is proved[7] under rather general conditions.

The amplitudes in (1.2) and (1.3) appear to be quite different. They, in fact, have some striking similarities. Both possess the general structure: (s resonances) $+$(s cut) for $s>4m^2$, $t<0$. In both amplitudes a square-root branch point in the energy plane is accompanied by a square-root branch point in the Regge plane. Our method of construction imposes, in both cases, the equations

$$\alpha(t)=b(t-t_R), \tag{1.5}$$

$$\alpha_c(t)=a(t-t_0), \tag{1.6}$$

where $t=t_R$ is the lowest resonance, $t=t_0$ ($=4m^2$) is the lowest normal threshold, and a and b are arbitrary constants. Equation (1.5) says that the lowest resonance lies on the leading Regge-pole trajectory. Equation (1.6) similarly asserts that the lowest branch point lies on the trajectory of the leading Regge branch point. Under the assumption that (1.6) holds for a $\pi\pi$ cut generated by double Regge ρ-pole exchange, one obtains

$$\alpha_\rho(m_\pi^2)=\tfrac{1}{2}. \tag{1.7}$$

Both amplitudes give the following form for the discontinuity across the Regge cut in the neighborhood of the branch point:

$$\frac{[\alpha_c(t)-\alpha]^{1/2}}{\Gamma(\alpha_c(t)+1)[\alpha_c(t)+\frac{1}{2}]}, \quad \alpha\approx\alpha_c(t). \tag{1.8}$$

Away from the branch point the discontinuities are different, but they both contain the exponential factor $e^{-[\alpha_c(t)-\alpha]}$. The expression (1.8) satisfies the general result of Bronzan and Jones[8] that the discontinuity of the partial-wave amplitude across the Regge cut is singular at the branch point and vanishes there.

Although the two models (1.2) and (1.3) are essentially asymptotically equivalent, the scale factor $1/s_0$ in the high-energy term $(s/s_0)^{\alpha(t)}$ is not the same in both; it is $1/b$ in the direct-duality model and $1/a$ in the cross-duality model. It would have been difficult to detect this difference experimentally if it were to appear only asymptotically. This particular difference, however, is maintained in the couplings of the resonances which are given in terms of α for (1.2) and in terms of α_c for (1.3). It is therefore a measurable difference exemplified in the ratio of the couplings, or widths, of adjacent resonances.

In the intermediate energy region the two models are, of course, quite different. This clearly follows from the fact that we do not impose any requirements besides crossing symmetry and asymptotic behavior. Differences in the intermediate energies are expected to be eliminated when unitarity is imposed. It is not in fact guaranteed that unitarity may be imposed while either form of duality is rigidly maintained. One observes that the duality, direct or crossed, that appears in these amplitudes is dictated by simplicity of construction. The two forms of duality presented give two very simple ways for satisfying crossing symmetry. It may well turn out that when crossing is imposed in such a special way, unitarity is always broken.

A notable difference between the models (1.2) and (1.3) is the relative coupling of the pole and cut terms; it is arbitrary in (1.2) and fixed in (1.3). In this respect, the cross-duality model is in line with models that generate Regge cuts by exchanging Regge poles,[9] where a single strength parameter determines the coupling of both the pole and cut terms.[10]

The $\pi\pi$ amplitude in the cross-duality model has now been constructed and is being studied by Mahanta and the present author.[11] It appears that simple constructions of the $I=2$ amplitude are possible which satisfy either (1.6), i.e.,

$$\alpha_c(4m_\pi^2)=0, \tag{1.9}$$

or the condition

$$\alpha_c(4m_\pi^2)=1. \tag{1.10}$$

Equation (1.9), as remarked above, is consistent with a leading $\rho\rho$ cut, while (1.10) corresponds to an amplitude with a leading PP cut. The slope of the Pomeranchuk trajectory is not predicted by (1.10), so that such an amplitude is also consistent with a trajectory of normal slope and nearly unit intercept.[10,12]

A term with the general structure of (1.3), i.e., of the form $f(\alpha(s),\alpha_c(t))+f(\alpha(t),\alpha_c(s))$ has previously been

[5] D. Sivers and J. Yellin, LRL Report No. 19418, 1970 (unpublished).

[6] C. Schmid, CERN Report No. TH. 1128, 1969 (unpublished); R. Oehme, Nucl. Phys. **B16**, 161 (1970).

[7] R. Jengo, Phys. Letters **28B**, 606 (1969); see also R. Oehme (Ref. 6).

[8] J. B. Bronzan and C. E. Jones, Phys. Rev. **160**, 1494 (1967).

[9] For a review, see J. D. Jackson, Rev. Mod. Phys. **42**, 12 (1970).

[10] See, e.g., S. Frautsci and B. Margolis, Nuovo Cimento **56A**, 1115 (1968).

[11] P. Mahanta and M. O. Taha (unpublished).

[12] J. S. Ball and G. Marchesini, Phys. Rev. **188**, 2508 (1969).

considered by Pinsky[13] as a unitarity correction to the Veneziano amplitude. In our work, the cut correction to Veneziano is given by the second term of (1.2), while (1.3) provides an alternative representation of the full amplitude.

We finally remark that several interesting questions remain to be studied, particularly in connection with the cross-duality model: the daughter and J-plane structure, N-point generalizations, and the form of the $\pi\pi$ amplitude. In both models one would like to introduce complex trajectories, complex Regge branch points, and resonances on the second sheet. The status of the Pomeranchuk singularity is also obscure in both amplitudes. We hope that the introduction of the cross-duality model enriches the context in which these questions are discussed.

II. ANALYTICITY AND ASYMPTOTIC BEHAVIOR

Let $M(s,t,u)$ be the scattering amplitude for a two-particle$\to$two-particle process in which the external particles are spinless and of equal mass m. Assume that there are no bound-state poles and that the amplitude satisfies the Mandelstam representation with no subtraction terms, i.e.,

$$M(s,t,u)=A(s,t)+A(u,t)+A(s,u)\,, \qquad (2.1)$$

$$A(s,t)=\frac{1}{\pi^2}\int_{t_0}^{\infty}\int_{s_0}^{\infty}\frac{\rho(s',t')}{(s'-s)(t'-t)}ds'dt'\,, \qquad (2.2)$$

where $s_0=t_0=4m^2$, $s+t+u=4m^2$, and $\rho(s,t)=\rho(t,s)$. We shall work with the amplitude $A(s,t)$ which is symmetric in s and t and possesses right-hand cuts (and resonance poles) in these variables.

Let $F(s,x)$ be the inverse Laplace transform of $A(s,-t)$ with respect to t, so that

$$F(s,x)=\lim_{\lambda\to\infty}\frac{1}{2\pi i}\int_{c-i\lambda}^{c+i\lambda}e^{-xt}A(s,t)dt\,,\quad c<t_0. \qquad (2.3)$$

$A(s,t)$ is then given by

$$A(s,t)=\int_0^{\infty}e^{xt}F(s,x)dx\,,\quad \mathrm{Re}t<t_0. \qquad (2.4)$$

The contour of integration in (2.3) may be transformed into one around the t cut, giving (on neglecting the contribution of the semicircle at infinity)

$$F(s,x)=\frac{1}{2\pi i}\int_{t_0}^{\infty}e^{-xt}\Delta_t(s,t)dt\,,\quad x>0 \qquad (2.5)$$

where $\Delta_t(s,t)$ is the discontinuity of $A(s,t)$ across the t cut. Introduce the function $f(s,x,a)=aF(s,ax)$, where a is an arbitrary positive constant. Equations (2.4) and (2.5) may then be written as

$$A(s,t)=\int_0^{\infty}e^{axt}f(s,x,a)dx\,,\quad \mathrm{Re}t<t_0 \qquad (2.6)$$

and

$$f(s,x,a)=a\int_{t_0}^{\infty}e^{-axt}D(s,t)dt\,,\qquad x>0 \qquad (2.7)$$

where

$$D(s,t)=(1/2\pi i)\Delta_t(s,t)\,. \qquad (2.8)$$

We now proceed to use Eq. (2.7) to determine the behavior of $f(s,x,a)$ corresponding to certain asymptotic forms assumed for $D(s,t)$. For this purpose we shall use Abelian theorems on Laplace transforms.[14] In particular, we consider, for large t, contributions to $D(s,t)$ from a Regge pole $\alpha(s)$ and a Regge cut with branch point at $\alpha_c(s)$. For small t ($\mathrm{Re}t\geqslant t_0$) we take the contributions from the neighborhood of the two-particle square-root branch point at the threshold $t=t_0$ and from the neighborhood of the lowest (zero-spin) resonance at $t=t_R$. For these contributions to $D(s,t)$, we take the following forms:

$$\text{Regge pole:}\quad \beta(s)\frac{t^{\alpha(s)}}{\Gamma(\alpha+1)}\qquad (t\to\infty)\,, \qquad (2.9a)$$

$$\text{Regge cut:}\quad \frac{\eta(s)t^{\alpha_c(s)}}{\Gamma(\alpha_c+1)(\ln t)^{\gamma}}\qquad (t\to\infty)\,, \qquad (2.9b)$$

$$\text{lowest resonance:}\quad A\delta(t-t_R)\qquad (t\approx t_R)\,, \qquad (2.9c)$$

$$\text{elastic cut:}\quad \frac{B(s)}{\Gamma(\frac{3}{2})}(t-t_0)^{1/2}\qquad (t\approx t_0)\,. \qquad (2.9d)$$

Equation (2.7) then determines the following corresponding behavior for $f(s,x,a)$:

$$\text{Regge pole:}\quad \frac{\beta(s)}{a^{\alpha(s)}}x^{-[\alpha(s)+1]}\qquad (x\to+0), \qquad (2.10a)$$

$$\text{Regge cut:}\quad \frac{\eta(s)x^{-[\alpha_c(s)+1]}}{a^{\alpha_c(s)}(-\ln x)^{\gamma}}\qquad (x\to+0), \qquad (2.10b)$$

$$\text{lowest resonance:}\quad Aae^{-axt_R},\ t_R>t_0\quad (x\to\infty)\,, \qquad (2.10c)$$

$$\text{elastic cut:}\quad \frac{B(s)}{a^{1/2}}e^{-axt_0}x^{-3/2}\qquad (x\to\infty)\,. \qquad (2.10d)$$

We remark here that (2.10a) and (2.10b) are valid for values of s such that $\mathrm{Re}\alpha(s)\geqslant 0$ and $\mathrm{Re}\alpha_c(s)\geqslant 0$, respectively. The integral representations which we shall construct for the amplitudes will therefore initially be valid in this domain and are given elsewhere by analytic continuation. We also note that at the asymptotic limit $x\to+0$ either (2.10a) or (2.10b) dominates, depending upon whether the leading singularity in

[13] S. Pinsky, Phys. Rev. Letters **22**, 677 (1969).

[14] D. V. Widder, *The Laplace Transform* (Princeton U. P., Princeton, N. J., 1946).

$D(s,t)$ comes from the pole or the cut, i.e., on whether $\text{Re}\alpha(s)>\text{Re}\alpha_c(s)$ or otherwise. In the limit $x\to\infty$ the elastic cut always dominates, so that (2.10c) and (2.10d) are additive with the resonance limit as a correction term to the cut contribution. Our assumption is that $t_0<t_R<t_1$, where t_1 is the next threshold branch point.

We now effect a change of variable from x to $y=e^{-x}$. Equation (2.6) becomes

$$A(s,t)=\int_0^1 y^{-at-1}g(s,y,a)dy\,,\quad \text{Re}t<t_0 \qquad (2.11)$$

where $g(s,y,a)=f(s,\,-\ln y,\,a)$. Near $x=0$, $y\approx 1-x$ so that in the forms (2.10a) and (2.10b) it is possible to make the substitution $y=1-x$. The contributions (2.10) then, respectively, take the following forms for $g(s,y,a)$:

$$\frac{\beta(s)}{a^{\alpha(s)}}(1-y)^{-[\alpha(s)+1]} \qquad (y\to 1,\, y<1), \qquad (2.12\text{a})$$

$$\frac{\eta(s)}{a^{\alpha_c(s)}}(1-y)^{-[\alpha_c(s)+1]}\times[-\ln(1-y)]^{-\gamma} \qquad (y\to 1,\, y<1), \qquad (2.12\text{b})$$

$$Aay^{at_R} \qquad (y\to +0), \qquad (2.12\text{c})$$

$$\frac{B(s)}{a^{1/2}}y^{at_0}(\ln y)^{-3/2} \qquad (y\to +0). \qquad (2.12\text{d})$$

The formalism we have developed so far does not make any extraneous assumptions. Our next step is to construct a crossing-symmetric amplitude $A(s,t)$ that satisfies (2.11) and (2.12). Such a construction will not in general satisfy even two-particle unitarity. It will, however, have cuts in both the Mandelstam and the Regge planes and will possess the expected asymptotic behavior. In addition, our method of constructing the amplitudes we obtain will impose restrictions on the parameters, namely, linearity of $\alpha(s)$ and $\alpha_c(s)$ and their relation to t_R and t_0. Such restrictions cannot be claimed as proved relations, since this is not an exact construction and uniqueness is not established. They give, however, strong hints for the type of connection one should be looking for among the parameters of the exact amplitude.

III. AMPLITUDES

It is now our intention to write down a function g satisfying (2.12) from which the amplitude is given by (2.11). We are not able to find a single function—i.e., one that is not naturally split as a sum of two terms—satisfying the conditions (2.12). A function consisting of the sum of two terms may, however, easily be constructed in two ways that readily suggest themselves. One may use either the combination $(a,c)+(b,d)$ of the terms in (2.12) or the combination $(a,d)+(b,c)$. In the first case the amplitude is written as

$$A(s,t)=A_1(s,t)+A_2(s,t)\,, \qquad (3.1)$$

where $A_1(s,t)$ interpolates the asymptotic forms of the leading Regge pole and of the lowest resonance, while $A_2(s,t)$ similarly links the Regge and threshold cuts. This is an extension of the Veneziano model in which exchanged Regge poles are dual to direct resonances, whereas exchanged Regge cuts are dual to direct-channel threshold cuts. This duality we shall hereafter refer to as "direct duality." If one uses the second combination, these links are crossed, and one obtains an amplitude in which Regge poles dualize unitarity cuts while Regge cuts dualize direct-channel resonances. We shall not call this property "interference" but shall refer to it as "cross duality." The two cases will now be treated consecutively.

A. Direct-Duality

It is immediately clear that in the direct-duality combination of the terms in (2.12) one needs to impose the condition

$$\gamma=\tfrac{3}{2}\,, \qquad (3.2)$$

which we do. Further, we make use of the arbitrariness of the positive constant a in (2.11) and (2.12) so as to construct the amplitude for this model as the sum (3.1) with

$$A_1(s,t)=\int_0^1 C_1(s,y)y^{-b(t-t_R)-1}(1-y)^{-\alpha(s)-1}dy\,, \qquad (3.3)$$

$$A_2(s,t)=\int_0^1 C_2(s,y)\frac{y^{-a(t-t_0)-1}(1-y)^{-\alpha_c(s)-1}}{\{1-\ln[y(1-y)]\}^{3/2}}dy\,, \qquad (3.4)$$

where a, b are arbitrary positive constants and $C_1(s,y)$, $C_2(s,y)$ are functions analytic in the neighborhood of $y=0$ and $y=1$, and which may also depend upon a and b. The simplest way to impose crossing is to require $C_1=\text{const}$, $C_2=\text{const}$, so that

$$\alpha(t)=b(t-t_R)\,, \qquad (3.5)$$

$$\alpha_c(t)=a(t-t_0)\,. \qquad (3.6)$$

Equation (3.5) gives the linearity of the leading trajectory, defines the constant b as its slope, and shows that R is the spin-zero resonance on this trajectory. Equation (3.6) similarly gives a linear trajectory for the leading branch point with the constant a as slope and *shows that the branch point passes through zero at the elastic threshold*. Moreover, comparison of (3.3) and (3.4) with the asymptotic forms in (2.21) enables us to write the contributions (2.9) to $D(s,t)$ in terms of the constants C_1 and C_2 as

$$C_1(bt)^{\alpha(s)}/\Gamma(\alpha+1) \qquad (t\to\infty), \qquad (3.7\text{a})$$

$$C_2(at)^{\alpha_c(s)}/\Gamma(\alpha_c+1)(\ln t)^{-3/2} \qquad (t\to\infty), \qquad (3.7\text{b})$$

$$(C_1/b)\delta(t-t_R)\equiv C_1\delta(\alpha(t)) \qquad (t\approx t_R), \qquad (3.7\text{c})$$

$$\frac{C_2}{\Gamma(\frac{3}{2})}a^{1/2}(t-t_0)^{1/2}\equiv\frac{C_2}{\Gamma(\frac{3}{2})}[\alpha_c(t)]^{1/2} \qquad (t\approx t_0). \qquad (3.7\text{d})$$

The representations (3.3) and (3.4) thus take the forms

$$A_1(s,t)=C_1\int_0^1 y^{-\alpha(t)-1}(1-y)^{-\alpha(s)-1}dy, \qquad (3.8)$$

$$A_2(s,t)=C_2\int_0^1 \frac{y^{-\alpha_c(t)-1}(1-y)^{-\alpha_c(s)-1}}{\{1-\ln[y(1-y)]\}^{3/2}}dy, \qquad (3.9)$$

supplemented by Eqs. (3.5) and (3.6). In (3.8) one immediately recognizes the Veneziano formula. The extra additive term (3.9) extends this model to include cuts in a manner that preserves the direct-duality structure of the original amplitude. Using the identity

$$\int_0^\infty x^{\nu-1}e^{-\mu x}dx=\Gamma(\nu)\mu^{-\nu}, \quad \mathrm{Re}\mu>0,\ \mathrm{Re}\nu>0$$

this additional term may be expressed as an integral over the beta function:

$$A_2(s,t)=\frac{C_2}{\Gamma(\frac{3}{2})}\int_0^\infty e^{-\lambda}\lambda^{1/2}B(\lambda-\alpha_c(t),\lambda-\alpha_c(s))d\lambda. \qquad (3.10)$$

By further changes of variable this integral may be made to look either like a cut contribution on the Regge plane,

$$A_2(s,t)=\frac{C_2}{\Gamma(\frac{3}{2})}\int_{-\infty}^{\alpha_c(t)}[\alpha_c(t)-\alpha]^{1/2} \times e^{-[\alpha_c(t)-\alpha]}B(-\alpha,\ -\alpha+\alpha_c(t)-\alpha_c(s))d\alpha, \qquad (3.11)$$

or like a right-hand unitarity integral

$$A_2(s,t)=\frac{C_2a^{3/2}}{\Gamma(\frac{3}{2})}\int_{s_0}^\infty (s'-s_0)^{1/2} \times e^{-a(s'-s_0)}B(a(s'-t),\,a(s'-s))ds'. \qquad (3.12)$$

One may remark here that the presence of the exponential damping factors in these integrals is a welcome feature, since these factors emphasize the contributions of the asymptotic regions at which the representations are a plausible approximation while they smoothly damp distant contributions. In the limit $|s|\to\infty$, Eq. (3.11) becomes

$$A_2(s,t)\sim\frac{C_2}{\Gamma(\frac{3}{2})}\int_{-\infty}^{\alpha_c(t)}[\alpha_c(t)-\alpha]^{1/2} \times e^{-[\alpha_c(t)-\alpha]}[-\alpha_c(s)]^\alpha\Gamma(-\alpha)d\alpha. \qquad (3.13)$$

If this is compared, in the neighborhood of the branch point $\alpha_c(t)$, to the usual form of a Regge-cut contribution to the amplitude, namely,

$$\frac{1}{2i}\int^{\alpha_c(t)}\frac{\alpha+\frac{1}{2}}{\sin\pi\alpha}\Delta(\alpha,t)(-s)^\alpha d\alpha, \qquad (3.14)$$

one obtains for the discontinuity of the partial-wave amplitude $\Delta(\alpha,t)$ across the cut

$$\Delta(\alpha,t)=\frac{-2iC_2}{\pi\Gamma(\frac{3}{2})}\frac{[\alpha_c(t)-\alpha]^{1/2}}{\Gamma(\alpha_c(t)+1)[\alpha_c(t)+\frac{1}{2}]}, \quad \alpha\approx\alpha_c(t). \qquad (3.15)$$

We now proceed to discuss the cross-duality construction.

B. Cross-Duality

As in (3.1), the amplitude is again expressed as the sum of two terms

$$A(s,t)=T_1(s,t)+T_2(s,t), \qquad (3.16)$$

where $T_1(T_2)$ links exchanged Regge-pole (Regge-cut) behavior with direct-channel threshold-cut (resonance) behavior. Explicitly, we take

$$T_1(s,t)=\int_0^1 D_1(s,y)\frac{y^{-a(t-t_0)-1}(1-y)^{-\alpha(s)-1}}{(1-\ln y)^{3/2}}dy, \qquad (3.17)$$

$$T_2(s,t)=\int_0^1 D_2(s,y)\frac{y^{-b(t-t_R)-1}(1-y)^{-\alpha_c(s)-1}}{[1-\ln(1-y)]^\gamma}dy. \qquad (3.18)$$

Crossing symmetry is then imposed by requiring

$$D_1(s,y)=D_2(s,y)=D,\ \text{a const}$$
$$\gamma=\tfrac{3}{2}, \qquad (3.19)$$
$$\alpha(t)=b(t-t_R),\quad \alpha_c(t)=a(t-t_0),$$

i.e., we end up with the same conditions as before. The basic differences between this model and the previous one are, however, quite striking. One first notices that there is only one over-all multiplicative constant D. In fact, under $s\leftrightarrow t$ crossing, $T_1\leftrightarrow T_2$ so that the amplitude $A(s,t)$ consists of one term which is then symmetrized:

$$A(s,t)=T(s,t)+T(t,s), \qquad (3.20)$$

$$T(s,t)=D\int_0^1\frac{y^{-\alpha_c(t)-1}(1-y)^{-\alpha(s)-1}}{(1-\ln y)^{3/2}}dy. \qquad (3.21)$$

Secondly, one observes that the Regge-pole contribution as $|t|\to\infty$ is of the form

$$D\Gamma(-\alpha(s))[-\alpha_c(t)]^{\alpha(s)}, \qquad (3.22)$$

where $\alpha_c(t)$ is the branch point of the leading cut. This means that, within this model, whenever one sees the leading Regge pole, one also sees something of the leading Regge cut. It also says that the scaling factors at high energy are different from those of the direct-duality model. The contributions (2.9) to $D(s,t)$ now read

$$\frac{D(at)^{\alpha(s)}}{\Gamma(\alpha(s)+1)} \qquad (t\to\infty), \quad (3.23a)$$

$$\frac{D(bt)^{\alpha_c(s)}}{\Gamma(\alpha_0(s)+1)(\ln t)^{3/2}} \qquad (t\to\infty), \quad (3.23b)$$

$$(D/b)\delta(t-t_R)\equiv D\delta(\alpha(t)) \qquad (t\approx t_R), \quad (3.23c)$$

$$\frac{D}{\Gamma(\frac{3}{2})}a^{1/2}(t-t_0)^{1/2}\equiv \frac{D}{\Gamma(\frac{3}{2})}[\alpha_c(t)]^{1/2} \quad (t\approx t_0). \quad (3.23d)$$

The presence of a single over-all constant means that the Regge cut is as strongly coupled as the Regge pole. On comparing (3.23a) and (3.23b) one sees that the dominance of pole over cut, or otherwise, is essentially determined by the slopes. This is in contrast with model A, in which $C_2 \ll C_1$ gives a weakly coupled cut. One may indeed say, in summary, that in the cross-duality model, resonances, unitarity cuts, Regge poles, and Regge cuts all stand or fall together.

The representation for $T(s,t)$ in (3.21) may again be expressed as an integral over the beta function. One obtains

$$T(s,t)=\frac{D}{\Gamma(\frac{3}{2})}\int_0^\infty e^{-\lambda}\lambda^{1/2} \times B(\lambda-\alpha_c(t),\, -\alpha(s))d\lambda \quad (t<t_0). \quad (3.24)$$

The representations which manifestly exhibit the cuts in the energy and angular momentum planes at fixed s are

$$T(s,t)=\frac{Da^{3/2}}{\Gamma(\frac{3}{2})}\int_{t_0}^\infty (t'-t_0)^{1/2} \times e^{-a(t'-t_0)}B(a(t'-t),\, -\alpha(s))dt' \quad (t<t_0) \quad (3.25)$$

and

$$T(s,t)=\frac{D}{\Gamma(\frac{3}{2})}\int_{-\infty}^{\alpha_c(t)} [\alpha_c(t)-\alpha]^{1/2} \times e^{-[\alpha_c(t)-\alpha]}B(-\alpha,\, -\alpha(s))d\alpha. \quad (3.26)$$

Taking $|s|\to\infty$, one obtains from (3.26)

$$T(s,t)\sim\frac{D}{\Gamma(\frac{3}{2})}\int_{-\infty}^{\alpha_c(t)} [\alpha_c(t)-\alpha]^{1/2} \times e^{-[\alpha_c(t)-\alpha]}\Gamma(-\alpha)[-\alpha(s)]^{-\alpha}d\alpha, \quad (3.27)$$

which, on comparison with (3.14), again produces (3.15) in the proximity of the branch point.

Using the identity

$$B(x,y)=\sum_{n=0}^\infty \frac{C_n(y)}{x+n}, \quad x>0 \quad (3.28)$$

where $C_n(y)$ is a polynomial in y of order n, one obtains from (3.24) a representation of $T(s,t)$ as an infinite sum of s-channel poles when t is below its threshold:

$$T(s,t)=\sum_{n=0}^\infty \frac{g_n(t)}{\alpha(s)-n}, \qquad t<t_0 \quad (3.29)$$

$$g_n(t)=\frac{D}{\Gamma(\frac{3}{2})}\int_0^\infty e^{-\lambda}\lambda^{1/2}C_n(\lambda-\alpha_c(t))d\lambda, \quad t<t_0. \quad (3.30)$$

The coefficients $g_n(t)$ are clearly polynomials of order n in t. One also notes from (3.30) that the positivity of the Legendre coefficients of the residues $C_n(-\alpha(t))$ of the beta function $B(-\alpha(s), -\alpha(t))$ ensures the same property for the residues $g_n(t)$.

We thus see that in the physical region of the s channel the first term, $T(s,t)$, in Eq. (3.20) represents the sum over all s resonances and behaves like a Regge cut as $s\to\infty$. The second term, $T(t,s)$, has the elastic cut in s and behaves like a Regge pole as $s\to\infty$. Of course, only one of these asymptotic forms is dominant at high energy. For t-channel scattering, the two terms are interchanged. One must, however, emphasize that there is no sense in writing the amplitude of (3.20) in the form

$$\sum_{n=0}^\infty \frac{g_n(t)}{\alpha(s)-n}+\sum_{n=0}^\infty \frac{g_n(s)}{\alpha(t)-n},$$

since this is not convergent in either of the two regions of scattering. This is in contrast with the typical interference model in which the sum over s poles converges for all t.[5] It may also be remarked that the amplitude does not consist of the sum (3.29) plus an entire function and that (3.29) does not possess Regge-pole behavior as $s\to\infty$.

We finally remark that in the cross-duality model (3.20) and (3.21) it appears that one may introduce an imaginary part in the trajectory function $\alpha(s)$, keeping $\alpha_c(t)$ linear and real, without introducing ancestors. This comes from the fact that the residues $g_n(t)$ are polynomials in $\alpha_c(t)$. The dynamical significance of complex poles but real branch points is, however, not clear to me. This question and the structure of the daughters in the model require further investigation.

IV. CONDITION $\alpha_c(t_0)=0$

In this section we briefly discuss the condition

$$\alpha_c(4m^2)=0, \quad (4.1)$$

under the assumption that the Regge cut is generated by the exchange of two linear Regge trajectories. If the pole trajectories are

$$\alpha_i(t)=\alpha_i(0)+\alpha_i't, \quad i=1,2 \quad (4.2)$$

then the branch point $\alpha_c(t)$ of the generated cut satisfies

$$\alpha_c(t)=\alpha_1(0)+\alpha_2(0)-1+\frac{\alpha_1'\alpha_2'}{\alpha_1'+\alpha_2'}t. \quad (4.3)$$

The condition (4.1) then implies that

$$\alpha_1(0)+\alpha_2(0)=1-4m^2\frac{\alpha_1'\alpha_2'}{\alpha_1'+\alpha_2'}. \quad (4.4)$$

If the mass m is assumed small, this equation gives

$$\alpha_1(0)+\alpha_2(0)\approx 1, \quad (4.5)$$

which clearly excludes a cut generated by double Pomeranchuk exchange (PP cut). If the leading cut is a PR cut, where R is another trajectory, then $\alpha_R(0)\approx 0$. This trajectory R cannot, however, be the leading trajectory of the system, since the leading trajectory passes through zero at the (mass)^2 of the lowest resonance which is above threshold.

Now the scattering process we have been discussing is an idealized one in which the particles and resonance do not possess any substantial structure. The significance of condition (4.1), if any, lies therefore not in its particular form for this process but in its general indication of "quantization rules" of this type to be looked for on more general grounds. In both the direct- and cross-duality models discussed, the condition (4.1) was imposed as a consistency requirement enabling the amplitudes to satisfy crossing symmetry in the special way imposed as well as the asymptotic forms of Eqs. (2.12). To see how this condition emerges, take the cross-duality amplitude (3.16)–(3.18) with $\gamma=\frac{3}{2}$. Crossing is imposed by requiring

$$T_1(s,t)=T_2(t,s). \quad (4.6)$$

Let $\alpha_c(t)$ be given by

$$\alpha_c(t)=a(t-t_0)+\beta, \quad (4.7)$$

where β is an arbitrary constant. By a change of variable on (3.18), Eq. (4.6) then becomes

$$\int_0^1 D_1(s,y)\frac{y^{-a(t-t_0)-1}(1-y)^{-b(s-s_R)-1}}{(1-\ln y)^{3/2}}dy$$
$$=\int_0^1 D_2(t,1-y)\frac{y^{-a(t-t_0)-1-\beta}(1-y)^{-b(s-s_R)-1}}{(1-\ln y)^{3/2}}dy. \quad (4.8)$$

Both sides of this equation have cuts in t and poles in s. Analytic comparison of the two sides gives

$$D_1(s_R,1)=\lim_{y\to 1}D_2(t,1-y), \quad (4.9)$$

$$D_1(s,0)=\lim_{y\to 0}y^{-\beta}D_2(t_0,1-y). \quad (4.10)$$

But Eqs. (2.12b) and (2.12d), respectively, require

$$D_2(s,1)=\eta(s)/b^{\alpha_c(s)}, \quad (4.11)$$

$$D_1(s,0)=B(s)/a^{1/2}. \quad (4.12)$$

Equations (4.10)–(4.12) then give

$$B(s)=a^{1/2}\eta(t_0), \quad (4.13)$$

$$\beta=0. \quad (4.14)$$

This last equation is condition (4.1).

If, besides (4.7), we also had

$$a(t)=b(t-t_R)+\alpha(t_R),$$

we would have similarly concluded from (4.9) and (2.12) that

$$\alpha(t_R)=0 \quad (4.15)$$

as a consistency condition on the low-energy resonance and Regge-pole behavior. Thus condition (4.1) appears as a restriction that guarantees the consistency of our assumptions on the energy-plane and Regge-plane cuts. Whether such conditions are required in general, i.e., when crossing is not necessarily imposed in a special way, is not clear.

The cross-duality amplitude for $\pi\pi$ scattering is presently being investigated by Mahanta and the present author.[11] It appears that constructions of the $I=2$ $\pi\pi$ amplitude are possible in which either

$$\alpha_c(4m_\pi^2)=0 \quad (4.16)$$

or

$$\alpha_c(4m_\pi^2)=1 \quad (4.17)$$

is satisfied. One notes that (4.16) is well satisfied by a $\rho\rho$ cut, in which case it gives

$$\alpha_\rho(0)=\tfrac{1}{2}-m_\pi^2\alpha_\rho', \quad (4.18)$$

a condition equivalent to[15]

$$\alpha_\rho(m_\pi^2)=\tfrac{1}{2}. \quad (4.19)$$

Condition (4.17), on the other hand, is well satisfied by a PP cut, i.e., by the assumption that the leading cut is generated by double Pomeranchuk exchange. It then gives

$$\alpha_P(0)=1-m_\pi^2\alpha_P', \quad (4.20)$$

so that the Pomeranchuk trajectory is given by

$$\alpha_P(t)=1+\alpha_P'(t-m_\pi^2). \quad (4.21)$$

In particular, the condition corresponding to (4.19) is, in this case,

$$\alpha_P(m_\pi^2)=1, \quad (4.22)$$

replacing the usual statement $\alpha_P(0)=1$. If one imposes the inequality

$$0<\alpha_P'/\alpha_\rho'\leqslant 1, \quad (4.23)$$

then Eqs. (4.19) and (4.20) give

$$0<2(m_\rho^2/m_\pi^2-1)[1-\alpha_P(0)]\leqslant 1. \quad (4.24)$$

This yields

$$0.98\leqslant\alpha_P(0)<1, \quad (4.25)$$

[15] C. Lovelace, Phys. Letters **28B**, 264 (1968).

which is a remarkable limitation on the range of variation of $\alpha_P(0)$ that also avoids the known analytic difficulties connected with an identically unit intercept for the Pomeranchuk trajectory.[16–19]

The trajectory of the leading branch point in an amplitude satisfying (4.16) is given by

$$\alpha_c(t)=\tfrac{1}{2}\alpha_\rho'(t-4m_\pi^2)=\frac{t-4m_\pi^2}{4(m_\rho^2-m_\pi^2)}. \tag{4.26}$$

For the amplitude satisfying (4.17), the leading branch-point trajectory is

$$\alpha_c(t)=\tfrac{1}{2}\alpha_P'(t-4m_\pi^2)+1, \tag{4.27}$$

where the slope is, in this case, undetermined since we do not know any particles on the Pomeranchuk trajectory.

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Method of Extending the Blankenbecler-Sugar-Logunov-Tavkhelidze Approximation to the Bethe-Salpeter Equation*†

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We propose a systematic method of obtaining accurate solutions to the Bethe-Salpeter (BS) equation, starting with the Blankenbecler-Sugar-Logunov-Tavkhelidze (BSLT) equation as the lowest-order approximation. For the equal-mass scattering problem, where the difference between the BS and the BSLT amplitudes is the most marked, the first-order correction we evaluate gives good agreement with the BS amplitude. We have also applied the method to the unequal-mass scattering problem, when the mass ratio is the pion-nucleon mass ratio. Here we find that the BSLT amplitude itself is a good approximation to the BS amplitude.

I. INTRODUCTION

In recent years, several authors have worked on the problem of obtaining solutions to the Bethe-Salpeter (BS) equation[1] for low-energy scattering problems.[2] However, it still requires a considerable amount of labor and computer time to obtain accurate numerical solutions to the BS equation, and it is desirable to find more efficient methods.

A convenient relativistic approximation to the BS equation was proposed by Blankenbecler and Sugar[3] and by Logunov and Tavkhelidze[4]; we shall refer to their equation as the Blankenbecler-Sugar-Logunov-Tavkhelidze (BSLT) approximation or equation. The advantage of the BSLT equation is that it reduces the dimension of the integral equation to be solved. Instead of the two-dimensional integral equation for the BS partial-wave amplitude, one now deals with a one-dimensional integral equation for the BSLT partial-wave amplitude. This enables accurate numerical solutions to be obtained with far greater ease.

However, on solving the BS and the BSLT equations for the equal-mass scattering problem, one finds that except for low energies and small interaction strengths,

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