Duality and the Adler Condition

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We have shown that the Adler condition is a natural consequence of duality. However, contrary to the usual expectations, it is not related to the existence of partially or totally conserved currents.

IN spite of all the difficulties encountered in constructing a reasonable dual-resonance model, the results obtained up to now seem to reflect some aspects of the physical world. We should like to report in this paper one further attractive property of the model, namely, that the Adler[1] condition is a natural consequence of duality.[2] As will be shown later in the paper, this result does not imply the existence of a partially conserved current. This is rather remarkable since our usual belief was that in soft-pion calculations, partial conservation of axial-vector current (PCAC) plays a dominant role. We think a deeper understanding of the situation is necessary before we can draw any general conclusions on the basis for the Adler condition.

Let us start with the simplest kind of dual model where there is only one trajectory for the internal and external lines and where the external particles of positive parity are in the ground state. (The external particles can also be taken of negative parity but this implies a parity doubling of the trajectory.) The Adler condition in this case means that the amplitude goes to zero when the energy-momentum four-vector of one of the external particles goes to zero except for possible pole terms. Let us first see that this is indeed the case for the four-point amplitude (Fig. 1, $N=4$):

$$\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}. \qquad (1)$$

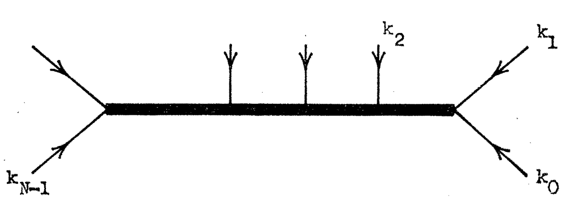

FIG. 1. N-point function $[s_i=(k_0+k_1+k_2+\cdots+k_i)^2,\ t=(k_1+k_2)^2]$.

The dangerous pole terms for this amplitude are

$$-\left[\frac{1}{\alpha(s)}+\frac{1}{\alpha(t)}\right] \qquad (2)$$

when $k_{1\mu}\to 0$, i.e., when $\alpha(s)$ and $\alpha(t)\to 0$. By explicit calculation we can show that

$$\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}+\left[\frac{1}{\alpha(s)}+\frac{1}{\alpha(t)}\right]\xrightarrow[\alpha(t)\to 0;\,\alpha(s)\to 0]{} 0. \qquad (3)$$

Next we consider the general N-point tree graphs[3] (Fig. 1). In the operator formalism of Fubini, Gordon, and Veneziano[4] this can be written as

$$\langle 0|G\Delta(s_2)\gamma_2\Delta(s_1)\gamma_1|0\rangle, \qquad (4a)$$

where the vertex

$$\gamma_i=\gamma(k_i)=e^{k_i a^\dagger}e^{k_i a}, \qquad (4b)$$

the propagator

$$\Delta(s_i)=\int z^{-\alpha(s_i)+R_a-1}(1-z)^{\alpha(0)-1}dz, \qquad (4c)$$

and G are defined in Ref. 4. Since we are allowing $k_{1\mu}\to 0$, we must subtract from (4) the two following pole terms:

$$-\frac{1}{\alpha(s_1)}\langle 0|G\Delta(s_2)\gamma_2|0\rangle \qquad (5)$$

and

$$-\frac{1}{\alpha(t)}\langle 0|G\Delta(s_2)\gamma_{(1+2)}|0\rangle, \qquad (6)$$

where $\gamma_{(1+2)}=e^{(k_1+k_2)a^\dagger}e^{(k_1+k_2)a}$. The first term is the scalar pole term in the s_1 channel and the second the scalar pole term in the t channel. Before proceeding, let us clarify our limiting procedure. Since γ_1 is not the

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[1] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

[2] Adler condition in connection with dual model was first considered by C. Lovelace, Phys. Letters **28B**, 264 (1968).

[3] K. Bardakçi and H. Ruegg, Phys. Letters **28B**, 342 (1968); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); C. Goebel and B. Sakita, *ibid.* **22**, 257 (1969); H. M. Chan and T. S. Tsun, Phys. Letters **28B**, 485 (1969); Z. Koba and H. B. Nielsen, Nucl. Phys. **10B**, 633 (1969).

[4] S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969); Y. Nambu, University of Chicago Report No. EFI 69-64 (unpublished).

correct vertex when k_1 goes off the mass shell, we should be careful to stay on the mass shell. This can be done by putting $\alpha(0)\to 0$ simultaneously as we let $k_1\to 0$. Also, since the propagator is singular in the limit of $\alpha(0)\to 0$, we could not have allowed $\alpha(0)\to 0$ before allowing $k_1\to 0$. Now let us use the following expansion:

$$\langle G\Delta(s_2)e^{k_2a\dagger}=\sum_{(N)} f_{(N)}\langle 0|P^{(N)}, \tag{7}$$

where $\langle 0|P^{(N)}$ means the partition states

$$\{i_1,i_2,\ldots|\sum_n ni_n=N\}.$$

We observe that in the limit $k_{1\mu}\to 0$ and $\alpha(0)\to 0$, the $f_{(N)}$'s are not singular. We must then show that

$$\langle 0|P^{(N)}e^{k_2a}\Delta(s_1)\gamma_1|0\rangle+\frac{1}{\alpha(t)}\langle 0|P^{(N)}e^{k_1a\dagger}|0\rangle +\frac{1}{\alpha(s_1)}\langle 0|P^{(N)}|0\rangle \xrightarrow[k_{1\mu}\to 0]{} 0. \tag{8}$$

If $\langle 0|P^{(N)}$ is the vacuum, this reduces to the four-point case and has already been proved. If $\langle 0|P^{(N)}\neq\langle 0|$, we have for the first term

$$\langle 0|P^{(N)}e^{k_2a}\Delta(s_1)\gamma_1|0\rangle =\int z^{-\alpha(s_1)-1}(1-z)^{\alpha(0)-1} \times\langle 0|P^{(N)}e^{k_2a}\exp(k_1a^{\dagger(n)}z^n/\sqrt{n})|0\rangle =(k_{1\mu})^{i_1}\cdots(k_{1\mu})^{i_p}\int z^{-\alpha(s_1)+N-1}(1-z)^{-\alpha(t)-1}dz, \tag{9}$$

where we used the explicit form for $\langle 0|P^{(N)}$:

$$\langle 0|P^{(N)}=\langle 0|(a_\mu{}^{(1)})^{i_1}\cdots(a_\mu{}^{(p)})^{i_p}\prod_1^p(j)^{i_j/2}.$$

The second term in (8) is simply

$$-(k_{1\mu})^{i_1}\cdots(k_{1\mu})^{i_p}\int(1-z)^{-\alpha(t)-1}dz, \tag{10}$$

and the third term is zero because $\langle 0|P^{(N)}\neq\langle 0|$. The left-hand side of (8) is therefore

$$(k_{1\mu})^{i_1}\cdots(k_{1\mu})^{i_p} \times\left\{\int z^{-\alpha(t)-1}[1-(1-z)^{-\alpha(s_1)+N-1}]dz\right\}. \tag{11}$$

When $k_{1\mu}\to 0$, the factor inside the curly bracket stays finite, and we have at least one $k_{1\mu}$ multiplying it because $\langle 0|P^{(N)}\neq\langle 0|$. The whole expression therefore

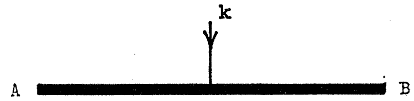

FIG. 2. Vertex function.

goes to 0 when $k_{1\mu}\to 0$. This proves the Adler condition for the N-point scalar function. This result is also clearly true if the N-point scalar function contains internal loops.[5]

Let us compare our results pertaining to the Adler condition with the usual results of the PCAC case. For this purpose let us consider the vertex function $\langle A|e^{ka\dagger}e^{ka}|B\rangle$ as shown in Fig. 2. Suppose we take A and B to be partition states. The vertex function then goes to zero when $k_\mu\to 0$ if $|A\rangle\neq|B\rangle$, but it stays finite if $|A\rangle=|B\rangle$. This is very different from the PCAC case, where all the one-particle matrix elements of $\phi^\pi(k)$ vanish in the limit $k\to 0$. The theory of conserved axial-vector current with massless pions also gives this result. Without this property, the soft pion could couple to the internal line and we would not have the Adler condition for the above usual theories. In our case, all the diagonal elements are finite. However, the Adler condition is not violated because we cannot distinguish here between a scalar coupled to the internal line and one coupled to the external line. Indeed, by duality, we can always shift the soft particle to the position where it couples only to the external line. It turned out that the diagonal elements give exactly the pole terms, which in any case do not vanish.

It was necessary to show this explicitly for the following reason: When we talk about the diagonal element, it is not precisely diagonal since $\langle 0|\Delta(s)|0\rangle =B(-\alpha(s),\alpha(0))$. This means that the vacuum contains an infinite number of scalar excitations as defined by Nambu.[4] It is the contribution of these scalar excitations which gives the crossed-channel poles in addition to the direct-channel poles.

We emphasize that we do not have the partially or totally conserved current because the vertex function does not vanish in the $k_1\to 0$ limit.

For the sake of interested readers, we show that our argument holds formally for the most general case shown in Fig. 3. The square stands for any configuration

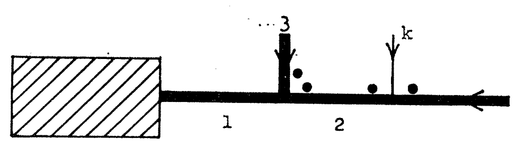

FIG. 3. General dual amplitude.

[5] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969); K. Bardakçi, M. Halpern, and J. Shapiro, *ibid.* **185**, 1910 (1969); D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters **2**, 399 (1969).

including loops.[6] This amplitude can be written in the notation of Ida, Matsumoto, and Yazaki[6]:

$$P\begin{pmatrix} \alpha & \dot{\gamma} \\ \dot{} & \dot{\beta} \end{pmatrix}^{P^\dagger\Omega^\dagger|3\rangle}_{P^\dagger\Delta\gamma(k)|2\rangle} \equiv \langle 3|\Omega P\begin{pmatrix} \gamma & \beta \\ & \dot{\alpha} \end{pmatrix}^{\Delta P\gamma(k)|2\rangle}_{P^\dagger}, \quad (12)$$

where

$$\begin{pmatrix} \alpha & \dot{\gamma} \\ \dot{} & \beta \end{pmatrix}$$

is a convenient notation for the symmetric vertex obtained first by Caneschi, Schwimmer, and Veneziano,[7] P is the projection operator to the physical state, and Ω is the twisting operator. We should not specify the structure of the operator multiplying line 1. We expand $P^\dagger|3\rangle$ and $P^\dagger|2\rangle$ in terms of partition states

$$P^\dagger|2\rangle = \sum_{(N_2)} |2_{(N_2)}\rangle, \quad (13a)$$

$$P^\dagger|3\rangle = \sum_{(N_3)} |3_{(N_3)}\rangle. \quad (13b)$$

Now, using the formulas

$$\int (1-z)^{L-\alpha(0)}\Omega\begin{pmatrix} \alpha & \dot{\gamma} \\ \dot{} & \dot{\beta} \end{pmatrix}^{d\Delta(z)}_{P^\dagger} = \Omega P\begin{pmatrix} \alpha & \dot{\gamma} \\ \dot{} & \beta \end{pmatrix}^{\Delta}_{P^\dagger}, \quad (14)$$

where $L=R-\frac{1}{2}p_1^2$ and $\Omega P=P\Omega P$,[6] we get

$$(12)=\sum_{(N_3)} \int \langle 3_{(N_3)}|(1-z)^{L-\alpha(0)} \times\Omega\begin{pmatrix} \gamma & \beta \\ & \dot{\alpha} \end{pmatrix}^{d\Delta(z)P\gamma(k)|2\rangle}_{P^\dagger}. \quad (15)$$

In the $k\to 0$ and $\alpha(0)\to 0$ limit, $\gamma(k)$ can be replaced by 1 in (15). Using (13a), each term of (15) becomes

$$B(-\alpha(P_3^2)+N_3,\ -\alpha(P_2^2)+N_2)\langle 3_{(N_3)}|\Omega\begin{pmatrix} \gamma & \dot{\beta} \\ & \dot{\alpha} \end{pmatrix}^{|2_{(N_2)}\rangle}_{P^\dagger}.$$

This completes the proof because when $\alpha(P_3^2)-N_3\to 0$ and $\alpha(P_2^2)-N_2\to 0$, we have

$$B(-\alpha(P_3^2)+N_3,\ -\alpha(P_2^2)+N_2) +\left[\frac{1}{\alpha(P_3^2)-N_3}+\frac{1}{\alpha(P_2^2)-N_2}\right]\to 0,$$

just as in Eq. (3). The crucial point in this proof is that we can have any value for P_3 and P_2.

Let us now discuss the case where the internal and the external trajectories are different. We assume the masses of the external scalar particles to be zero from the beginning. We assume that the intercept of the n-body resonance in the channels $k_i+k_{i+1}+\cdots+k_{i+n-1}$ $(1\leqslant i,\ i+n\leqslant N-1)$ (see Fig. 1) is $\alpha_n(0)$ and that

$$\alpha_n=(n-2)\alpha_3-(n-3)\alpha_2\leqslant 1. \quad (16)$$

The N-point scalar amplitude corresponding to Fig. 1 can be written[8]

$$\int dzdy\langle 0|\tilde{G}y^{-\alpha_3(s_2)+R-1}(1-y)^{-\alpha_2(0)-1}A(c^\dagger y)\gamma_2 \times z^{-\alpha_2(s_1)+R-1}(1-z)^{-\alpha_2(0)-1}\gamma_1|0\rangle \times(1-yz)^{2\alpha_2(0)-\alpha_3(0)}, \quad (17a)$$

where

$$A(x)=\sum (C_n/n!)^{1/2}x^n, \quad (17b)$$

and the C_n are defined by

$$(1-x)^{2\alpha_2-\alpha_3}=\sum_0^\infty C_nx^n, \quad (18)$$

and $[c,c^\dagger]=1$. When $k_1\to 0$, we have $\gamma_1=1$. Expression (17) then becomes

$$\int dy\langle 0|\tilde{G}y^{-\alpha_3(s_2)+R-1}(1-y)^{-\alpha_2(0)-1}A(c^\dagger y)\gamma_2|0\rangle \times\int dz\, z^{-\alpha_2(0)-1}(1-z)^{-\alpha_2(0)-1}(1-yz)^{2\alpha_2(0)-\alpha_3(0)}. \quad (19)$$

The Adler condition can be satisfied if the last integral vanishes independently of y. This can be achieved if and only if

$$\alpha_2(0)=\tfrac{1}{2} \quad\text{and}\quad \alpha_3(0)=0. \quad (20)$$

[The $y=0$ point gives $\alpha_2(0)=\frac{1}{2}$ and the $y=1$ point gives $\alpha_3(0)=0$. For the points between zero and 1, we use the formula $\int_0^1 t^{x-1}(1-t)^{y-1}(1+bt)^{-x-y}dt=(1+b)^{-x}B(x,y)$.] Of course, this is not the most general solution, since we started with the constraint (16). Note also that, in general, the Adler condition will hold only for $k_1\to 0$, not for $k_i\to 0$ $(i\neq 1)$. In our model, because of the nonvanishing soft scalar vertex, we have no reason to expect the pole terms to vanish. We have to force it. This is achieved by the Lovelace-type condition (20). Once again we stress that it does not lead to the partially or totally conserved current because it does not say anything about the vertex function. We are not clear

[6] M. Ida, H. Matsumoto, and S. Yazaki, University of Tokyo report, 1970 (unpublished). This work is based on the following contributions: S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 881 (1969); K. Bardakçi and S. Mandelstam, Phys. Rev. **183**, 1456 (1969); **184**, 1640 (1969); D. Amati, M. LeBellac, and D. Olive, Nuovo Cimento **66A**, 315 (1970); **66A**, 831 (1970); C. B. Chiu, S. Matsuda, and C. Rebbi, Phys. Rev. Letters **23**, 1526 (1969); C. B. Thorn, Phys. Rev. D **2**, 1071 (1970); R. C. Brower and J. H. Weis, Nuovo Cimento Letters **3**, 285 (1970); F. Gliozzi, *ibid.* **2**, 846 (1969), and other papers quoted therein. See also K. Kikkawa and H. Sato, Phys. Letters **32B**, 280 (1970).

[7] L. Caneschi, A. Schwimmer, and G. Veneziano, Phys. Letters **30B**, 356 (1969); L. Caneschi and A. Schwimmer, Nuovo Cimento Letters **3**, 213 (1970).

[8] See second paper of Ref. 6. Using the definitions (17b) and (4b), the amplitude can be written in a completely factorized form, $\hat{A}=\langle 0|\cdots\Delta(2i+1)V(2i+1)\Delta(2i)V(2i)\cdots|0\rangle$, with the following notations: $\Delta(j)=[R_a+R_b+R_c+R_d-\alpha(s_j)]^{-1}$, $V(2i)=\gamma_{2i}V_bV_c$, $V(2i+1)=\gamma_{2i+1}V_bV_d$, $V_b=f(b\dagger)|0\rangle_b\ {}_b\langle 0|f(b)$, $V_c=A(c\dagger)|0\rangle_c\ {}_c\langle 0|A(c)$, $V_d=A(d\dagger)|0\rangle_d\ {}_d\langle 0|A(d)$, $f(b)=\sum_n (B_n/n!)^{1/2}b^n$, and $(1-x)^{-\alpha_2(0)-1}=\sum_0^\infty B_nx^n$. It is not difficult to generalize $\hat{A}$ when the intercepts of the trajectories are not constrained by (16).

about the deeper meaning of the Lovelace condition, if there is any.

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K_{l3} Decays and $SU(3)$ Breaking in the Brandt-Preparata Approach

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A discussion of the decay $K^+ \to \pi^0+l+\nu$ is presented, which makes clear that a recent calculation of the ξ parameter in $K^0{}_3$ decay depends upon the introduction of large $SU(3)$ violation and not upon a modification of the hypothesis of partial conservation of the axial-vector current. In addition, a formula is derived giving a means of estimating $\xi(m_K{}^2)$ and its dependence upon $SU(3)$ breaking.

I. INTRODUCTION

In a recent paper,[1] Brandt and Preparata presented a treatment of the decay $K\to\pi+l+\nu$ which yielded a value for the ratio of the two form factors which determine the matrix element of the axial-vector current between single K- and π-meson states (the so-called ξ parameter), consistent with minus one. Since, as they point out, at least some of the experimental evidence points to the fact that ξ could be negative and on the order of unity, there is no *a priori* reason to argue that their treatment of this process is not in fact correct (although there is certainly room for discussion of this point). However, this is not the point to which I would like to address myself in this paper. Rather, I would like to limit myself to presenting a clarification of the physical content of the assumptions implicit in the authors' derivation of this large negative value for ξ. I believe that such a clarification is important, since this prediction for ξ does not come from a modification of the hypothesis of partial conservation of the axial-vector current (PCAC), but rather in the introduction (in an indirect way) of large $SU(3)$ breaking. Since the entire result depends upon this assumption, it seems worthwhile to present a treatment of K_{l3} decays which explicitly separates the assumptions involved in applying the PCAC hypothesis from all other assumptions.

In order to present the arguments leading to this conclusion in the clearest possible way, I shall first state and discuss an exact, on-mass-shell formula, giving the value of ξ at $t=m_K{}^2$. This formula has the advantage that it is written in terms of physically measurable quantities and the so-called "PCAC-correction terms" are explicitly exhibited. Before going on to prove this formula, I will show how it can be used to obtain the result of Brandt and Preparata. The derivation is easily compared with that given by Brandt and Preparata, but has the advantage that at each stage it clearly distinguishes between independent physical assumptions. The formula to be derived has the additional interesting property that it explicitly exhibits the dependence of ξ upon the introduction of small $SU(3)$ breaking.

II. CONVENTIONS

Prior to stating the theorem to be discussed, it is useful to establish the notation to be used throughout. My conventions are as follows.

The symbols $V_\alpha{}^\mu(x)$ and $A_\alpha{}^\mu(x)$ ($\alpha=1,\ldots,8$; $\mu=0,\ldots,3$) denote the octet of vector and axial-vector currents assumed to satisfy the Gell-Mann current algebra,

$$\begin{aligned}
[V_\alpha{}^0(x),V_\beta{}^\mu(y)]_{x^0=y^0}&=if_{\alpha\beta\gamma}V_\gamma{}^\mu(x)\delta^3(x-y)+\text{S.T.}\,,\\
[V_\alpha{}^0(x),A_\beta{}^\mu(y)]_{x^0=y^0}&=if_{\alpha\beta\gamma}A_\gamma{}^\mu(x)\delta^3(x-y)+\text{S.T.}\,,\\
[A_\alpha{}^0(x),A_\beta{}^\mu(y)]_{x^0=y^0}&=if_{\alpha\beta\gamma}V_\gamma{}^\mu(x)\delta^3(x-y)+\text{S.T.}\,,
\end{aligned}\tag{1}$$

where S.T. stands for possible Schwinger terms. I shall also denote by A_{K^+} and A_{π^0} the linear combinations

$$\begin{aligned}
A_{K^+}{}^\mu(x)&=A_4{}^\mu(x)+iA_5{}^\mu(x)\,,\\
A_{\pi^0}{}^\mu(x)&=A_3{}^\mu(x)\,,
\end{aligned}\tag{2}$$

etc.

If one believes that the decay $K^+\to\pi^0+l+\nu$ can be described by the usual theory of weak interactions, then it measures the matrix element

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[1] R. Brandt and G. Preparata, Nuovo Cimento Letters 4, 80 (1970).