Ward Identities for η Decay in Perturbation Theory*

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The process $\eta \to 3\pi$ is known to violate a simple prediction of partial conservation of axial-vector current. In many models, $\eta \to 3\pi$ proceeds through a fermion loop with electromagnetic corrections. The first radiative corrections to fermion loops give rise to divergent double-loop integrals with forms like $\int d^4r d^4k / [r^2-m^2]^2[(r-k)^2-m^2]^2$. As with single linearly divergent integrals, when a meaning is ascribed to them it may not always be possible to shift the origin of integration without changing the value of these integrals. Such integrals appear when one tries to check, in perturbation theory, Ward identities and low-energy theorems which follow from the formal manipulation of the equations of motions. They can cause anomalies similar to the one in the axial-vector-current two-photon vertex. We study some applications to the $\eta\pi\sigma$ vertex, to the process $\eta \to 3\pi$, and to the corrections of order α^2 to $\pi^0 \to 2\gamma$. No anomalies which can be related to η decay are discovered.

I. INTRODUCTION

THE anomalous behavior of some matrix elements of the divergence of the neutral axial-vector current and the Ward identities in which they occur have been receiving a good deal of attention.[1–8] It has become apparent that, because of the singular behavior of currents built out of fermion fields, the equations of motion which follow from the formal manipulations of Lagrangian field theory cannot always be maintained in perturbation theory.

Furthermore, this anomalous behavior is not just a formal curiosity associated with the infinities in the perturbation expansion. It enables one to reconcile, at least qualitatively, the observed rate of π^0 decay with the hypothesis of partially conserved axial-vector current (PCAC).[2,4] The large $\pi^0 \to 2\gamma$ rate had seemed to be a glaring failure of current algebra and PCAC.[9]

Another possible failure is the discrepancy between the nonzero value of the Dalitz-plot extrapolation to zero π^0 four-momentum in $\eta \to \pi^+\pi^-\pi^0$ and the zero prediction of the "naive" theory.[10,11] Our principal purpose here is to investigate the matrix elements which contribute to $\eta \to 3\pi$ and similar processes, to see whether or not they too have an anomalous behavior in terms of the predictions of the formal equations. We show that there do indeed exist formally divergent terms in the perturbation expressions for the axial-vector Ward identity, which are "anomalous" in the sense that they cannot be absorbed into renormalization effects. They all have the form of eight-dimensional integrals, and would be zero if arbitrary shifts in the origin of integration—legal for convergent integrals—were permissible. Thus their existence, as with anomalies previously studied, depends on the possibility of performing formal manipulations on singular expressions.

In the model we study here, careful calculation shows that such an anomalous term is really absent in the $\eta \to 3\pi$ amplitude, given to lowest order by a four-vertex fermion loop, although this result seems to depend on a detail of the renormalization prescription. In a more interesting version, we study $\eta \to \pi\sigma$, which has a three-vertex fermion loop. (σ is a scalar meson, and the subsequent decay $\sigma \to 2\pi$ relates this calculation to the physical $\eta \to 3\pi$ process.) The loop integration is more singular, and the value of the anomalous term depends upon the method used to give meaning to a divergent integral. As might be expected, electromagnetic gauge invariance also depends on the value of

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[1] The impossibility of maintaining gauge invariance and PCAC in perturbation theory was noted by J. S. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969).

[2] The puzzle of Ref. 1 was traced to a linearly divergent Feynman graph, and a correction to PCAC proposed, by S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[3] The result of Ref. 2 was derived from a split-point definition of the axial-vector current by C. R. Hagen, Phys. Rev. **177**, 2622 (1969). It had been obtained earlier in another context by J. Schwinger, *ibid.* **82**, 664 (1951).

[4] The split-point method was discussed in detail by R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969).

[5] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969). This paper elaborates on Ref. 2. In Sec. VI we corroborate their result that a matrix element of the anomalous correction to the divergence of the axial-vector current is known to all orders.

[6] S. L. Adler and D. G. Boulware, Phys. Rev. **184**, 1740 (1969).

[7] I. S. Gerstein and R. Jackiw, Phys. Rev. **181**, 1955 (1969). This work calculates one-loop contributions to Ward identities, and this is closely related to our paper.

[8] See also R. A. Brandt, Phys. Rev. **180**, 1490 (1969); K. G. Wilson, *ibid.* **181**, 1909 (1969); W. A. Bardeen, *ibid.* **184**, 1848 (1969); R. W. Brown, C. C. Shih, and B. L. Young, *ibid.* **186**, 1491 (1969); C. W. Kim, W. W. Rapko, and A. Sato, *ibid.* **187**, 1966 (1969); W. Bardeen, in Proceedings of the Trieste Conference on Renormalization Theory, 1969 (unpublished).

[9] D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967). Sutherland's proof is incomplete, but the result is correct.

[10] D. G. Sutherland, Phys. Letters **23**, 384 (1966).

[11] For a discussion, see S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968), p. 137.

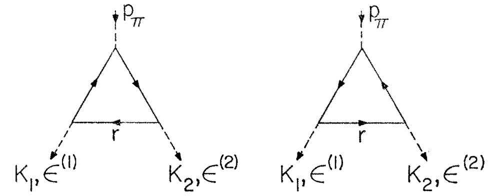

FIG. 1. Lowest-order graphs for π^0 decay, and the related axial-vector, two-photon vertex.

the same integral, and requiring gauge invariance assigns it exactly the value needed to satisfy the formal axial-vector Ward identity.

Our principal result is therefore that we can find no anomalies to explain the observed rate of η decay (without violating electromagnetic gauge invariance). It is not clear whether their absence is accidental to the process we study.

Before coming to the question of η decay it is instructive to review briefly the status of π^0 decay. The history of the anomaly related to π^0 decay is this: Let k_1 and k_2 be the momenta of the final photons, and $\epsilon^{(1)}$ and $\epsilon^{(2)}$ their polarization vectors. The most general kinematically allowed form of the amplitude is $\epsilon_{\rho_1\rho_2\alpha\beta}k_1^{\rho_2}k_2^{\rho_1}\epsilon^{(1)\alpha}\epsilon^{(2)\beta}F$. Many years ago, Steinberger[12] computed F in Born approximation from the graphs of Fig. 1 and obtained $F=(\alpha/\pi)(g/m)$, where m is the proton mass and g is the πN coupling constant. At the time neither g nor the π^0 decay rate were well known, but recent measurements show that Steinberger's formula gives the right order of magnitude.[2] One must assume that the PCAC formula,

$$\partial_\mu j_{5,3}{}^\mu = m_\pi{}^2 F_\pi \phi_3(x)\,, \qquad (1.1)$$

where $j_{5,3}{}^\mu$ is the neutral isovector axial-vector current and ϕ_3 is the π^0 field, is exact in the presence of electromagnetism even to order α. This assumption goes slightly beyond the original PCAC, which applied to strong interactions alone. However, it is easy to see that in the σ model,[13] for example, where Eq. (1.1) holds as an equation of motion in the absence of electromagnetism, as a formal equation of the theory Eq. (1.1) still holds when photons are added with minimal coupling.

To apply PCAC, one considers the off-mass-shell amplitude

$$\int d^4x\, e^{-i(k_1+k_2)\cdot x}[\Box^2+m_\pi{}^2]\langle 2\gamma|\phi_3(x)|\rangle\,.$$

Its general form is

$$\epsilon_{\rho_1\rho_2\alpha\beta}k_1^{\rho_2}k_2^{\rho_1}\epsilon^{(1)\alpha}\epsilon^{(2)\beta}F((k_1+k_2)^2)\,.$$

Physical pion decay is obtained from $F(m_\pi{}^2)$. Equation (1.1) implies (the argument is not completely trivial) that $F(0)=0$. The argument can be generalized to the $\pi^0\gamma\gamma$ vertex where the photons are also off the mass shell. The amplitude is then proportional to

$$\alpha\langle|T[j_{\rm em}{}^{\rho_1}(y)j_{\rm em}{}^{\rho_2}(z)\phi_3(x)]|\rangle\,.$$

Using Eq. (1.1) and the identity

$$\langle|T[j_{\rm em}{}^{\rho_1}(y)j_{\rm em}{}^{\rho_2}(z)\partial_\mu j_{5,3}{}^\mu(x)]|\rangle = \frac{\partial}{\partial x^\mu}\langle|T[j_{\rm em}{}^{\rho_1}(y)j_{\rm em}{}^{\rho_2}(z)j_{5,3}{}^\mu(x)]|\rangle\,, \qquad (1.2)$$

one learns that F is α times a presumably smooth function which vanishes when $k_1+k_2=0$.

The identity (1.2) follows provided that the equal-time commutation relations

$$[j_{5,3}{}^0(x),j_{\rm em}{}^\rho(y)]\delta(x^0-y^0)=0\,, \qquad (1.3)$$

which follow from formal manipulation of the currents in simple models, are indeed true. An error of order e or α in either (1.3) or (1.1) would invalidate the general result, which is that $F(0)$ is zero to order α and therefore provided that F is indeed a smooth function, that the π^0 decay rate should be much smaller than observed. [Clearly, violations of (1.3) and (1.1) cannot be independent.]

Bell and Jackiw[1] were the first to point out that Eq. (1.1) cannot be maintained in perturbation theory except at the expense of electromagnetic gauge invariance. The contradiction can be traced to the singular behavior of the term in $j_{5,3}{}^\mu(x)$ which is bilinear in the fermion field.[3,4] In graphical language, it is sufficient to consider the triangle graphs of Fig. 1, with the pseudoscalar vertex replaced by $\gamma^\mu\gamma_5$. The graphs are then linearly divergent, and contribute a violation of (1.1) of the form

$$\int d^4r[f(r)-f(r+k_1+k_2)]\,. \qquad (1.4)$$

Such a difference would be zero if the integral were convergent or even logarithmically divergent; but in this case it is ambiguous. Thus neither gauge invariance nor PCAC is satisfied automatically. If expression (1.4) is assigned the value required by gauge invariance, the triangle graph has a fixed value which can be described by[2–4]

$$\partial_\mu j_{5,3}{}^\mu = m_\pi{}^2F_\pi\phi_3 + (\alpha/4\pi)\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \qquad (1.5)$$

instead of (1.1).

Can Eq. (1.5) be used to explain the electromagnetic decay $\eta\to 3\pi$? It follows from (1.1) that this amplitude vanishes when a final π^0 has zero 4-momentum, a result which seems clearly violated by experiment.[11] The observed η width of about 2.6×10^{-3} MeV is clearly of order α^2 and the decay distribution shows no tendency to vanish at zero 4-momentum of the π^0. Therefore, experimentally one knows that there must be a correction to Eq. (1.1) of order α.

The correction (1.5) will not do the job, however,

[12] J. Steinberger, Phys. Rev. **76**, 1180 (1949).
[13] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

due to its tensor structure. This can easily be seen by reducing out the final π^0 and using Eq. (1.5). The amplitude $\eta \to 3\pi$ is proportional to

$$\frac{1}{F_\pi m_\pi^2}\int d^4x \exp[-ip_{\pi^0}\cdot x](\Box^2+m_\pi^2)[\langle 2\pi|\,\partial_\mu j_{5,3}{}^\mu(x) - (\alpha/4\pi)\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}(x)F^{\alpha\beta}(x)\,|\,\eta\rangle]. \quad (1.6)$$

At $p_{\pi^0}=0$, the first term vanishes as usual. To evaluate the second term, insert a complete set of states between the two F's. Only states with one photon can contribute to order α, but these give zero due to the antisymmetric symbol.

In the next three sections we study the $\eta\pi^0\sigma$ vertex, having in mind a model field theory like the $SU(3)$ version of the σ model[14] (although we compute only the spinor loops; meson loops are always more convergent). What we actually evaluate is the lowest nonvanishing order of the matrix element

$$\langle|\,T[j_{5,3}{}^\mu(x)j_\sigma(y)j_\eta(z)]\,|\rangle \quad (1.7)$$

and its divergence, which in a theory with PCAC should be proportional to the $\eta\pi^0\sigma$ vertex. It is easy to see that the off-mass-shell vertex is proportional, at $p_{\pi^0}=0$, to the commutators of $j_{5,3}{}^0(x)$ with $j_{\rm em}{}^\mu$, and with j_σ and j_η, the sources of the η and σ.

We call the equation satisfied by the matrix elements of $\partial_\mu j_{5,3}{}^\mu(x)$ the axial-vector Ward identity. The identity seems to be violated by an eight-dimensional integral of the form

$$\int d^4r d^4k[f(r,k)-f(r-k,\,-k)]. \quad (1.8)$$

The variable r is the fermion loop momentum, while k is the momentum of an internal photon. The integrals are convergent if the r integration is done first for fixed k, and logarithmically divergent in k for fixed r. For single four-dimensional integrals, convergence or logarithmic divergence is a sufficient condition to permit the variable shifts necessary to prove that (1.8) is zero. However, the integrals of the form (1.8) which occur are not unconditionally convergent or even logarithmically divergent in eight-dimensional space. They have denominators of the general form $[(r-k)^2-m^2]$ which, even after performing the Wick rotation on r^0 and k^0, do not get large in every direction.

We find that if one introduces Lorentz-invariant cutoffs or regulators to make the integrals finite, the value of the anomalous term depends on the cutoff in a complicated way. Gauge invariance, however, requires that an integral identical to the one occurring in the anomalous term be zero. Therefore, if the ambiguous integral has a unique meaning whenever it occurs, we conclude that it must be zero. Obviously, this result is not quite as clean as the analogous $\pi^0 \to 2\gamma$ anomaly.

In Sec. V we study the process $\eta \to 3\pi^0$ without an intermediate σ. The most divergent contribution comes from the fermion box graph. Again, there seem to be terms left over, with eight-dimensional integrals in them, which are not predicted by the axial-vector Ward identity. The "normal" terms, however, are also of this form, and their explicit expressions depend on the way the internal photon momentum is allowed to run around the fermion loop. After some algebraic manipulation it can be seen that the photon momentum does not go the same way in the normal terms on both sides. Correcting this "error" turns out to cancel exactly the extra terms that should not be there, *independently* of the cutoff procedure, so that the naive axial-vector Ward identity is unambiguously satisfied.

There is a problem associated with $\eta \to 3\pi$ when we check gauge invariance of the internal photon. The amplitude is not manifestly gauge invariant but depends on an integral of the form of (1.8) being zero. Again the integral depends on the cutoff procedure. However, the integral is of the same type as the integral that was required to be zero in $\eta \to \sigma\pi$.

Our techniques are immediately applicable to the α^2 corrections to $\pi^0 \to 2\gamma$. Here we find that besides the α^2 terms from (1.5) there is an additional correction to F of the form (1.8) but that this integral is zero by symmetric integration independently of the method of cutoff. This integral in the more difficult massless fermion case has been studied by Adler and Sen.[15] Unlike $\eta \to 3\pi$, we find no problem with internal gauge invariance, so there is no new α^2 anomaly in $\pi^0 \to 2\gamma$.

Finally, although we seem to have shown that the axial-vector Ward identities hold up in perturbation theory in spinor electrodynamics, there is a formal question about the applicability of our calculations to to the $SU(3)$ "σ model" or any similar theory. Without electromagnetism, the σ model breaks chiral invariance in a well-defined way and has $SU(3)$ symmetry. (It is presumably renormalizable, although the proof depends on the symmetry expressed by the axial-vector Ward identity.) Now we add photons in a minimal way. Before there was no η-3π vertex. Now, this vertex can be computed in terms of the old (bare) coupling constants and turns out to be one of our divergent eight-dimensional integrals. The usual renormalization approach is to add a counter term of the form $\lambda\phi_\eta\phi_\pi{}^3$ to the Lagrangian. But this is not a counterterm to anything which was in the theory to start with. The $\eta \to 3\pi$ vertex is of order e^2, but may be essentially uncalculable from the theory if one takes this approach. The situation is similar to the impossibility of computing the four-pion vertex $\lambda\phi^4$ in γ_5 theory in terms of the πN coupling constant.[16]

[14] See Ref. 11, p. 24.

[15] K. Johnson (private communication).

[16] The idea that η decay proceeds by an effective direct η-3π interaction term, rather than by an anomalous term in the PCAC equation, appears frequently. See, e.g., K. Wilson, Phys. Rev. **179**, 1499 (1969); R. Brandt and G. Preparata (unpublished); N. Cabibbo and L. Maiani, in *Evolution in Particle Physics*, edited by M. Conversi (Academic, New York, to be published); R. Brandt, M. Goldhaber, G. Preparata, and C. Orzalesi, Phys. Rev. Letters **24**, 1517 (1970).

In short, it might be that to assert PCAC to all orders in α just does not make sense in models, and that trying to compute an anomalous contribution to the η-3π vertex from the Fenyman graphs of these models is simply not right.

II. $\eta\pi\sigma$ VERTEX

We begin with a study of the $\eta\pi\sigma$ vertex and the Ward identity satisfied by the axial-vector current. Since we expect any anomalous behavior to come from closed fermion loops, it is not necessary to complicate the problem with the full paraphernalia of the σ model, or with $SU(3)$ matrices. It is sufficient to calculate the three-point vertex function of the *sources* of the η, π^0, and σ in spinor electrodynamics.

Therefore we introduce a fermion doublet with field ψ, and define the usual isovector and isoscalar currents

$$\begin{aligned} j_i^\mu &= \tfrac{1}{2}\bar{\psi}\gamma^\mu\tau_i\psi\,, \\ j^\mu &= \tfrac{1}{2}\bar{\psi}\gamma^\mu\psi\,, \\ j_{5i}^\mu &= \tfrac{1}{2}\bar{\psi}\gamma^\mu\gamma_5\tau_i\psi\,, \\ j_5^\mu &= \tfrac{1}{2}\bar{\psi}\gamma^\mu\gamma_5\psi\,, \end{aligned} \tag{2.1}$$

as well as the analogous scalar and pseudoscalar densities

$$\begin{aligned} j_i &= \tfrac{1}{2}\bar{\psi}\tau_i\psi\,, \quad & j &= \tfrac{1}{2}\bar{\psi}\psi\,, \\ j_{5i} &= \tfrac{1}{2}i\bar{\psi}\gamma_5\tau_i\psi\,, \quad & j_5 &= \tfrac{1}{2}i\bar{\psi}\gamma_5\psi\,. \end{aligned} \tag{2.2}$$

The isovector axial current is generated by

$$\delta\psi = -i\epsilon_i(\tfrac{1}{2}\tau_i)\gamma_5\psi\,, \quad \delta\bar{\psi} = -i\bar{\psi}(\tfrac{1}{2}\tau_i)\epsilon_i\gamma_5\,. \tag{2.3}$$

The Lagrange density is the usual kinetic and mass terms, plus the electromagnetic interaction

$$\mathfrak{L}_{\text{em}} = -eA_\mu j_{\text{em}}^\mu\,, \tag{2.4}$$

where

$$j_{\text{em}}^\mu = \bar{\psi}\gamma^\mu Q\psi = 2\bar{Q}j^\mu + j_3^\mu\,. \tag{2.5}$$

Here $\bar{Q}$ is the average charge of the doublet. The part of the Lagrangian which is not invariant under the transformation (2.3) is (m is the fermion bare mass)

$$-2mj - eA_\mu j_{\text{em}}^\mu \tag{2.6}$$

and the "naive" divergence of j_{5i}^μ is

$$\partial_\mu j_{5i}^\mu = \frac{\delta L}{\delta\epsilon_i} = 2mj_{5i} - e\epsilon_{3ik}A_\mu j_{5k}^\mu\,. \tag{2.7}$$

The neutral current satisfies

$$\partial_\mu j_{5,3}^\mu = 2mj_{5,3}\,, \tag{2.8}$$

while the charged currents satisfy

$$(\partial_\mu \mp ieA_\mu)j_{5\pm}^\mu = 2mj_{5\pm}\,. \tag{2.9}$$

We want to study the vertex

$$F_5 = 2m\int d^4x d^4y d^4z\, e^{-ip_\eta\cdot z}e^{+ip_\sigma\cdot y}e^{+ip_\pi\cdot x} \times\langle|T[j_{53}(x)j(y)j_5(z)]|\rangle \tag{2.10}$$

and its relationship to

$$F = \int d^4x d^4y d^4z\, e^{-ip_\eta\cdot z}e^{+ip_\sigma\cdot y}e^{+ip_\pi\cdot x} \times\frac{\partial}{\partial x^\mu}\langle|T[j_{53}^\mu(x)j(y)j_{53}(z)]|\rangle\,. \tag{2.11}$$

To get (2.10) from (2.11), the $\partial/\partial x^\mu$ must be brought inside the time ordering. One term is (2.10), and in addition there are the equal-time commutators. The result can be written

$$F = F_5 + F_1 + F_2\,, \tag{2.12}$$

where

$$F = -ip_{\pi\mu}\int d^4x d^4y d^4z\, e^{-ip_\eta\cdot z}e^{ip_\sigma\cdot y}e^{ip_\pi\cdot x} \times\langle|T[j_{53}^\mu(x)j(y)j_5(z)]|\rangle \tag{2.13}$$

from (2.11), F_5 is given by (2.10), and

$$F_1+F_2 = \int d^4x d^4y d^4z\, e^{-ip_\eta\cdot z}e^{ip_\sigma\cdot y}e^{ip_\pi\cdot x} \times\langle|T(\delta(x^0-y^0)[j_{5,3}^0(x),j(y)]j_5(z) \times\delta(x^0-z^0)[j_{5,3}^0(x),j_5(z)]j(y))|\rangle\,. \tag{2.14}$$

The naive values for the two commutators are

$$\delta(x^0-y^0)[j_{53}^0(x),j(y)] = i\delta_4(x-y)j_{53}(y)\,, \tag{2.15}$$

$$\delta(x^0-z^0)[j_{53}^0(x),j_5(z)] = -i\delta_4(x-z)j_3(z)\,, \tag{2.16}$$

so that

$$F_1 = i\int d^4x d^4z\, e^{-ip_\eta\cdot z}e^{i(p_\sigma+p_\pi)\cdot y} \times\langle|T[j_{53}(y)j_5(z)]|\rangle\,, \tag{2.17}$$

$$F_2 = -i\int d^4y d^4z\, e^{i(p_\pi-p_\eta)\cdot z}e^{ip_\sigma\cdot y} \times\langle|T[j(y)j_3(z)]|\rangle\,. \tag{2.18}$$

We shall call (2.12) the axial-vector Ward identity. It ought to be true in every order of perturbation theory. In zeroth order, each term vanishes because of isospin conservation, but it is useful to write down the formal expressions. The graphs are displayed in Fig. 2. The commutator terms correspond to bubbles which look like off-diagonal terms in a mass matrix. The algebraic expressions are obtained by replacing the currents in the expressions for F, F_5, F_1, and F_2 by their noninteracting values $[s(r)=[\gamma\cdot r-m]^{-1}]$:

$$F = -\tfrac{1}{4}i\delta_4(p_\eta-p_\pi-p_\sigma) \times\int d^4r\,\text{Tr}\{s(r)\gamma_5 s(r-p_\eta)\gamma\cdot p_\pi\gamma_5\tau_3 s(r-p_\sigma) + s(r)s(r+p_\sigma)\gamma\cdot p_\pi\gamma_5\tau_3 s(r+p_\eta)\gamma_5\}\,, \tag{2.19}$$

$$F_1 = \tfrac{1}{2}i\delta_4(p_\eta-p_\pi-p_\sigma)\int d^4r \times\text{Tr}\{s(r)\gamma_5 s(r+p_\eta)\gamma_5\tau_3\}\,, \tag{2.20}$$

$$F_2=\tfrac{1}{2}i\delta_4(p_\eta-p_\pi-p_\sigma)\int d^4r\,\mathrm{Tr}\{s(r)s(r+p_\sigma)\tau_3\}\,. \quad (2.21)$$

F_5 can be obtained from (2.19) by replacing $\gamma\cdot p_\pi$ by $-2m$.

An alternative derivation of (2.12) is instructive. Let us abandon pure spinor electrodynamics and include in our theory canonical σ and η fields (denoted σ, η), whose coupling to the ψ field is described by an interaction Lagrangian

$$\mathfrak{L}_I=2g(j\sigma+j_5\eta)\,. \quad (2.22)$$

(The equality of the three coupling constants is not necessary for what follows.)

Now the divergence equation (2.6) becomes replaced by

$$\partial_\mu j_{5i}{}^\mu=2mj_{5i}-2g\sigma j_{5i}+2g\eta j_i-e\epsilon_{3ik}A_\mu j_{5k}{}^\mu\,. \quad (2.23)$$

Let us compute the function

$$\int d^4xd^4yd^4z\,e^{-ip_\eta\cdot z}e^{ip_\sigma\cdot y}e^{ip_\pi\cdot x} \times\frac{\partial}{\partial x^\mu}\langle|T[\eta(z)j_{53}{}^\mu(x)\sigma(y)]|\rangle\,, \quad (2.24)$$

which is $-ip_\pi{}^\mu$ multiplied by the η and σ propagators, times the σ-η axial-vector vertex function. The derivative can be brought inside the T product, since the current commutes with the canonical η and σ field operators. Expression (2.24) is proportional to (2.11); the proportionality constant is g^2 times the σ and η *bare* propagators. Now inserting the right-hand side of (2.23) for $\partial_\mu j_{53}{}^\mu$ into (2.24), only the first three terms contribute, and give, except for the same over-all factor, the three terms on the right-hand side of the axial-vector Ward identity (2.12).[17]

The point to notice is that the divergence of a current and the commutator of its time component with other densities cannot be independent.[6] An anomaly in (2.23) must be associated with an anomaly in the commutators (2.15) and (2.16), since either can be used to compute a correction in the identity (2.12). The relationship should not be surprising, since to calculate the divergence one computed $\delta\mathfrak{L}/\delta\epsilon_i$, and the variation of each piece of L depends on its commutator with the current's time component.

We need to calculate each term in (2.12) up to order e^2, which means to treat the currents as free currents and do perturbation theory twice with $\mathfrak{L}_{\mathrm{em}}$. In the first method we must compute

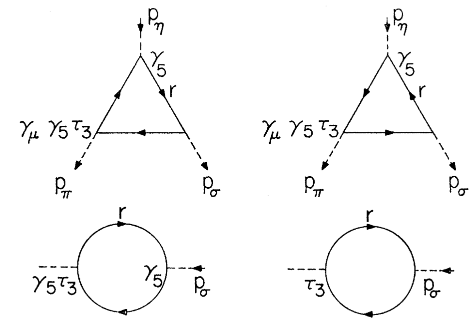

FIG. 2. Zeroth-order graphs for the axial-vector Ward identity for the $\eta\pi\sigma$ vertex.

$$\frac{\partial}{\partial x^\mu}\langle|T[j_{53}{}^\mu(x)j(y)j_5(z)j_{\mathrm{em}}{}^\alpha(x_1)j_{\mathrm{em}}{}^\beta(x_2) \times A_\alpha(x_1)A_\beta(x_2)]|\rangle\,, \quad (2.25)$$

which will involve the commutator

$$\delta(x^0-y^0)[j_{53}{}^0(x),j_{\mathrm{em}}{}^\alpha(y)]\,. \quad (2.26)$$

The fact that there are no extra terms in the naive Ward identity (2.12) can be ascribed, on the one hand, to the absence of a term in e on the right-hand side of Eq. (2.23) for $i=3$, or, on the other hand, to the vanishing of the commutator (2.26).

III. EVALUATION OF $\eta\sigma\pi$ VERTEX TO ORDER α

In this section we compute F, F_5, F_1, and F_2 to order e^2, in order to check whether the axial-vector Ward identity (2.12) holds in perturbation theory. All the quantities are unrenormalized. The zeroth-order terms (2.18)–(2.21) vanish only because the charge-space factor is $\mathrm{Tr}\tau_3=0$. The terms to order e^2 all have a common charge-space factor $\mathrm{Tr}\tau_3Q^2=2\bar{Q}$. For convenience we take $2\bar{Q}$ to be 1, as for a nucleon doublet, and ignore this factor from now on.

F has six terms, corresponding to the six graphs in Fig. 3, and six more from graphs in which the fermion line runs around the other way. The two sets are equal by charge-conjugation invariance. We write

$$F=\frac{\alpha}{8\pi^3}\delta_4(p_\pi+p_\sigma-p_\eta)\int d^4rd^4k\,D_{\mu\nu}(k)F^{\mu\nu}(r,k)\,, \quad (3.1)$$

where $D_{\mu\nu}(k)$ is the photon propagator and

$$\begin{aligned}F^{\mu\nu}=\mathrm{Tr}\{&s(r)\gamma_5s(r-p_\eta)\gamma\cdot p_\pi\gamma_5s(r-p_\sigma)\gamma^\mu s(r-p_\sigma-k)\gamma^\nu s(r-p_\sigma)\\&+s(r)\gamma_5s(r-p_\eta)\gamma^\mu s(r-p_\eta-k)\gamma^\nu s(\sigma-p_\eta)\gamma\cdot p_\pi\gamma_5s(r-p_\sigma)\\&+s(r)\gamma^\mu s(r-k)\gamma^\nu s(r)\gamma_5s(r-p_\eta)\gamma\cdot p_\pi\gamma_5s(r-p_\sigma)\\&+s(r)\gamma^\mu s(r-k)\gamma_5s(r-p_\eta-k)\gamma^\nu s(r-p_\eta)\gamma\cdot p_\pi\gamma_5s(r-p_\sigma)\\&+s(r)\gamma_5s(r-p_\eta)\gamma^\mu s(r-p_\eta-k)\gamma\cdot p_\pi\gamma_5s(r-p_\sigma-k)\gamma^\nu s(r-p_\sigma)\\&+s(r)\gamma_5s(r-p_\eta)\gamma\cdot p_\pi\gamma_5s(r-p_\sigma)\gamma^\mu s(r-p_\sigma-k)s(r-k)\gamma^\nu\}\,. \quad (3.2)\end{aligned}$$

[17] If Eq. (2.23) is used instead of (2.8) in working out the axial-vector Ward identity from (2.11), there will be two additional terms in (2.12), coming from the extra terms $2g(\eta j_5-\sigma j_{53})$ in the divergence. Therefore, there is not a one-to-one correspondence between the terms obtained from the two methods. The new terms are of higher order in g and do not enter into our lowest-order calculation.

Similarly, to this order

$$F_5=\frac{\alpha}{8\pi^3}\delta_4(p_\pi+p_\sigma-p_\eta)\int d^4r d^4k\, D_{\mu\nu}(k)F_5{}^{\mu\nu}(r,k)\,,\quad (3.3)$$

where $F_5{}^{\mu\nu}$ can be obtained from $F^{\mu\nu}$ by replacing $\gamma\cdot p_\pi$ with $-2m$. Finally, F_1 and F_2 come from the three graphs in Fig. 4.

$$F_{1,2}=\frac{\alpha}{8\pi^3}\delta_4(p_\pi+p_\sigma-p_\eta)\int d^4r d^4k\, D_{\mu\nu}(k)F_{1,2}{}^{\mu\nu}(r,k)\,,\quad (3.4)$$

where

$$F_{1,2}{}^{\mu\nu}=\mathrm{Tr}\{s(r)\Gamma s(r-p_{\eta,\sigma})\gamma^\mu s(r-p_{\eta,\sigma}-k)\gamma^\nu s(r-p_{\eta,\sigma})\Gamma +s(r)\gamma^\mu s(r-k)\gamma^\nu s(r)\Gamma s(r-p_{\eta,\sigma})\Gamma+s(r)\gamma^\mu \times s(r-k)\Gamma s(r-p_{\eta,\sigma}-k)\gamma^\nu s(r-p_{\eta,\sigma})\Gamma\}\,,\quad (3.5)$$

where Γ is γ_5 in $F_1{}^{\mu\nu}$ and 1 in $F_2{}^{\mu\nu}$.

To evaluate $F^{\mu\nu}$, we use the algebraic identity

$$-\gamma\cdot p_\pi\gamma_5=2m\gamma_5+s^{-1}(p-p_\pi)\gamma_5+\gamma_5 s^{-1}(p)\quad (3.6)$$

in each term of Eq. (3.2) to obtain a decomposition of $F^{\mu\nu}$ and therefore of F [$s(p)$ is the propagator $(\gamma\cdot p-m)^{-1}$]. The six $2m\gamma_5$ terms simply give $F_5{}^{\mu\nu}$. Of the remaining 12 terms, six are $F_1{}^{\mu\nu}+F_2{}^{\mu\nu}$. Thus we obtain

$$F^{\mu\nu}=F_5{}^{\mu\nu}+F_1{}^{\mu\nu}+F_2{}^{\mu\nu}+A^{\mu\nu}\,,\quad (3.7)$$

where $A^{\mu\nu}$ (the anomaly) is the remaining six terms in the decomposition of $F^{\mu\nu}$ using (3.6):

$$\begin{aligned}A^{\mu\nu}=\mathrm{Tr}\{&s(r)\gamma_5 s(r-p_\eta)\gamma_5\gamma^\mu s(r-p_\sigma-k)\gamma^\nu s(r-p_\sigma)\\&+s(r)\gamma_5 s(r-p_\eta)\gamma^\mu s(r-p_\eta-k)\gamma^\nu\gamma_5 s(r-p_\sigma)\\&+s(r)\gamma^\mu s(r-k)\gamma_5 s(r-p_\eta-k)\gamma^\nu\gamma_5 s(r-p_\sigma)\\&+s(r)\gamma_5 s(r-p_\eta)\gamma^\mu\gamma_5 s(r-p_\sigma-k)\gamma^\nu s(r-p_\sigma)\\&+s(r)\gamma_5 s(r-p_\eta)\gamma^\mu s(r-p_\eta-k)\gamma_5\gamma^\nu s(r-p_\sigma)\\&+s(r)\gamma_5 s(r-p_\eta)\gamma_5\gamma^\mu s(r-p_\sigma-k)s(r-k)\gamma^\nu\}\,.\end{aligned}\quad (3.8)$$

If $A^{\mu\nu}=0$, then (3.7) implies (2.12) and there is no anomaly. In expression (3.8) the first term cancels the fourth, and the second term cancels the fifth, leaving

$$A^{\mu\nu}=\mathrm{Tr}\{s(r)\gamma^\mu s(r-k)\gamma_5 s(r-p_\eta-k)\gamma^\nu\gamma_5 s(r-p_\sigma) +s(r)\gamma_5 s(r-p_\eta)\gamma_5\gamma^\mu s(r-p_\sigma-k)s(r-k)\gamma^\nu\}\,,\quad (3.9)$$

which is not obviously zero. From the cyclic property of the trace and the symmetry of $D_{\mu\nu}$, the second term of (3.9) can be slightly rewritten; the anomaly, which is an integral over $D_{\mu\nu}A^{\mu\nu}$, is proportional to

$$A=\int d^4r d^4k\, D_{\mu\nu}(k) \times\mathrm{Tr}\{s(r)\gamma^\mu s(r-k)\gamma_5 s(r-p_\eta-k)\gamma^\nu\gamma_5 s(r-p_\sigma) -s(r-k)\gamma^\mu s(r)\gamma_5 s(r-p_\eta)\gamma^\mu\gamma_5 s(r-p_\sigma-k)\}\,.\quad (3.10)$$

This would be zero if one could make, in the second term, the changes in the integration variables $k\to -k$, $r\to r-k$.

Each term in the integral (3.10) is convergent in r for fixed k, and logarithmically divergent in k for fixed r. In single integrals like

$$\int\frac{d^4k}{[(k-p)^2-m^2]^2}\quad (3.11)$$

the origin in k may be shifted without changing the integral's value, independently of the cutoff method used to define it. However, we shall see that the analogous remark is apparently not true for integrals of the form of each term in (3.10); the value of A depends on the cutoff procedure and on the *order* of doing the integrations.

To simplify the algebra, let us evaluate A at the point where it might contribute to a low-energy theorem, namely, $p_\eta=p_\sigma=p$ or $p_\pi=0$. First, we evaluate A in the Feynman gauge, $D_{\mu\nu}(k)=-g_{\mu\nu}$. We return to the question of gauge invariance in Sec. IV. Then

$$A=16m^2\int d^4r\frac{d^4k}{k^2}\frac{2r\cdot k+2(r-k)\cdot k-k\cdot p}{(r^2-m^2)[(r-k)^2-m^2][(r-p-k)^2-m^2][(r-p)^2-m^2]}\,.\quad (3.12)$$

At $p=0$, one may easily do the r integration first and get, by the Feynman parameter method,

$$A=32m^2\int\frac{d^4k}{k^2}\int_0^1 dz\frac{z(1-z)(2z-1)}{z(1-z)k^2-m^2}=0,\quad (3.13)$$

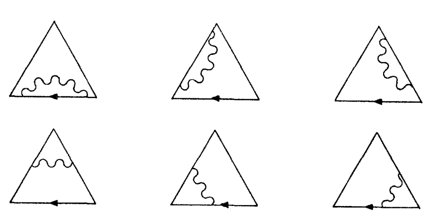

FIG. 3. Graphs for radiative corrections to F.

as one must, since A is convergent in r for fixed k. However, let us instead do the k integration first, as suggested by the renormalization prescription of computing all radiative corrections to propagators and vertices, and then inserting these into skeleton graphs. (We do not discuss the renormalizability of the axial-vector vertex here.) Then, again at $p=0$,[18]

$$A=32m^2\int\frac{d^4r}{[r^2-m^2]^2}f(r)\,,\quad (3.14)$$

[18] This is sufficiently general, since $A(p_\sigma,p_\eta)-A(0,0)$ is convergent in either r or k, and so is zero because shifting the origins is now legal. The same is true of the analogous choice made just before Eq. (5.10).

where

$$f(r)=\int\frac{d^4k}{k^2}\frac{2r\cdot k-k^2}{[(k-r)^2-m^2]^2}$$
$$=-\int\frac{d^4k}{(k^2-m^2)^2}+\int d^4k\times\left(\frac{1}{(k^2-m^2)^2}-\frac{1}{[(k-r)^2-m^2]^2}\right)+2\int\frac{d^4k}{k^2}\frac{r\cdot k}{[(k-r)^2-m^2]^2}. \quad (3.15)$$

The divergent constant is isolated in the first term in (3.15). The second term is unambiguously zero. The third term is finite and may be computed by standard techniques. Thus

$$-f(r)=\int\frac{d^4k}{(k^2-m^2)^2}+2i\pi^2\left(1+(r^2-m^2)\int_0^1\frac{dz}{[r^2(1-z)-m^2]}\right)$$
$$=\int\frac{d^4k}{(k^2-m^2)^2}+2i\pi^2\left[1-\frac{r^2-m^2}{r^2}\ln\left(\frac{m^2-r^2}{m^2}\right)\right]. \quad (3.16)$$

To define the divergent term, and the now-divergent r integration, cutoffs must be introduced. First let us define them by rotating the k^0 and r^0 contours in the usual way to get Euclidean metrics, and then integrate the magnitudes of the two four-vectors up to the cutoffs:

$$\int d^4k\, f(k^\mu k_\mu)\to 2i\pi^2\int_0^\Lambda k^3dk\, f(-k^2), \quad (3.17)$$

$$\int d^4r\, f(r^\mu r_\mu)\to 2i\pi^2\int_0^M r^3dr\, f(-r^2). \quad (3.18)$$

We use this prescription only for divergent integrals, and keep only terms which do not vanish as Λ, $M\to\infty$. Then

$$\int\frac{d^4k}{(k^2-m^2)^2}\to D(\Lambda) \quad (3.19)$$

and

$$\int\frac{d^4r}{(r^2-m^2)^2}\to D(M), \quad (3.20)$$

where

$$D(x)=i\pi^2[\ln(x^2/m^2)-1]. \quad (3.21)$$

Both cutoffs are taken as constants; e.g., Λ is not a function of r. Therefore, the cutoff double integration, though finite for finite Λ and M, is not obviously zero under the substitutions $k\to -k$, $r\to r-k$. In fact,

$$A=32m^2[D(M)D(\Lambda)+2i\pi^2D(M)+D'], \quad (3.22)$$

where[19]

$$D'=2i\pi^2\int\frac{d^4r}{(r^2-m^2)r^2}\ln\frac{m^2}{m^2-r^2}$$
$$=-4\pi^4\int_0^M\frac{x^3dx}{(x^2+m^2)x^2}\ln\frac{m^2}{x^2+m^2}$$
$$=\pi^4\ln^2(M^2/m^2). \quad (3.23)$$

As $\Lambda,M\to\infty$, $A\to\infty$ unless Λ is a particular function of M. For example, if $\Lambda=M$, $A=-32\pi^4m^2$. If Λ^2 is asymptotically proportional to $M^2\exp[1/\ln M^2]$, A is finite, and can be made exactly zero by choosing $\Lambda^2=M^2\exp[\ln(M^2/m^2)-1]^{-1}$.

This result does not depend on the cutoff procedure (3.17) and (3.18). If instead the divergent integrals are defined by

$$\int d^4k\, f(k)\to\int d^4k\frac{\Lambda^2}{k^2+\Lambda^2}f(k), \quad (3.24)$$

$$\int d^4r\, f(r)\to\int d^4r\frac{M^2}{r^2+M^2}f(r), \quad (3.25)$$

the functions D and D' are the same for large Λ and M (see the Appendix).

We conclude, therefore, that without further restrictions on the definition of A, A is not in general zero, and is in fact generally infinite.

Finally, one might think that the correct way to define these integrals is to cut off the k integration alone,[5] and do it first; we show in the Appendix that A is still ambiguous and not generally zero even with this procedure.

IV. GAUGE INVARIANCE IN $\eta\pi\sigma$ VERTEX

We argue next that F, F_5, F_1, and F_2 ought to be gauge invariant, and therefore that the anomaly A, defined by (3.10), must be gauge invariant also. The quantities F and F_5 are off-shell matrix elements of neutral densities, and F_1 and F_2 are electromagnetic contributions to combinations of mass shifts and mixing angles. To the extent that all these quantities are observable, they should be gauge invariant.

Now A has the remarkable property that, if the same cutoff method is used in the gauge-independent and

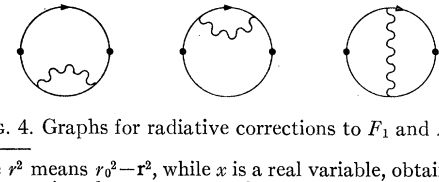

FIG. 4. Graphs for radiative corrections to F_1 and F_2.

[19] Here r^2 means $r_0^2-\mathbf{r}^2$, while x is a real variable, obtained from $(r^2)^{1/2}$ by rotating the r_0 contour. The extra terms present in $\eta\to\pi^+\pi^-\pi^0$ are convergent and therefore unlikely to have anomalous behavior.

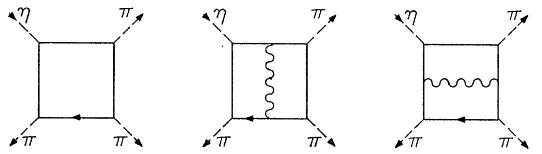

FIG. 5. Zeroth-order graph for $\eta\to 3\pi$, and new-type second-order graphs.

gauge-dependent terms, A must be zero if it is gauge independent. Let us see how this comes about. It will be sufficient to discuss covariant gauges,

$$D_{\mu\nu}(k)=(-g_{\mu\nu}+ak_\mu k_\nu/k^2)/k^2 \tag{4.1}$$

and to restrict a to be a constant. Then $A=A_1+aA_2$, where

$$A_1=-\int d^4r\frac{d^4k}{k^2}A^\mu{}_\mu \tag{4.2}$$

and

$$A_2=\int d^4r\frac{d^4k}{k^2}\frac{k_\mu k_\nu}{k^2}A^{\mu\nu}. \tag{4.3}$$

Gauge invariance requires $A_2=0$. Let us first examine the relationship between A_1 and A_2 at $p_\eta=p_\sigma=0$. Then $A^{\mu\nu}$ must be a sum of scalar functions of r and k multiplied by tensors $g^{\mu\nu}$, $r^\mu r^\nu$, $k^\mu k^\nu$, $r^\mu k^\nu$, and $k^\mu r^\nu$, respectively. It is easy to see from (3.10) with $p_\eta=p_\sigma=0$ that only the $g^{\mu\nu}$ term occurs. Therefore,

$$A^\mu{}_\mu=(4k_\mu k_\nu/k^2)A^{\mu\nu} \tag{4.4}$$

and $A_1=-4A_2$; both pieces vanish simultaneously.

The argument can be extended to p_η, $p_\sigma\neq 0$. For simplicity let us take $p_\eta=p_\sigma=p$. It follows from (3.10) that the only additional tensors which occur are proportional to p^μ or p^ν, i.e., there are no terms in $r^\mu r^\nu$, $r^\mu k^\nu$, $k^\mu r^\nu$, or $k^\mu k^\nu$. Now $A^{\mu\nu}$ has the form

$$A^{\mu\nu}=g^{\mu\nu}[A_2(p,r,r-k)-A_2(p,r-k,r)] +[B^{\mu\nu}(p,r,r-k)-B^{\mu\nu}(p,r-k,r)], \tag{4.5}$$

where the tensors occurring in $B^{\mu\nu}$ are $p^\mu r^\nu$, $r^\mu p^\nu$, $k^\mu p^\nu$, and $p^\mu k^\nu$. The tensor $B^{\mu\nu}$ is sufficiently convergent that

$$\int d^4r\frac{d^4k}{k^2}[B^\mu{}_\mu(p,r,r-k)-B^\mu{}_\mu(p,r-k,r)] =\int d^4r\frac{d^4k}{k^2}\frac{k^\mu k^\nu}{k^2}[B^{\mu\nu}(p,r,r-k) -B^{\mu\nu}(p,r-k,r)]=0, \tag{4.6}$$

i.e, only the term in $g^{\mu\nu}$ is so divergent that it contributes to the anomaly; arguing as above, $A_1=-4A_2$ still, and gauge invariance implies $A=0$. The proof of (4.6) is easy, although not trivially immediate. The point is that dimensionally a coefficient of p^μ must be more convergent than a coefficient of $g^{\mu\nu}$. The argument can also be extended to $p_\eta\neq p_\sigma$, and is therefore general.

For completeness, we display the gauge-dependent parts of the right-hand side of the axial-vector Ward identity (2.12). Like A, the value of the individual pieces depends on the method by which the divergent integrals are defined. Quantities (like A_2 above) proportional to the gauge-dependent parts of F, F_5, F_1, and F_2 may be defined as

$$G=\int d^4r\frac{d^4k}{k^2}\frac{k_\mu k_\nu}{k^2}F^{\mu\nu}(p,r,k), \tag{4.7}$$

$$G_5=\int d^4r\frac{d^4k}{k^2}\frac{k_\mu k_\nu}{k^2}F_5{}^{\mu\nu}(p,r,k), \tag{4.8}$$

$$G_{1,2}=\int d^4r\frac{d^4k}{k^2}\frac{k_\mu k_\nu}{k^2}F_{1,2}{}^{\mu\nu}(p,r,k). \tag{4.9}$$

Then

$$G=G_5+G_1+G_2+A_2 \tag{4.10}$$

is an algebraic identity.

G_5 ought to be zero, since it is the gauge-dependent part of a more-or-less observable quantity. It can be obtained from Eq. (3.2) by replacing $\gamma\cdot p_\pi$ with $-2m$ and contracting with $k^\mu k^\nu/k^2$. The resulting trace looks formidable, but can be unraveled using the identities

$$s(p)\gamma\cdot ks(p-k)=s(p-k)-s(p)=s(p-k)\gamma\cdot ks(p) \tag{4.11}$$

and

$$\int f(k^2)s(p)\gamma\cdot ks(p-k)\gamma\cdot ks(p)d^4k =\int[s(p-k)-s(p)]f(k)d^4k. \tag{4.12}$$

The second identity (4.12) is not completely general but depends on the validity of symmetric integration in the k variable. After some algebra, one obtains

$$\begin{aligned}G_5=-2m\int d^4r\frac{d^4k}{k^4}\mathrm{Tr}\{&[s(r-k)\gamma_5 s(r-p_\eta-k)\gamma_5 s(r-p_\sigma)\\&-s(r)\gamma_5 s(r-p_\eta)\gamma_5 s(r-p_\sigma-k)]\\&+[s(r)\gamma_5 s(r-p_\eta-k)\gamma_5 s(r-p_\sigma-k)\\&-s(r-k)\gamma_5 s(r-p_\eta)\gamma_5 s(r-p_\sigma)]\\&+[s(r-k)\gamma_5 s(r-p_\eta)\gamma_5 s(r-p_\sigma-k)\\&-s(r)\gamma_5 s(r-p_\eta-k)\gamma_5 s(r-p_\sigma)]\}.\end{aligned} \tag{4.13}$$

Each term in brackets would vanish if the change of variables $r-k\leftrightarrow r$ were permissible. Thus, vector gauge invariance is subject to the same ambiguity as the axial-vector Ward identity. In particular, G_5 will not be zero under any of the cutoff procedures discussed above except those which also give $A=0$.

Similarly, the gauge-dependent parts of F_1 and F_2 are

$$G_1=\int d^4r\frac{d^4k}{k^4}\mathrm{Tr}\{s(r-k)\gamma_5 s(r-p_\eta-k)\gamma_5 -s(r)\gamma_5 s(r-p_\eta)\gamma_5\} \tag{4.14}$$

and

$$G_2=\int d^4r\frac{d^4k}{k^4}\,\mathrm{Tr}\{s(r-k)s(r-p_\sigma-k)-s(r)s(r-p_\sigma)\}\,. \quad (4.15)$$

V. $\eta\to 3\pi$ AMPLITUDE

In this section we study the analogous calculation for the process of $\eta\to 3\pi$, without an intermediate σ meson. In order to avoid having to consider photon-pion vertices, we restrict our calculations to $\eta\to 3\pi^0$ in a model similar to the one considered in the previous sections; again, we look for anomalies in the fermion loops; i.e., we wish to compute graphs like Fig. 5, and their lowest-order radiative corrections.[20]

Analogously to the second method described in Sec. II, we could obtain the relevant axial-vector Ward identity by adding to the interaction Lagrangian (2.22) a term of the form $2g\phi_i j_{5i}$, which would add an explicit term $2g\phi_i j$ to $\partial_\mu j_{5i}{}^\mu$ in (2.23). However, it is just as simple to use the commutator method of the first part of Sec. II, so that the divergence is still given by Eq. (2.8), and compute the four-point function for the sources of the four mesons:

$$T_5=2m\int d^4w d^4x d^4y d^4z\; e^{ip_i\cdot w}e^{ip_j\cdot x}e^{ip_k\cdot y}e^{-ip_\eta\cdot z}\times\langle|T[j_{53}(w)j_{53}(x)j_{53}(y)j_5(z)]|\rangle\,, \quad (5.1)$$

where p_i, p_j, and p_k are the momenta of the three final π^0's.

We need the relation between T_5 and T, defined by

$$T=-ip_{i\mu}\int d^4w d^4x d^4y d^4z\; e^{ip_i\cdot w}e^{ip_j\cdot x}e^{ip_k\cdot y}e^{-ip_\eta\cdot z}\times\langle|T[j_{53}{}^\mu(w)j_{53}(x)j_{53}(y)j_5(z)]|\rangle\,. \quad (5.2)$$

The factor $-ip_{i\mu}$ is equivalent to $\partial/\partial w^\mu$ acting on the vacuum-expectation value. Bringing this derivative through the time ordering, we obtain

$$T=T_5+T^{(\eta)}+T^{(\pi)}\,. \quad (5.3)$$

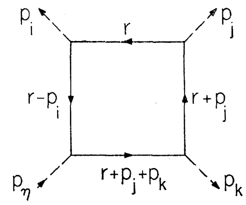

FIG. 6. Labeling of momenta for graphs which enter $T^{\mu\nu}$.

The first term on the right, T_5, comes from $\partial_\mu j_{53}{}^\mu(w)$ and is essentially the $\eta\to 3\pi$ amplitude. We have arbitrarily chosen to write the PCAC equation for the first pion (with momentum p_i). Any anomalies in a perturbation-theory evaluation of (5.3) ought to carry over into a more realistic field theory, e.g., one with PCAC, and could explain the η-decay puzzle.

The term $T^{(\eta)}$ comes from the commutator (2.16), and looks like a *vertex*, scalar isovector $\to 2\pi^0$, rather than a two-point function as in the previous case. $T^{(\pi)}$ is really two terms, coming from the commutator of $j_{53}{}^0(w)$ with $j_{53}(x)$ and $j_{53}(y)$. This commutator is a scalar, isoscalar density, so $T^{(\pi)}$ is proportional to the $\eta\pi\sigma$ vertex already discussed. All of the terms violate G parity, and so vanish without electromagnetic corrections. We shall evaluate them to order e^2.

The topologically distinct graphs in the vertex terms consist of two distinct fermion traiangles, each with six photon insertions, as before. The graphs contributing to T and T_5 have four-vertex fermion loops, each with ten possible distinct photon insertions; there are six possible permutations of the external momenta on the fermion loops, for a total of 60 graphs. Of the ten photon insertions, eight are propagator and vertex corrections; the two others are the skeleton graphs of Fig. 5. These graphs are the new feature of this amplitude.

It will be sufficient to compute the insertions to one of the six permutations of the meson momenta. The remaining ones follow from symmetry considerations. We use the order indicated in Fig. 6, and omit the charge-space factor, as before. The integral for T is proportional to

$$\int d^4r\frac{d^4k}{k^2}D_{\mu\nu}(k)T^{\mu\nu}\,, \quad (5.4)$$

where

$$\begin{aligned}T^{\mu\nu}=-\mathrm{Tr}\{&s(r)\gamma^\mu s(r-k)\gamma^\nu s(r)\gamma_5 s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma_5 s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma^\nu s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma^\mu s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma^\mu s(r-p_i-k)\gamma^\nu s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma^\mu s(r-k)\gamma_5 s(r+p_j-k)\gamma^\nu s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma_5 s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma^\mu s(r+p_j+p_k-k)\gamma_5 s(r-p_i-k)\gamma^\nu s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma^\mu s(r-p_i-k)\gamma\cdot p_i\gamma_5 s(r-k)\gamma^\nu\\
&+s(r)\gamma^\mu s(r-k)\gamma_5 s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma\cdot p_i\gamma_5\\
&+s(r)\gamma_5 s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma_5 s(r-p_i-k)\gamma^\nu s(r-p_i)\gamma\cdot p_i\gamma_5\}\,. \quad (5.5)\end{aligned}$$

[20] The extra terms present in $\eta\to\pi^+\pi^-\pi^0$ are convergent and therefore unlikely to have anomalous behavior. Here r^2 means $r_0{}^2-r^2$, while x is a variable, obtained from $(r^2)^{1/2}$ by rotating the r_0 contour.

The first four terms are the propagator corrections. The next four are the vertex corrections and the final two are graphs like Fig. 5.

These terms can be unraveled using

$$-s(r-p_i)\gamma\cdot p_i\gamma_5 s(r)=2m\gamma_5+\gamma_5 s(r)+s(r-p_i)\gamma_5\,, \tag{5.6}$$

which is just another form of (3.6). The $2m\gamma_5$ term gives the contribution to T_5 on the right-hand side of (5.3) appropriate to the chosen permutation of external momenta. The remaining terms should simply be proportional to (one-sixth of) $T^{(\eta)}+T^{(\pi)}$.

Each of these vertex functions has six terms to this order, while there are 20 terms obtained by replacing $-s(r-p_i)\gamma\cdot p_i\gamma_5 s(r)$ by $\gamma_5 s(r)+s(r-p_i)\gamma_5$ in each of the ten terms in (5.5). Six of these 20 terms contribute to $T^{(\eta)}$, and are simply the radiative corrections to Fig. 7(a):

$$\begin{aligned}\mathrm{Tr}\{&s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma^\nu s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\\&+s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma^\mu s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\gamma_5 s(r-p_i)\\&+s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma^\mu s(r-p_i-k)\gamma^\nu s(r-p_i)\\&+s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\gamma_5 s(r-p_i)\\&+s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma^\mu s(r+p_j+p_k-k)\gamma_5 s(r-p_i-k)\gamma^\nu s(r-p_i)\\&+s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma_5 s(r-p_i-k)\gamma^\nu s(r-p_i)\}\,.\end{aligned} \tag{5.7}$$

Similarly, six of the 20 terms contribute to $T^{(\pi)}$, and are the radiative corrections to Fig. 7(b):

$$\begin{aligned}\mathrm{Tr}\{&s(r)\gamma^\mu s(r-k)\gamma^\nu s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\\&+s(r)\gamma_5 s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma^\nu s(r+p_j)\gamma_5 s(r+p_j+p_k)\\&+s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma^\mu s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\\&+s(r)\gamma^\mu s(r-k)\gamma_5 s(r+p_j-k)\gamma^\nu s(r+p_j)\gamma_5 s(r+p_j+p_k)\\&+s(r)\gamma_5 s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\\&+s(r)\gamma^\mu s(r-k)\gamma_5 s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\}\,.\end{aligned} \tag{5.8}$$

The necessary symmetry in p_j and p_k will not appear until we add together the permutations of the external legs.

In both (5.7) and (5.8), the sixth term in the trace is the vertex correction to the scalar vertex and, unlike the other five terms, does not have the standard form of a vertex insertion; the loop momentum k runs around the long way instead of around the vertex. If the integrals into which (5.7) and (5.8) are to be inserted were convergent, the result would be the same; but here we must be more careful. The expected form can be obtained in each case by the now-familiar substitutions $k\to -k$, $r\to r-k$.

Thus, the "error" in $T^{(\pi)}$ is

$$\begin{aligned}\mathrm{Tr}\{&s(r)\gamma^\mu s(r-k)\gamma_5 s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\\&\times\gamma^\nu s(r+p_j+p_k)-s(r-k)\gamma^\mu s(r)\gamma_5 s(r+p_j)\\&\times\gamma_5 s(r+p_j+p_k)\gamma^\nu s(r+p_j+p_k-k)\}\,.\end{aligned} \tag{5.9}$$

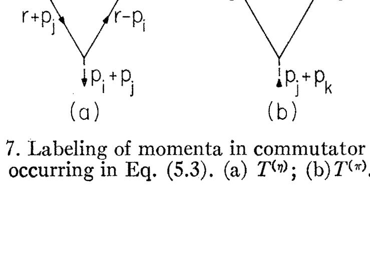

FIG. 7. Labeling of momenta in commutator terms occurring in Eq. (5.3). (a) $T^{(\eta)}$; (b) $T^{(\pi)}$.

Since the difference (5.9) does not unambiguously contribute zero to $T^{(\pi)}$, we must consider it as a contribution to the anomaly, provided that the correct way to define all these corrections is indeed to let the momentum k go around the short loop. This is an arbitrary prescription without a complete renormalization scheme, but let us tentatively adopt it and investigate the consequences.

We evaluate (5.9) in the Feynman gauge, i.e., contract with $-g_{\mu\nu}$, and set $p_i=p_j=p_k=0$.[18] Then the "error" in $T^{(\pi)}$ becomes

$$\begin{aligned}\mathrm{Tr}\{&s(r-k)\gamma^\mu s(r)\gamma_5 s(r)\gamma_5 s(r)\gamma_\mu s(r-k)\\&-s(r)\gamma^\mu s(r-k)\gamma_5 s(r-k)\gamma_5 s(r-k)\gamma_\mu s(r)\}\\&=\frac{16m(k^2-2r\cdot k)}{[(r-k)^2-m^2]^2(r^2-m^2)^2}\,.\end{aligned} \tag{5.10}$$

Equation (5.10) has the same form as the $\eta\pi\sigma$ anomaly, so we know it is not unambiguously zero.

Another anomalous contribution to the right-hand side of (5.3) comes from the similar "error" in the last term of (5.7). Replacing this term by the correct scalar vertex correction in $T^{(\eta)}$, we obtain the difference

$$\begin{aligned}\mathrm{Tr}\{&s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma_5 s(r+p_j-k)\gamma_5 s(r-p_i-k)\\&\times\gamma^\nu s(r-p_i)-s(r+p_j-k)\gamma^\mu s(r+p_j)\gamma_5\\&\times s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma^\nu s(r-p_i-k)\}\,.\end{aligned} \tag{5.11}$$

Again, contracting with $-g_{\mu\nu}$ at $p_i=p_j=p_k=0$, expression (5.11) becomes

$$\frac{-16m(k^2-2r\cdot k)}{[(r-k)^2-m^2]^2(r^2-m^2)^2}, \tag{5.12}$$

as before.

There remain eight terms in the expansion of (5.5) still unaccounted for. Four of them cancel algebraically, leaving an additional contribution to the anomaly

$$\begin{aligned}\mathrm{Tr}\{&s(r-k)\gamma_5 s(r+p_j-k)\gamma^\nu s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma_5\gamma^\mu \\ &+s(r)\gamma_5 s(r+p_j)\gamma^\mu s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma_5 s(r-p_i-k)\gamma^\nu\gamma_5 \\ &+s(r)\gamma_5 s(r+p_j)\gamma_5 s(r+p_j+p_k)\gamma^\mu s(r+p_j+p_k-k)\gamma_5 s(r-p_i-k)\gamma^\nu\gamma_5 \\ &+s(r-k)\gamma_5 s(r+p_j-k)\gamma_5 s(r+p_j+p_k-k)\gamma^\nu s(r+p_j+p_k)\gamma_5 s(r-p_i)\gamma_5\gamma^\mu\}. \end{aligned} \tag{5.13}$$

Because of the symmetry of $D_{\mu\nu}(k)$, γ^μ and γ^ν may be interchanged in the second and fourth terms of (5.13). Then it is easy to see that (5.13) would contribute zero to the integral over r and k if $k\to -k$, $r\to r-k$, were permissible.

As before, we evaluate the trace in (5.13) with the external momenta set to zero, and contract with $-g_{\mu\nu}$. The result is

$$\frac{32m(k^2-2r\cdot k)}{[(r-k)^2-m^2]^2(r^2-m^2)^2}. \tag{5.14}$$

Thus, the apparent errors in the scalar vertex corrections to $T^{(\eta)}$ and $T^{(\pi)}$ cancel exactly the explicit terms remaining from the expansion of $s(r-p_i)\gamma\cdot p_i\gamma_5 s(r)$, and the anomaly is zero independently of the cutoff procedure in the Feynman gauge.

Finally, let us check the gauge invariance of T_5, $T^{(\eta)}$, and $T^{(\pi)}$. Each is a physical quantity, and so should be gauge invariant.

We use the general gauge (4.1) in Eq. (5.5), with $\gamma\cdot p_i$ replaced by $2m$, and in Eqs. (5.7) and (5.8), and, as in Sec. IV, reduce the result using the identities (4.11) and (4.12). In each term, the coefficient of a turns out to depend upon the integral

$$\int d^4r\int \frac{d^4k}{k^4}(m^2-r^2+r\cdot k)\frac{2r\cdot k-k^2}{(r^2-m^2)^2[(r-k)^2-m^2]^2} \tag{5.15}$$

for $p_i=p_j=p_k=0$. But this integral is exactly the same as the one occurring in expression (4.13) for G_5, evaluted at $p_\sigma=p_\eta=0$, and so is the same type as the integral (3.12) for A. In particular, therefore, the same cutoff methods which make A vanish make the integral (5.16) zero also.

Thus, although there is no ambiguity in the $\eta\to 3\pi$ amplitude, there is an ambiguity in the gauge-dependent parts of T, $T^{(\pi)}$, and $T^{(\eta)}$, which must be defined to be zero.

VI. ORDER-α^2 CONTRIBUTION TO AXIAL-VECTOR WARD IDENTITY FOR $\pi^0\to 2\gamma$

Thus far we have studied one case, $\eta\to\sigma\pi$, where the physical amplitude and the gauge-dependent part were each ambiguous by a divergent integral of the form

$$\int d^4r d^4k[f(r,k)-f(r-k,\,-k)], \tag{6.1}$$

and another case, $\eta\to 3\pi$, where only the gauge-dependent part was ambiguous by a term like (6.1). Finally we want to consider $\pi^0\to 2\gamma$; here we will find that there is no ambiguity of the form of (6.1).

In order α^2, the relevant axial-vector Ward identity relates the physical amplitude

$$\begin{aligned}\epsilon^{\rho_2\rho_1\alpha\sigma}k_{1\alpha}k_{2\sigma}T_5(k_1,k_2) &=\int d^4k\,D_{\mu\nu}(k)\int d^4x d^4y e^{ik\cdot x}e^{i(k_1+k_2)\cdot y} \\ &\times\langle\gamma,k_{1\rho_1};\gamma,k_{2\rho_2}|T[j_{53}(y)j_{\mathrm{em}}{}^\mu(x)j_{\mathrm{em}}{}^\nu(0)]|\rangle \end{aligned} \tag{6.2}$$

to the quantity

$$\begin{aligned}\epsilon^{\rho_2\rho_1\alpha\sigma}k_{1\alpha}k_{2\sigma}T(k_1,k_2) &=\int d^4k\,D_{\mu\nu}(k)\int d^4x d^4y\; e^{ik\cdot x}e^{i(k_1+k_2)\cdot y} \\ &\times\frac{\partial}{\partial y^\eta}\langle\gamma,k_{1\rho_1};\gamma,k_{2\rho_2}|T[j_{53}{}^\eta(y)j_{\mathrm{em}}{}^\mu(x)j_{\mathrm{em}}{}^\nu(0)]|\rangle. \end{aligned} \tag{6.3}$$

If the commutator (2.25) vanishes, then these two expressions must be equal; this is what we want to check by evaluating (6.3) in perturbation theory. We calculate the order α^2 part of the graphs shown in Fig. 1 [where p_π now connects to $\gamma\cdot(k_1+k_2)$] and choose the momentum as shown there. Then

$$\epsilon^{\rho_1\rho_2\alpha\sigma}k_{1\alpha}k_{2\sigma}T(k_1,k_2)=\int d^4r\int d^4k \times D_{\mu\nu}(k)T^{\rho_2\rho_1\mu\nu}(r,k), \tag{6.4}$$

where

$$T^{\rho_2\rho_1\mu\nu}(r,k)=\mathrm{Tr}\{s(r)\gamma^\mu s(r-k)\gamma^\nu s(r)\gamma^{\rho_2}s(r-k_2)\gamma\cdot(k_1+k_2)\gamma_5 s(r+k_1)\gamma^{\rho_1}$$
$$+s(r)\gamma^\mu s(r-k)\gamma^{\rho_2}s(r-k_2-k)\gamma^\nu s(r-k_2)\gamma\cdot(k_1+k_2)\gamma_5 s(r+k_1)\gamma^{\rho_1}$$
$$+s(r)\gamma^{\rho_2}s(r-k_2)\gamma^\mu s(r-k_2-k)\gamma^\nu s(r-k_2)\gamma\cdot(k_1+k_2)\gamma_5 s(r+k_1)\gamma^{\rho_1}$$
$$+s(r)\gamma^{\rho_2}s(r-k_2)\gamma^\mu s(r-k_2-k)\gamma\cdot(k_1+k_2)\gamma_5 s(r+k_1-k)\gamma^\nu s(r+k_1)\gamma^{\rho_1}$$
$$+s(r)\gamma^{\rho_2}s(r-k_2)\gamma\cdot(k_1+k_2)\gamma_5 s(r+k_1)\gamma^\mu s(r+k_1-k)\gamma^\nu s(r+k_1)\gamma^{\rho_1}$$
$$+s(r)\gamma^{\rho_2}s(r-k_2)\gamma\cdot(k_1+k_2)\gamma_5 s(r+k_1)\gamma^\mu s(r+k_1-k)\gamma^{\rho_1}s(r-k)\gamma^\nu+(k_1,\rho_1\leftrightarrow k_2,\rho_2)\}\,. \quad (6.5)$$

From the identity (5.6), the $2m\gamma_5$ term is precisely the desired quantity (6.2), but again there is something left over. The part left over, $A^{\rho_2\rho_1}$, must have the form

$$A^{\rho_2\rho_1}(k_1,k_2)=\epsilon^{\rho_2\rho_1\alpha\sigma}k_{1\alpha}k_{2\sigma}A(k_1,k_2) \quad (6.6)$$

and is given by

$$A^{\rho_2\rho_1}(k_1,k_2)=\int d^4r\int d^4k\, D_{\mu\nu}(k)$$
$$\times\mathrm{Tr}\{s(r)\gamma^\mu s(r-k)\gamma^{\rho_2}s(r-k_2-k)\gamma^\nu\gamma_5 s(r+k_1)\gamma^{\rho_1}$$
$$+s(r)\gamma^{\rho_2}s(r-k_2)\gamma_5\gamma^\mu s(r+k_1-k)\gamma^{\rho_1}s(r-k)\gamma^\nu$$
$$+(k_1,\,\rho_1\leftrightarrow k_2,\,\rho_2)\}\,. \quad (6.7)$$

Again this would be zero if we could let $k\to -k$, $r\to r-k$ in one of the terms.

If we now expand the propagators which depend on k_1 or k_2, using

$$s(r+k_1)=s(r)-s(r)\mathbf{k}_1 s(r)+\text{higher order in } k_1, \quad (6.8)$$

where $s(r)=(\mathbf{r}-m)^{-1}$, then only the second term contributes. The first term cannot have the form (6.6) while the higher-order terms are convergent integrals with the change of variable allowed. But if we put the second term of (6.8) in (6.7), and introduce a Feynman parameter to make the denominators functions of k^2 and r^2 only, then the integral $A^{\rho_1\rho_2}$ is zero by symmetric integration independently of how it is regulated. Thus we find no ambiguity in the axial-vector Ward identity for the $O(\alpha^2)$ contribution to $\pi^0\to 2\gamma$.

In addition, the perturbation-theory expansion of (6.2), i.e., (6.5) with $\gamma\cdot(k_1+k_2)$ replaced by $2m$, is both externally and internally gauge invariant, as can be shown by multiplying by $k_{1\rho_1}$, $k_{2\rho_2}$ or setting $D_{\mu\nu}(k)=k_\mu k_\nu/k^2$, and using the identities (4.11) and (4.12). Thus we find no ambiguity of the form (6.1) in $\pi^0\to 2\gamma$.

VII. CONCLUSIONS

We have searched for an anomaly in the $\eta\pi^0\sigma$ vertex and in the $\eta\pi^0\pi^0\pi^0$ vertex, and found ambiguous expressions whose most natural values are zero. In the first case, we conclude that if all ambiguous integrals are to be evaluated in the way which ensures electromagnetic gauge invariance, the "naive" axial-vector Ward identity is satisfied. In the second case, we reached the same conclusion under the assumption that divergent integrals for certain graphs are to be defined by requiring the internal loop momenta to appear in a consistent way.

It is evident that there is no clean result of the sort presented by Jackiw and Johnson[4] and by Adler.[2] Although our results suggest that PCAC should hold to order e^2 for the vertices we have studied, in models of the sort discussed, it is apparent that a serious study of the renormalizability of the axial-vector vertex in such theories is necessary to understand the nature of the η-decay paradox. It would certainly be useful to reexamine these questions using the split-point definition of densities bilinear in fermion fields[3,4]—although perhaps the effect would be simply to translate our results into another language.

We have also used our methods to compute the gauge-dependent terms of the $\eta\to 3\pi^0$ vertex and other quantities which appear in Eq. (5.3), as well as the order α^2 corrections to the anomaly responsible for $\pi^0\to 2\gamma$. As for the $\eta\pi\sigma$ vertex, the gauge-dependent terms of (5.3) are not individually unambiguously zero, even though they are more or less observable quantities. In contrast, the corrections to the axial-vector–2γ vertex are individually internally gauge invariant, without any special definition of the cutoff procedure.

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APPENDIX

First we show that with the cutoff procedure defined by (3.24) and (3.25), expression (3.22) is obtained for A. It is necessary only to calculate D and D'. Instead of (3.19), we have

$$D(\Lambda)=+\int\frac{d^4k}{(k^2-m^2)^2}\frac{\Lambda^2}{k^2+\Lambda^2}. \quad (\mathrm{A1})$$

We introduce the Feynman parameters as usual:

$$D(\Lambda)=2\Lambda^2\int d^4k\int_0^1 dx\int_0^1 dy\, x\delta(1-x-y)$$
$$+[k^2-xm^2+y\Lambda^2]^{-3}$$
$$=+i\pi^2\Lambda^2\int_0^1 x[(1-x)\Lambda^2-xm^2]^{-1}$$
$$=-i\pi^2\ln(\Lambda^2/m^2) \quad (\mathrm{A2})$$

plus terms which vanish as $\Lambda\to\infty$, in agreement with (3.21); and similarly

$$D'=+2i\pi^2\int\frac{d^4r}{(r^2-m^2)}\int_0^1\frac{dz}{r^2(1-z)-m^2}\frac{M^2}{r^2+M^2}$$

$$=-2\pi^4M^2\int_0^1 dz\int_0^1 dx\int_0^{1-x}dy\,(1-zy)^{-2}\times[x(m^2+M^2)-m^2]^{-1}. \quad (A3)$$

The y and z integrations are elementary:

$$D'=-\frac{2\pi^4M^2}{m^2+M^2}\int_0^1\frac{\ln x\,dx}{x+m^2/(m^2+M^2)}. \quad (A4)$$

Substitute $\xi=x+m^2/(m^2+M^2)$, and expand the new limits of integration in powers of m^2/M^2, keeping only first-order terms:

$$D'=+2\pi^4\int_{+m^2/M^2}^{1+m^2/M^2}\frac{\ln\xi+\ln(1-m^2/M^2\xi)}{\xi}d\xi. \quad (A5)$$

The second term is a constant as $M^2\to\infty$. The first is

$$-\pi^4\int_{+m^2/M^2}^{1+m^2/M^2}\frac{d}{d\xi}(\ln^2\xi)d\xi=\pi^4\ln^2\left(\frac{M^2}{m^2}\right), \quad (A6)$$

in agreement with (3.23).

Finally, let us define A according to the procedure suggested at the end of Sec. II:

$$A=\int d^4r\frac{d^4k}{k^4}\frac{\Lambda^2}{k^2-\Lambda^2}\frac{1}{(r^2-m^2)}\frac{1}{(k-r)^2-m^2}\times\left(\frac{1}{(r-k)^2-m^2}-\frac{1}{r^2-m^2}\right) \quad (A7)$$

and do the k integration first. From (A7),

$$A=24\int d^4r d^4k\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\int_0^{1-\alpha-\beta}\frac{d\gamma(\alpha-\gamma)}{[(1-\gamma)k^2-\beta\Lambda^2+(1-\alpha-\beta)(r^2-m^2)-(1-\alpha+\beta-\gamma)(2k\cdot r)]^5}. \quad (A8)$$

Integrating over k, and then over r, and making some algebraic substitutions, one obtains from (A8)

$$A=\pi^4\Lambda^2\int_0^1 dx\int_0^1 dy\int_0^1 dz\times\frac{(2z-1)}{[xy(z^2-z+1)-1]^2}\frac{1}{\Lambda^2(1-y)+m^2y}. \quad (A9)$$

Substituting $z\to 1-z$, one obtains $A=-A$, and therefore apparently $A=0$. This is not correct, however, since the integral diverges near $xy=1$, $z=0$, or $z=1$. To define the integral, replace the z integral by

$$\int_0^1 dz\to\lim_{\epsilon\to0;\,\epsilon'\to0}\int_\epsilon^{1-\epsilon'}dz. \quad (A10)$$

Then A is in general a logarithmically divergent function of ϵ and ϵ', and is zero only under the arbitrary rule $\epsilon=\epsilon'$. The divergence is due to our failure to introduce an r cutoff.