General Properties of q-Number Schwinger Terms*

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A systematic study of properties of q-number Schwinger terms in the algebra of currents is carried out on the basis of Lorentz covariance. It is found that these terms can be expressed by means of a second-rank Lorentz tensor. As an application, the existence of a covariant time-ordered product of two currents is proven for a wide class of Schwinger terms.

I. INTRODUCTION

The study of q-number Schwinger terms in the algebra of currents is of considerable interest. Experimentally, its presence may be manifest in sum rules[1] involving cross sections for high-energy electron-proton scattering and electron-positron annihilation reactions. Also, its Lorentz property is very important in constructing Lorentz-covariant time-ordered products of two or more currents, as has been shown by many authors.[2–5]

There are several interesting attempts[5–11] to determine the structure of Schwinger terms. In this paper, we present a systematic discussion of the problem based on rather general (and plausible) assumptions. To make our *Ansatz* in a clear fashion, let $j_\mu{}^a(x)$ $(a=1,2,\ldots,n)$ be a set of n local vector and/or axial-vector currents. The total number (n) of the currents depends upon the specific group under consideration. For example, we have $n=1$ if we are only considering the electromagnetic current, while we get $n=3$, 6, 8, or 16 if we are considering the $SU(2)$, $SW(2)$, $SU(3)$, or $SW(3)$ group, respectively. Moreover, we shall use the standard Pauli notation with $x_4=ix_0$ and $j_4{}^a(x)=ij_0{}^a(x)$ throughout this paper. Also, we shall specify Lorentz indices by Greek subscripts $\mu,\nu,\lambda=1,2,3,4$, while Latin indices $(k,l,s$, etc.) represent their space components with values 1,2,3.

Now we shall state our *Ansätze*.

Ansatz I

This is nothing but the ordinary algebra of currents, i.e.,

$$\delta(x_0-y_0)[j_4{}^a(x),j_4{}^b(y)]=Q_4{}^{ab}(x)\delta^{(4)}(x-y)\,,$$
$$\delta(x_0-y_0)[j_4{}^a(x),j_k{}^b(y)]=Q_k{}^{ab}(x)\delta^{(4)}(x-y)-S_{lk}{}^{ab}(y)\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)\,.$$

Although in many applications we need not assume that $Q_4{}^{ab}(x)$ and $Q_k{}^{ab}(x)$ are components of a single Lorentz vector $Q_\mu{}^{ab}(x)$, it gives an unnecessary complication,[12] and for simplicity we assume hereafter that $Q_\mu{}^{ab}(x)$ is another local vector (and/or axial-vector) current. In the case of the $SU(3)$ group, we have $Q_\mu{}^{ab}(x)=-f_{abc}j_\mu{}^c(x)$. However, we do not assume any specific form for $Q_\mu{}^{ab}(x)$ in this paper.

We may remark that the absence of Schwinger terms with derivatives higher than the second order has been formally proved in accordance with our *Ansatz* by Gross and Jackiw[6] by means of a Jacobi identity among $J_0{}^a(x)$, $J_0{}^b(y)$ and the energy density $\Theta_{00}(z)$. This fact is also consistent with a work by Levin,[10] who reached the same conclusion for electromagnetic currents. However, our *Ansatz* may not be valid in some models, as indicated by recent work of Boulware and Jackiw[11] on anomalous commutators. If the conclusion of these authors is accepted, then we would have Schwinger terms with derivatives up to the third order. In this paper, we shall not consider such complications.

For convenience, let us set

$$j_{\mu\nu}{}^a(x)=\frac{\partial}{\partial x_\mu}j_\nu{}^a(x)-\frac{\partial}{\partial x_\nu}j_\mu{}^a(x)\,,$$
$$D^a(x)=\frac{\partial}{\partial x_\mu}j_\mu{}^a(x)\,. \tag{1.1}$$

We are interested in equal-time commutators among $j_\mu{}^a(x)$, $j_{\mu\nu}{}^a(x)$, and $D^a(x)$ and we shall assume *Ansatz* II.

Ansatz II

Let $A(x)$ and $B(x)$ to be any two of $j_\mu{}^a(x)$, $j_{\mu\nu}{}^a(x)$, and $D^a(x)$. Then the equal-time commutator between

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[1] See, e.g., J. M. Cornwall, D. Corrigan, and R. E. Norton, Phys. Rev. Letters 24, 1141 (1970); J. Pestieau and H. Terazawa, *ibid.* 24, 1149 (1970).
[2] R. F. Dashen and S. Y. Lee, Phys. Rev. 187, 2017 (1969).
[3] T. C. Yang, Phys. Rev. D2, 2312 (1970).
[4] M. Tonin, Nuovo Cimento 47, 919 (1967).
[5] D. J. Gross and R. Jackiw, Nucl. Phys. B14, 269 (1969).
[6] D. J. Gross and R. Jackiw, Phys. Rev. 163, 1688 (1967).
[7] T. K. Kuo and M. Sugawara, Phys. Rev. 163, 1716 (1967).
[8] R. Jackiw, Phys. Letters 27B, 96 (1968); 27B, 394 (1968).
[9] R. Jackiw, Phys. Rev. 175, 2058 (1968).
[10] D. Levin, Phys.Rev. (to be published).
[11] D. Boulware and R. Jackiw, Phys. Rev. 186, 1442 (1969).

[12] Some consequences without this assumption are given by Kuo and Sugawara (Ref. 7). Also, when the electromagnetic or weak interaction is taken into account, this assumption will not be valid, as has been noted by Jackiw (Ref. 9). Hence, in the present paper, we are implicitly assuming the absence of these interactions.

$A(x)$ and $B(y)$ can be expressed as a finite sum of $\delta^{(3)}(x-y)$ and its space derivatives:

$$\delta(x_0-y_0)[A(x),B(y)] = \sum_{m=0}^{N} \sum_{k_1,\ldots,k_m=1}^{3} \varphi_{k_1,\ldots,k_m}(y) \frac{\partial^m}{\partial x_{k_1}\cdots\partial x_{k_m}} \delta^{(4)}(x-y).$$

We define the order of the commutator to be the integer associated with the highest derivatives of the δ function. Hence the order of the commutator $[A(x),B(y)]$ in the above example is N, that of $[j_4{}^a(x),j_4{}^b(y)]$ is 0, and that of $[j_4{}^a(x),j_k{}^a(y)]$ is 1, since the Schwinger term $S_{lk}{}^{aa}(x)$ (no summation over a) is known to be nonzero.[13,14]

Ansatz III (Lorentz Invariance)

$$[K_s,j_\mu{}^a(x)]=\delta_{\mu 4}j_s{}^a(x)-\delta_{\mu s}j_4{}^a(x)+L_{s4}{}^{(x)}j_\mu{}^a(x),$$
$$[K_s,Q_\mu{}^a(x)]=\delta_{\mu 4}Q_s{}^a(x)-\delta_{\mu s}Q_4{}^a(x)+L_{s4}{}^{(x)}Q_\mu{}^a(x),$$

where K_s is the Lorentz-boost operator in the sth spatial direction and $L_{s4}{}^{(x)}$ is the differential operator

$$L_{s4}{}^{(x)}=x_4\frac{\partial}{\partial x_s}-x_s\frac{\partial}{\partial x_4}.$$

Notice that we do not assume any specific Lorentz-transformation property for the Schwinger term $S_{lk}{}^{ab}(x)$ except for the fact that it must be a second-rank tensor with respect to the spatial rotation subgroup, O_3.

Ansatz IV

$$[K_s,\delta(x_0-y_0)[A(x),B(y)]] = \delta(x_0-y_0)[[K_s,A(x)],B(y)] + \delta(x_0-y_0)[A(x),[K_s,B(y)]],$$
$$[K_s,[K_t,A(x)]]-[K_t,[K_s,A(x)]]=[[K_s,K_t],A(x)].$$

These are just Jacobi identities. Note that we do *not* assume a Jacobi identity among three density operators [$A(x)$, $B(y)$, and $C(z)$], since that may lead to a contradiction.[15] Also, we notice that $[K_s,K_t]$ is a pure spatial rotation operator; therefore, $[[K_s,K_t],A(x)]$ is calculable from the rotation property of $A(x)$ without any knowledge of its Lorentz transformation property.

Then from *Ansätze* I–IV, we can prove first that the order of the commutator $\delta(x_0-y_0)[D^a(x),J_4{}^b(y)]$ is at most 2, with the form

$$\delta(x_0-y_0)[D^a(x),j_4{}^b(y)] =\sigma^{ab}(x)\delta^{(4)}(x-y)+\Sigma_{lk}{}^{ab}(y)\frac{\partial^2}{\partial x_l\partial x_k}\delta^{(4)}(x-y). \quad (1.2)$$

Second, we have the identities

$$\Sigma_{lk}{}^{ab}(x)=\Sigma_{kl}{}^{ab}(x)=-\Sigma_{lk}{}^{ba}(x), \quad (1.3)$$

$$2\Sigma_{lk}{}^{ab}(x)=S_{lk}{}^{ab}(x)-S_{kl}{}^{ba}(x), \quad (1.4)$$

$$S_{lk}{}^{ab}(x)-S_{kl}{}^{ba}(x)=S_{kl}{}^{ab}(x)-S_{lk}{}^{ba}(x), \quad (1.5)$$

$$\sigma^{ba}(x)-\sigma^{ab}(x)=\frac{\partial}{\partial x_\mu}Q_\mu{}^{ab}(x)-\frac{\partial^2}{\partial x_l\partial x_k}\Sigma_{lk}{}^{ab}(x), \quad (1.6)$$

$$Q_\mu{}^{ab}(x)=-Q_\mu{}^{ba}(x). \quad (1.7)$$

These identities reduce to well-known results[5–7] when we set $\Sigma_{lk}{}^{ab}(x)=0$. To derive more interesting results, next we assume *Ansatz* V.

Ansatz V

The order of the commutator $\delta(x_0-y_0)[D^a(x),D^b(y)]$ is equal to or less than 2; i.e., we can wirte

$$\delta(x_0-y_0)[D^a(x),D^b(y)] =F^{ab}(x)\delta^{(4)}(x-y)+F_k{}^{ab}(x)\frac{\partial}{\partial y_k}\delta^{(4)}(x-y) +F_{kl}{}^{ab}(x)\frac{\partial^2}{\partial y_k\partial y_l}\delta^{(4)}(x-y). \quad (1.8)$$

Then, of course, we must have

$$F_{kl}{}^{ab}(x)=F_{lk}{}^{ab}(x)=-F_{kl}{}^{ba}(x),$$
$$F_k{}^{ab}(x)-F_k{}^{ba}(x)=-2\frac{\partial}{\partial x_l}F_{kl}{}^{ab}(x), \quad (1.9)$$
$$F^{ab}(x)+F^{ba}(x)=-\frac{1}{2}\frac{\partial}{\partial x_k}[F_k{}^{ab}(x)+F_k{}^{ba}(x)].$$

When we assume *Ansatz* V in addition, we can prove that the orders of the commutators $\delta(x_0-y_0)[j_l{}^a(x),j_k{}^b(y)]$ and $\delta(x_0-y_0)[D^a(x),j_k{}^b(y)]$ are at most 1 and 2, respectively, and that we can write

$$\delta(x_0-y_0)[j_l{}^a(x),j_k{}^b(y)] =f_{lk}{}^{ab}(x)\delta^{(4)}(x-y)-f_{slk}{}^{ab}(y)\frac{\partial}{\partial x_s}\delta^{(4)}(x-y), \quad (1.10)$$

$$\delta(x_0-y_0)[D^a(x),j_k{}^b(y)]=\sigma_k{}^{ab}(x)\delta^{(4)}(x-y) +G_{slk}{}^{ab}(y)\frac{\partial^2}{\partial x_l\partial x_s}\delta^{(4)}(x-y). \quad (1.11)$$

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[13] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959); T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955); K. Johnson, Nucl. Phys. **25**, 431 (1961).

[14] G. Pócsik, Nuovo Cimento **43A**, 541 (1966); S. Okubo, *ibid.* **44A**, 1015 (1966).

[15] K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37-38**, 74 (1966); G. Konishi and K. Yamamoto, *ibid.* **37**, 1314 (1967). We remark that, if the Jacobi identity among spatial components of $j_l{}^a(x)$ is valid, then we must have q-number Schwinger terms for the quark model. See F. Buccella, G. Veneziano, R. Gatto, and S. Okubo, Phys. Rev. **149**, 1268 (1966).

Furthermore, one has the relation

$$G_{slk}{}^{ab}(x)=G_{lsk}{}^{ab}(x)=-G_{slk}{}^{ba}(x)\,, \tag{1.12}$$

$$f_{slk}{}^{ab}(x)=f_{skl}{}^{ba}(x)\,, \tag{1.13}$$

$$\frac{\partial}{\partial x_s}f_{slk}{}^{ab}(x)=f_{lk}{}^{ab}(x)+f_{kl}{}^{ba}(x)\,, \tag{1.14}$$

$$[K_s,S_{lk}{}^{ab}(x)]-L_{s4}{}^{(x)}S_{lk}{}^{ab}(x)=-f_{slk}{}^{ab}(x)-2G_{slk}{}^{ab}(x)\,, \tag{1.15}$$

$$F_{sl}{}^{ab}(x)=\frac{\partial}{\partial x_k}G_{slk}{}^{ab}(x)+\frac{\partial}{\partial x_4}\Sigma_{sl}{}^{ab}(x)\,, \tag{1.16}$$

$$[K_s,\sigma^{ab}(x)]-L_{s4}{}^{(x)}\sigma^{ab}(x)=F_s{}^{ba}(x)\,. \tag{1.17}$$

Ansatz VI

The order of the commutator $\delta(x_0-y_0)[j_{4k}{}^a(x),D^b(y)]$ is at most 2; i.e., we can write

$$\delta(x_0-y_0)[j_{4k}{}^a(x),D^b(y)]=E_k{}^{ab}(x)\delta^{(4)}(x-y)+E_{lk}{}^{ab}(x)\frac{\partial}{\partial y_l}\delta^{(4)}(x-y)+E_{slk}{}^{ab}(x)\frac{\partial^2}{\partial y_l\partial y_s}\delta^{(4)}(x-y)\,. \tag{1.18}$$

When we assume *Ansatz* VI in addition, we can prove the existence of a symmetric, traceless Lorentz tensor $H_{\mu\nu}{}^{ab}(x)$ $(\mu,\nu=1,2,3,4)$, satisfying

$$H_{\mu\mu}{}^{ab}(x)=0\,,\quad H_{\mu\nu}{}^{ab}(x)=H_{\nu\mu}{}^{ab}(x)=-H_{\mu\nu}{}^{ba}(x)\,, \tag{1.19}$$

$$[K_s,H_{\mu\nu}{}^{ab}(x)]-L_{s4}{}^{(x)}H_{\mu\nu}{}^{ab}(x)=\delta_{\mu4}H_{s\nu}{}^{ab}(x)+\delta_{\nu4}H_{\mu s}{}^{ab}(x)-\delta_{\mu s}H_{4\nu}{}^{ab}(x)-\delta_{\nu s}H_{\mu4}{}^{ab}(x)\,. \tag{1.20}$$

Also, we can express $G_{slk}{}^{ab}(x)$, $\Sigma_{sl}{}^{ab}(x)$, and $E_{slk}{}^{ab}(x)$ as

$$\begin{aligned}G_{slk}{}^{ab}(x)&=\delta_{sk}H_{4l}{}^{ab}(x)+\delta_{lk}H_{4s}{}^{ab}(x)\,,\\ \Sigma_{sl}{}^{ab}(x)&=H_{sl}{}^{ab}(x)-\delta_{sl}H_{44}{}^{ab}(x)\,,\\ E_{slk}{}^{ab}(x)&=\frac{\partial}{\partial x_4}G_{slk}{}^{ab}(x)-\frac{\partial}{\partial x_k}\Sigma_{ls}{}^{ab}(x)\,,\end{aligned} \tag{1.21}$$

and $E_{lk}{}^{ab}(x)$ is given by Eq. (2.28) in terms of $\sigma_k{}^{ab}(x)$. Moreover, we can prove that the orders of $\delta(x_0-y_0)\times[j_l{}^a(x),j_{4k}{}^b(y)]$ and $\delta(x_0-y_0)[j_4{}^a(x),j_{4k}{}^b(y)]$ are at most 2. Hence we can write

$$\delta(x_0-y_0)[j_l{}^a(x),j_{4k}{}^b(y)]=X_{lk}{}^{ab}(y)\delta^{(4)}(x-y)+X_{slk}{}^{ab}(y)\frac{\partial}{\partial x_s}\delta^{(4)}(x-y)+X_{mslk}{}^{ab}(y)\frac{\partial^2}{\partial x_m\partial x_s}\delta^{(4)}(x-y)\,. \tag{1.22}$$

We can compute $\delta(x_0-y_0)[j_4{}^a(x),j_{4k}{}^b(y)]$ in terms of other quantities [see Eq. (2.51)] without introducing new ones.

Ansatz VII

The order of the commutator $\delta(x_0-y_0)[j_{4l}{}^a(x),j_{4k}{}^b(y)]$ is at most 2; i.e., one can write

$$\delta(x_0-y_0)[j_{4l}{}^a(x),j_{4k}{}^b(y)]=Y_{lk}{}^{ab}(x)\delta^{(4)}(x-y)+Y_{slk}{}^{ab}(y)\frac{\partial}{\partial x_s}\delta^{(4)}(x-y)+Y_{smlk}{}^{ab}(y)\frac{\partial^2}{\partial x_s\partial x_m}\delta^{(4)}(x-y)\,. \tag{1.23}$$

If we assume *Ansatz* VII in addition,[16] we must have a second-rank Lorentz tensor $R_{\mu\nu}{}^{ab}(x)$ satisfying conditions

$$\begin{aligned}R_{\mu\mu}{}^{ab}(x)&=R_{\mu\mu}{}^{ba}(x)\,,\\ R_{\mu\nu}{}^{ab}(x)+R_{\mu\nu}{}^{ba}(x)&=R_{\nu\mu}{}^{ab}(x)+R_{\nu\mu}{}^{ba}(x)\,,\\ 2H_{\mu\nu}{}^{ab}(x)&=R_{\mu\nu}{}^{ab}(x)-R_{\nu\mu}{}^{ba}(x)\,,\end{aligned} \tag{1.24}$$

and in terms of this tensor, we can express $S_{lk}{}^{ab}(x)$, $f_{slk}{}^{ab}(x)$, and $X_{mslk}{}^{ab}(x)$ as

$$\begin{aligned}S_{lk}{}^{ab}(x)&=R_{lk}{}^{ab}(x)-\delta_{lk}H_{44}{}^{ab}(x)\,,\\ f_{slk}{}^{ab}(x)&=\delta_{sl}R_{4k}{}^{ab}(x)+\delta_{sk}R_{4l}{}^{ba}(x)\,,\\ 2X_{mslk}{}^{ab}(x)&=\delta_{ml}R_{sk}{}^{ab}(x)+\delta_{sl}R_{mk}{}^{ab}(x)-(\delta_{ml}\delta_{sk}+\delta_{sl}\delta_{mk})R_{44}{}^{ab}(x)\,.\end{aligned} \tag{1.25}$$

Finally, we can prove that if we demand that the commutators $\delta(x_0-y_0)[j_{4l}{}^a(x),j_{4k}{}^b(y)]$ and $\delta(x_0-y_0)\times[j_l{}^a(x),j_{4k}{}^b(y)]$ have orders at most 1, then we must have $R_{\mu\nu}{}^{ab}(x)=\delta_{\mu\nu}R^{ab}(x)$. This implies $f_{slk}{}^{ab}(x)=f_{lk}{}^{ab}(x)=0$; i.e., the commutation relation $[j_\mu{}^a(x),j_\nu{}^b(y)]$ must satisfy that of the field algebra. Therefore, if we demand the validity of the nonzero space-space current commutators, then we conclude that at least some of the commutators which we studied must have the order 2, i.e., nonzero second-order derivatives of the δ function.

We remark that for the $SU(2)$ case we have $\partial_\mu j_\mu{}^a(x)=D^a(x)=0$. Hence the *Ansätze* V and VI are automatically satisfied with $H_{\mu\nu}{}^{ab}(x)=0$. In general, without assuming *Ansatz* VII, we can show that, if one of $\Sigma_{lk}{}^{ab}(x)$, $G_{lks}{}^{ab}$, and $F_{lk}{}^{ab}(x)$ is zero, then all of them must be zero with $H_{\mu\nu}{}^{ab}(x)=0$. As we see from Eq. (1.4), in this case we have the familiar formula $S_{lk}{}^{ab}(x)=S_{kl}{}^{ba}(x)$. The condition that $\sigma^{ab}(x)$ be a Lorentz scalar is that $F_s{}^{ab}(x)=0$, as we see from Eq. (1.17).

If the Schwinger term $S_{lk}{}^{ab}(x)$ is a c number independent of the coordinate x, then it must be propor-

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[16] Possibly, the validity of *Ansatz* VII is most debatable in comparison to the other postulates. Unfortunately, it is rather difficult to give a *raison d'être* for its validity, except for the fact that it is necessary to derive some more interesting results and that it is satisfied in the field algebra.

tional to δ_{lk}, which leads to $H_{\mu\nu}{}^{ab}(x)=0$. In addition, if we assume *Ansatz* VII, then $R_{\mu\nu}{}^{ab}$ is proportional to $\delta_{\mu\nu}$; then Eq. (1.25) gives the result $f_{slk}{}^{ab}(x)=0$.

II. DERIVATION OF MAIN RESULTS

We base our calculation on the following identity[17]:

$$[K_s,\delta(x_0-y_0)[A(x),B(y)]]$$
$$=\left\{L_{s4}{}^{(x)}+L_{s4}{}^{(y)}+(x_s-y_s)\frac{\partial}{\partial x_4}\right\}\delta(x_0-y_0)[A(x),B(y)]$$
$$-(x_s-y_s)\delta(x_0-y_0)\left[\frac{\partial}{\partial x_4}A(x),B(y)\right]$$
$$+\delta(x_0-y_0)[\Delta_sA(x),B(y)]$$
$$+\delta(x_0-y_0)[A(x),\Delta_sB(y)],\quad(2.1)$$

where $\Delta_sA(x)$ and $\Delta_sB(y)$ are defined by

$$\Delta_sA(x)=[K_s,A(x)]-L_{s4}{}^{(x)}A(x),$$
$$\Delta_sB(y)=[K_s,B(y)]-L_{s4}{}^{(y)}B(y).\quad(2.2)$$

In the derivation of Eq. (2.1), we used the Jacobi identity, *Ansatz* IV.

We also notice the following relation:

$$\left\{L_{s4}{}^{(x)}+L_{s4}{}^{(y)}+(x_s-y_s)\frac{\partial}{\partial x_4}\right\}\varphi(y)\frac{\partial}{\partial x_{k_1}}\cdots\frac{\partial}{\partial x_{k_m}}\delta^{(4)}(x-y)$$
$$=\{L_{s4}{}^{(y)}\varphi(y)\}\frac{\partial}{\partial x_{k_1}}\cdots\frac{\partial}{\partial x_{k_m}}\delta^{(4)}(x-y),\quad(2.3)$$

where $\varphi(y)$ is an arbitrary function of y (but not x). Equations (2.1)–(2.3) are our starting point.

For a while, we shall assume only *Ansätze* I–IV. First, let us set $A(x)=j_4{}^a(x)$ and $B(y)=j_4{}^b(y)$ in Eq. (2.1). Using *Ansätze* I and III, we then obtain

$$(x_s-y_s)\delta(x_0-y_0)[D^a(x),j_4{}^b(y)]$$
$$=\{S_{sk}{}^{ba}(y)-S_{ks}{}^{ab}(y)\}\frac{\partial}{\partial x_k}\delta^{(4)}(x-y).\quad(2.4)$$

From this together with the *Ansatz* II, we conclude that $\delta(x_0-y_0)[D^a(x),j_4{}^b(y)]$ cannot contain derivatives of $\delta^{(4)}(x-y)$ higher than second order. In other words, the order of the commutator $[D^a(x),j_4{}^b(y)]$ is at most 2, and one can set

$$\delta(x_0-y_0)[D^a(x),j_4{}^b(y)]$$
$$=\sigma^{ab}(x)\delta^{(4)}(x-y)+\Sigma_l{}^{ab}(y)\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)$$
$$+\Sigma_{lk}{}^{ab}(y)\frac{\partial^2}{\partial x_l\partial x_k}\delta^{(4)}(x-y).\quad(2.5)$$

Without loss of generality, one can assume

$$\Sigma_{lk}{}^{ab}(y)=\Sigma_{kl}{}^{ab}(y).\quad(2.6)$$

Inserting Eq. (2.5) into Eq. (2.4), one obtains

$$\Sigma_l{}^{ab}(y)=0,\quad(2.7)$$
$$2\Sigma_{lk}{}^{ab}(y)=S_{lk}{}^{ab}(y)-S_{kl}{}^{ba}(y).\quad(2.8)$$

From Eqs. (2.6) and (2.8), we find also

$$\Sigma_{lk}{}^{ab}(y)=-\Sigma_{lk}{}^{ba}(y),\quad(2.9)$$
$$S_{lk}{}^{ab}(y)-S_{kl}{}^{ba}(y)=S_{kl}{}^{ab}(y)-S_{lk}{}^{ba}(y).\quad(2.10)$$

These are nothing but Eqs. (1.2)–(1.5). To prove Eq. (1.7), we notice that we must have

$$Q_4{}^{ab}(x)=-Q_4{}^{ba}(x)$$

because of *Ansatz* I. Then, the Lorentz covariance of $Q_\mu{}^{ab}(x)$ (see *Ansatz* III) gives the desired relation,

$$Q_\mu{}^{ab}(x)=-Q_\mu{}^{ba}(x).\quad(2.11)$$

In order to derive Eq. (1.6), we note the following identity:

$$\delta(x_0-y_0)[D^a(x),j_4{}^b(y)]+\delta(x_0-y_0)[j_4{}^a(x),D^b(y)]$$
$$=\left(\frac{\partial}{\partial x_4}+\frac{\partial}{\partial y_4}\right)\delta(x_0-y_0)[j_4{}^a(x),j_4{}^b(y)]$$
$$+\frac{\partial}{\partial x_k}\delta(x_0-y_0)[j_k{}^a(x),j_4{}^b(y)]$$
$$+\frac{\partial}{\partial y_k}\delta(x_0-y_0)[j_4{}^a(x),j_k{}^b(y)].\quad(2.12)$$

Together with *Ansatz* I, this yields

$$\sigma^{ab}(x)-\sigma^{ba}(x)-\frac{\partial}{\partial x_l}\Sigma_l{}^{ba}(x)+\frac{\partial^2}{\partial x_l\partial x_k}\Sigma_{lk}{}^{ab}(x)$$
$$=\frac{\partial}{\partial x_\mu}Q_\mu{}^{ab}(x),\quad(2.13)$$
$$\Sigma_l{}^{ab}(x)+\Sigma_l{}^{ba}(x)+2\frac{\partial}{\partial x_k}\Sigma_{lk}{}^{ab}(x)$$
$$=\frac{\partial}{\partial x_k}[S_{kl}{}^{ab}(x)-S_{lk}{}^{ba}(x)],\quad(2.14)$$
$$2[\Sigma_{lk}{}^{ab}(x)-\Sigma_{lk}{}^{ba}(x)]$$
$$=S_{lk}{}^{ab}(x)-S_{lk}{}^{ba}(x)+S_{kl}{}^{ab}(x)-S_{kl}{}^{ba}(x),\quad(2.15)$$

where we have used Eqs. (2.5) and (2.11). In view of Eqs. (2.6)–(2.8), one can easily check that Eqs. (2.14) and (2.15) are automatically satisfied while Eq. (2.13) gives the desired relation Eq. (1.6). We may remark

[17] The Jacobi identity with the boost operator K_s has been used by many authors. See Refs. 2, 3, 6, and 7 as well as E. Kazes, Phys. Rev. **157**, 1309 (1967).

that Eqs. (2.13)–(2.15) also follow from the identity

$$\frac{\partial^2}{\partial x_\mu \partial y_\nu} T(j_\mu{}^a(x) j_\nu{}^b(y)) = \frac{\partial^2}{\partial y_\nu \partial x_\mu} T(j_\mu{}^a(x) j_\nu{}^b(y)).$$

Next, setting $A(x)=j_4{}^a(x)$ and $B(y)=D^b(y)$ in Eq. (2.1) and using *Ansatz* III, we find

$$\begin{aligned}\frac{\partial}{\partial x_k}\{(x_s-y_s)\delta(x_0-y_0)[j_k{}^a(x),D^b(y)]\}\\ =-\{[K_s,\sigma^{ba}(x)]-L_{s4}{}^{(x)}\sigma^{ba}(x)\}\delta^{(4)}(x-y)\\ -\{[K_s,\Sigma_{lk}{}^{ba}(x)]-L_{s4}{}^{(x)}\Sigma_{lk}{}^{ba}(x)\}\frac{\partial^2}{\partial x_l\partial x_k}\delta^{(4)}(x-y)\\ -2\frac{\partial}{\partial x_4}\Sigma_{sl}{}^{ba}(x)\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)\\ +(x_s-y_s)\delta(x_0-y_0)[D^a(x),D^b(y)]. \quad (2.16)\end{aligned}$$

Therefore, if the order of $\delta(x_0-y_0)[D^a(x),D^b(y)]$ is at most 3, then the right-hand side of Eq. (2.16) contains at most the second-order derivatives of $\delta^{(4)}(x-y)$. Thus, following the argument of Dashen and Lee,[2] one concludes that the order of $\delta(x_0-y_0)[j_k{}^a(x),D^b(y)]$ is at most 2. Hence one can write

$$\begin{aligned}\delta(x_0-y_0)[D^a(x),j_k{}^b(y)]\\ =\sigma_k{}^{ab}(x)\delta^{(4)}(x-y)+G_{lk}{}^{ab}(y)\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)\\ +G_{lsk}{}^{ab}(y)\frac{\partial^2}{\partial x_l\partial x_s}\delta^{(4)}(x-y), \quad (2.17)\end{aligned}$$

where without loss of generality one can assume

$$G_{lsk}{}^{ab}(y)=G_{slk}{}^{ab}(y). \quad (2.18)$$

Moreover, if we assume *Ansatz* V, i.e., if we can write the commutator $\delta(x_0-y_0)[D^a(x),D^b(y)]$ as in Eq. (1.8), then Eq. (2.16) gives us

$$G_{slk}{}^{ba}(x)+G_{skl}{}^{ba}(x) =-[K_s,\Sigma_{lk}{}^{ba}(x)]+L_{s4}{}^{(x)}\Sigma_{lk}{}^{ba}(x), \quad (2.19)$$

$$2\frac{\partial}{\partial x_k}G_{lsk}{}^{ba}(x)-G_{sl}{}^{ba}(x) =-2\frac{\partial}{\partial x_4}\Sigma_{sl}{}^{ba}(x)+2F_{sl}{}^{ba}(x), \quad (2.20)$$

$$-\frac{\partial}{\partial x_k}G_{sk}{}^{ba}(x) =-[K_s,\sigma^{ba}(x)]+L_{s4}{}^{(x)}\sigma^{ba}(x)+F_s{}^{ab}(x). \quad (2.21)$$

Similarly, when we set $A(x)=j_4{}^a(x)$ and $B(y)=j_k{}^b(y)$ in Eq. (2.1), we get

$$\begin{aligned}\frac{\partial}{\partial x_l}\{(x_s-y_s)\delta(x_0-y_0)[j_l{}^a(x),j_k{}^b(y)]\}\\ =-\{[K_s,S_{lk}{}^{ab}(y)]-L_{s4}{}^{(y)}S_{lk}{}^{ab}(y)\}\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)\\ +(x_s-y_s)\delta(x_0-y_0)[D^a(x),j_k{}^b(y)]. \quad (2.22)\end{aligned}$$

Inserting Eq. (2.17) into this equation, we find that the order of $\delta(x_0-y_0)[j_l{}^a(x),j_k{}^b(y)]$ must be at most 1. We may remark that Eq. (2.22) reduces to Eq. (2.2) of Dashen and Lee[2] when we set $y=0$ and assume the conservation law $D^a(x)=\partial_\mu j_\mu{}^a(x)=0$. At any rate, inserting Eqs. (1.10) and (2.17) into Eq. (2.22), we derive

$$G_{lk}{}^{ab}(y)=0, \quad (2.23)$$

$$[K_s,S_{lk}{}^{ab}(y)]-L_{s4}{}^{(y)}S_{lk}{}^{ab}(y)=-f_{slk}{}^{ab}(y) -2G_{slk}{}^{ab}(y). \quad (2.24)$$

This establishes Eqs. (1.11) and (1.15). Notice that Eqs. (1.13) and (1.14) follow directly from the definition Eq. (1.10). Again, Eq. (2.24) reduces to that of Dashen and Lee if $G_{slk}{}^{ab}(y)=0$.

Interchanging (a,b) and (l,k) in Eq. (2.24) and subtracting, we find

$$[K_s,\Sigma_{lk}{}^{ab}(y)]-L_{s4}{}^{(y)}\Sigma_{lk}{}^{ab}(y)=G_{skl}{}^{ba}(y)-G_{slk}{}^{ab}(y),$$

where we used Eqs. (1.4) and (1.13). Comparing this with Eq. (2.19), we get

$$G_{skl}{}^{ab}(y)=-G_{skl}{}^{ba}(y). \quad (2.25)$$

This result, together with Eq. (2.18), establishes the validity of Eq. (1.12). Equations (1.16) and (1.17) are nothing but Eqs. (2.20) and (2.21) because of $G_{lk}{}^{ab}(y)=0$ [see Eq. (2.23)].

Before going into further detail, we remark that $\Sigma_{lk}{}^{ab}(y)=0$ leads to $G_{slk}{}^{ab}(y)=0$. This is because, if we have $\Sigma_{lk}{}^{ab}(x)=0$, then Eq. (2.19) gives us $G_{slk}{}^{ba}(x)=-G_{skl}{}^{ba}(x)$. Then, repeated use of this equation, together with $G_{lsk}{}^{ab}=G_{slk}{}^{ab}(x)$ [Eq. (2.18)], leads to $G_{slk}{}^{ab}(x)=0$, since

$$\begin{aligned}G_{slk}{}^{ab}=-G_{skl}{}^{ab}=-G_{ksl}{}^{ab}=G_{kls}{}^{ab}\\ =G_{lks}{}^{ab}=-G_{lsk}{}^{ab}=-G_{slk}{}^{ab}.\end{aligned}$$

Next, setting $A(x)=j_k{}^a(x)$ and $B(y)=D^b(y)$ in Eq. (2.1), we compute

$$\{[K_s,\sigma_k{}^{ba}(x)]-L_{s4}{}^{(x)}\sigma_k{}^{ba}(x)\}\delta^{(4)}(x-y)$$
$$+\{[K_s,G_{lmk}{}^{ba}(x)]-L_{s4}{}^{(x)}G_{lmk}{}^{ba}(x)\}\frac{\partial^2}{\partial x_l\partial x_m}\delta^{(4)}(x-y)$$
$$=-2\frac{\partial}{\partial x_4}G_{slk}{}^{ba}(x)\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)$$
$$+(x_s-y_s)\delta(x_0-y_0)[j_{4k}{}^a(x),D^b(y)]$$
$$+\frac{\partial}{\partial x_k}\{(x_s-y_s)\delta(x_0-y_0)[j_4{}^a(x),D^b(y)]\}$$
$$=-2\frac{\partial}{\partial x_4}G_{slk}{}^{ba}(x)\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)$$
$$+2\left\{\frac{\partial}{\partial x_k}\Sigma_{sl}{}^{ba}(x)\frac{\partial\delta^{(4)}(x-y)}{\partial x_l}+\Sigma_{ls}{}^{ba}(x)\frac{\partial^2\delta^{(4)}(x-y)}{\partial x_l\partial x_k}\right\}$$
$$+(x_s-y_s)\delta(x_0-y_0)[j_{4k}{}^a(x),D^b(y)]. \quad (2.26)$$

This proves that the order of $\delta(x_0-y_0)[j_{4k}{}^a(x),D^b(y)]$ is at most 3. If we assume *Ansatz* VI, i.e., that its order is at most 2, then Eq. (2.26) gives

$$[K_s,G_{lmk}{}^{ba}(x)]-L_{s4}{}^{(x)}G_{lmk}{}^{ba}(x)$$
$$=\delta_{lk}\Sigma_{sm}{}^{ba}(x)+\delta_{mk}\Sigma_{ls}{}^{ba}(x)\,, \quad (2.27)$$
$$[K_s,\sigma_k{}^{ba}(x)]-L_{s4}{}^{(x)}\sigma_k{}^{ba}(x)=+E_{sk}{}^{ab}(x)\,, \quad (2.28)$$
$$E_{slk}{}^{ab}(x)=-\frac{\partial}{\partial x_4}G_{slk}{}^{ba}(x)+\frac{\partial}{\partial x_k}\Sigma_{ls}{}^{ba}(x)\,, \quad (2.29)$$

where we have assumed, without loss of generality,

$$E_{slk}{}^{ab}(x)=E_{lsk}{}^{ab}(x)\,. \quad (2.30)$$

Now, notice that because of Eqs. (2.19) and (2.27), a set of $G_{lmk}{}^{ab}(x)$ and $\Sigma_{lk}{}^{ab}(x)$ is closed under commutation with the boost operator K_s. This implies that $G_{lmk}{}^{ab}(x)$ and $\Sigma_{lk}{}^{ab}(x)$ form an invariant subspace under the Lorentz group. To determine their Lorentz-tensor property, first let us set

$$G_{lkk}{}^{ba}(x)\equiv 4G_l{}^{ba}(x)\,. \quad (2.31)$$

Then Eq. (2.27) gives us

$$[K_s,G_l{}^{ba}(x)]-L_{s4}{}^{(x)}G_l{}^{ba}(x)=\Sigma_{ls}{}^{ba}(x)\,. \quad (2.32)$$

When we set

$$\tilde{G}_{lmk}{}^{ba}(x)=G_{lmk}{}^{ba}(x)-\delta_{lk}G_m{}^{ba}(x)-\delta_{mk}G_l{}^{ba}(x)\,, \quad (2.33)$$

Eqs. (2.27) and (2.32) lead to

$$[K_s,\tilde{G}_{lmk}{}^{ba}(x)]-L_{s4}{}^{(x)}\tilde{G}_{lmk}{}^{ba}(x)=0\,. \quad (2.34)$$

This implies that $\tilde{G}_{lmk}{}^{ba}(x)$ is a Lorentz scalar. However, $\tilde{G}_{lmk}{}^{ba}(x)$ is obviously a third-rank tensor with respect to the spatial rotation subgroup O_3. Hence the only possibility is that we have

$$\tilde{G}_{lmk}{}^{ba}(x)=\epsilon_{lmk}\tilde{G}^{ba}(x)\,,$$

where $\tilde{G}^{ba}(x)$ is a Lorentz scalar. But $\tilde{G}_{lmk}{}^{ba}(x)$ is symmetric with respect to the interchange of l and m, as we see from Eqs. (2.18) and (2.33). On the other hand, ϵ_{lmk} is antisymmetric with respect to l and m. Therefore, we conclude that we have $\tilde{G}_{lmk}{}^{ba}(x)=0$ or

$$G_{lmk}{}^{ba}(x)=\delta_{lk}G_m{}^{ba}(x)+\delta_{mk}G_l{}^{ba}(x)\,. \quad (2.35)$$

If we do not use this Lorentz-invariance argument then in order to arrive at the same conclusion we have to make repeated applications of the Jacobi identity among K_s, K_t, and $\tilde{G}_{lmk}{}^{ba}(x)$ and note that $[K_s,K_t]$ is a purely spatial rotation operator.

Since $G_{mkl}{}^{ba}(x)=-G_{mkl}{}^{ab}(x)$, we find

$$G_m{}^{ab}(x)=-G_m{}^{ba}(x)\,. \quad (2.36)$$

Also, Eq. (2.19) is now written as

$$[K_s,\Sigma_{lk}{}^{ba}(x)]-L_{s4}{}^{(x)}\Sigma_{lk}{}^{ba}(x)$$
$$=-\delta_{sk}G_l{}^{ba}(x)-\delta_{sl}G_k{}^{ba}(x)-2\delta_{kl}G_s{}^{ba}(x)\,. \quad (2.37)$$

Setting

$$\tilde{\Sigma}_{lk}{}^{ba}(x)=\Sigma_{lk}{}^{ba}(x)-\tfrac{1}{4}\delta_{lk}\Sigma_{ss}{}^{ba}(x)\,, \quad (2.38)$$

we find

$$[K_s,\tilde{\Sigma}_{lk}{}^{ba}(x)]-L_{s4}{}^{(x)}\tilde{\Sigma}_{lk}{}^{ba}(x)$$
$$=-\delta_{sk}G_l{}^{ba}(x)-\delta_{sl}G_k{}^{ba}(x) \quad (2.39)$$

as well as

$$[K_s,\Sigma_{mm}{}^{ba}(x)]-L_{s4}{}^{(x)}\Sigma_{mm}{}^{ba}(x)=-8G_s{}^{ba}(x)\,. \quad (2.40)$$

If we set

$$H_{lk}{}^{ba}(x)=\tilde{\Sigma}_{lk}{}^{ba}(x)\,,$$
$$H_{4l}{}^{ba}(x)=H_{l4}{}^{ba}(x)=G_l{}^{ba}(x)\,, \quad (2.41)$$
$$H_{44}{}^{ba}(x)=-\tfrac{1}{4}\Sigma_{mm}{}^{ba}(x)\,,$$

then the commutation relations (2.32), (2.39), and (2.40) can be rewritten as a single equation:

$$[K_s,H_{\mu\nu}{}^{ba}(x)]-L_{s4}{}^{(x)}H_{\mu\nu}{}^{ba}(x)$$
$$=\delta_{\mu 4}H_{s\nu}{}^{ba}(x)+\delta_{\nu 4}H_{\mu s}{}^{ba}(x)$$
$$-\delta_{\mu s}H_{4\nu}{}^{ba}(x)-\delta_{\nu s}H_{\mu 4}{}^{ba}(x) \quad (2.42)$$

for all $\mu,\nu=1,2,3,4$. This implies that $H_{\mu\nu}{}^{ba}(x)$ is indeed a second-rank Lorentz tensor. From the definition Eq. (2.41), we find

$$H_{\mu\mu}{}^{ba}(x)=0\,,$$
$$H_{\mu\nu}{}^{ba}(x)=H_{\nu\mu}{}^{ba}(x)=-H_{\mu\nu}{}^{ab}(x)\,. \quad (2.43)$$

In terms of $H_{\mu\nu}{}^{ab}(x)$, one can express

$$G_{lsk}{}^{ab}(x)=\delta_{lk}H_{4s}{}^{ab}(x)+\delta_{sk}H_{4l}{}^{ab}(x)\,,\qquad \Sigma_{sl}{}^{ab}(x)=H_{sl}{}^{ab}(x)-\delta_{sl}H_{44}{}^{ab}(x)\,. \tag{2.44}$$

These expressions prove Eqs. (1.19)–(1.21). In terms of $H_{\mu\nu}{}^{ab}(x)$, we can rewrite $F_{ls}{}^{ab}(x)$ as

$$F_{ls}{}^{ab}(x)=T_{4ls}{}^{ab}(x)-\tfrac{1}{3}\delta_{ls}T_{444}{}^{ab}(x)\,, \tag{2.45}$$

where a completely symmetric tensor, $T_{\mu\nu\lambda}{}^{ab}(x)$, is defined by

$$T_{\mu\nu\lambda}{}^{ab}(x)=\frac{\partial}{\partial x_\lambda}H_{\mu\nu}{}^{ab}(x)+\frac{\partial}{\partial x_\mu}H_{\nu\lambda}{}^{ab}(x)+\frac{\partial}{\partial x_\nu}H_{\lambda\mu}{}^{ab}(x)\,, \tag{2.46}$$

and where we have used Eqs. (1.16) and (2.44).

From these considerations we first notice that, if we have $\Sigma_{lk}{}^{ab}(x)=0$, this leads to $H_{sl}{}^{ab}(x)=\delta_{sl}H_{44}{}^{ab}(x)$. Because of the Lorentz covariance of $H_{\mu\nu}{}^{ab}(x)$, this is possible only if we have $H_{\mu\nu}{}^{ab}(x)=\delta_{\mu\nu}H^{ab}$. However, since $H_{\mu\nu}{}^{ab}(x)$ is traceless, we must have $H_{\mu\nu}{}^{ab}(x)=0$ identically and hence $G_{slk}{}^{ab}(x)=F_{lk}{}^{ab}(x)=0$. On the other hand, if we have $G_{slk}{}^{ab}(x)=0$, then it gives $H_{4k}{}^{ab}(x)=0$, which in turn shows $H_{\mu\nu}{}^{ab}(x)\propto\delta_{\mu\nu}$. Hence we have $H_{\mu\nu}{}^{ab}(x)=0$ again; i.e., $\Sigma_{kl}{}^{ab}(x)=F_{lk}{}^{ab}(x)=0$. However, if we have $F_{lk}{}^{ab}(x)=0$, then Eq. (2.45) implies $T_{4ls}{}^{ab}=\frac{1}{3}\delta_{ls}T_{444}{}^{ab}$. Since $T_{\mu\nu\lambda}{}^{ab}$ is a completely symmetric Lorentz tensor, this is possible only if $T_{\mu\nu\lambda}{}^{ab}$ has the form

$$T_{\mu\nu\lambda}{}^{ab}(x)=\delta_{\mu\nu}T_\lambda{}^{ab}(x)+\delta_{\mu\lambda}T_\nu{}^{ab}(x)+\delta_{\lambda\nu}T_\mu{}^{ab}(x)\,,$$

where $T_\lambda{}^{ab}(x)$ is a Lorentz vector. Setting $\mu=\nu=\lambda=1$, for example, we get $(\partial/\partial x_1)H_{11}{}^{ab}(x)=T_1{}^{ab}(x)$. Again, due to Lorentz covariance and the tracelessness condition $H_{\mu\mu}{}^{ab}(x)=0$, the solution is given by

$$\frac{\partial}{\partial x_\lambda}H_{\mu\nu}{}^{ab}(x)=\tfrac{2}{3}[\delta_{\lambda\mu}T_\nu{}^{ab}(x)+\delta_{\lambda\nu}T_\mu{}^{ab}(x)]-\tfrac{1}{3}\delta_{\mu\nu}T_\lambda{}^{ab}(x)\,.$$

When we notice the integrability condition

$$\frac{\partial^2}{\partial x_\lambda\partial x_\rho}H_{\mu\nu}{}^{ab}(x)=\frac{\partial^2}{\partial x_\rho\partial x_\lambda}H_{\mu\nu}{}^{ab}(x)\,,$$

then it follows that $(\partial/\partial x_\mu)T_\nu{}^{ab}(x)=\delta_{\mu\nu}\varphi^{ab}(x)$ for some Lorentz scalar function $\varphi(x)$. Therefore, if we further demand that $H_{\mu\nu}{}^{ab}(x)$ is a local operator which does not explicitly depend upon the coordinate x, then the only possibility is that $T_\nu{}^{ab}(x)=0$ and $H_{\mu\nu}{}^{ab}(x)=0$. Therefore, summarizing our results, we find that, if one of $\Sigma_{sk}{}^{ab}(x)$, $G_{slk}{}^{ab}(x)$, and $F_{lk}{}^{ab}(x)$ is zero, then all of them must be zero identically.

Returning to the original problem, let us choose $A(x)=j_k{}^a(x)$ and $B(y)=j_l{}^b(y)$ in Eq. (2.1). After some calculations, we obtain

$$\begin{aligned}
(x_s-y_s)\delta(x_0-y_0)[j_{4k}{}^a(x),j_l{}^b(y)]
&=\Big\{-\delta_{ls}Q_k{}^{ab}(y)-\delta_{ls}\frac{\partial}{\partial y_m}S_{mk}{}^{ba}(y)\\
&\quad-[K_s,f_{kl}{}^{ab}(y)]+L_{s4}{}^{(y)}f_{kl}{}^{ab}(y)\Big\}\delta^{(4)}(x-y)\\
&\quad+\{\delta_{ls}S_{mk}{}^{ba}(y)-\delta_{km}S_{sl}{}^{ab}(y)\\
&\quad+[K_s,f_{mkl}{}^{ab}(y)]-L_{s4}{}^{(y)}f_{mkl}{}^{ab}(y)\}\frac{\partial}{\partial x_m}\delta^{(4)}(x-y)\\
&=\Big\{-\delta_{ls}Q_k{}^{ab}(x)-\frac{\partial}{\partial x_k}S_{sl}{}^{ab}(x)+\frac{\partial}{\partial x_4}f_{skl}{}^{ab}(x)\\
&\quad+[K_s,f_{lk}{}^{ba}(x)]-L_{s4}{}^{(x)}f_{lk}{}^{ba}(x)\Big\}\delta^{(4)}(x-y)\\
&\quad-\{\delta_{ls}S_{mk}{}^{ba}(x)-\delta_{km}S_{sl}{}^{ab}(x)+[K_s,f_{mkl}{}^{ab}(x)]\\
&\quad-L_{s4}{}^{(x)}f_{mkl}{}^{ab}(x)\}\frac{\partial}{\partial y_m}\delta^{(4)}(x-y)\,.
\end{aligned}\tag{2.47}$$

From this, we conclude that the order of $\delta(x_0-y_0)\times[j_{4k}{}^a(x),j_l{}^b(y)]$ is at most 2. Thus one can write it as Eq. (1.22), and, inserting the expression, we get

$$[K_s,f_{lk}{}^{ab}(y)]-L_{s4}{}^{(y)}f_{lk}{}^{ab}(y)+\delta_{ls}Q_k{}^{ab}(y)-\frac{\partial}{\partial y_k}S_{sl}{}^{ba}(y)+\frac{\partial}{\partial y_4}f_{skl}{}^{ba}(y)=-X_{slk}{}^{ab}(y)\,, \tag{2.48}$$

$$[K_s,f_{mkl}{}^{ba}(y)]-L_{s4}{}^{(y)}f_{mkl}{}^{ba}(y)-\delta_{km}S_{sl}{}^{ba}(y)+\delta_{ls}S_{mk}{}^{ab}(y)=2X_{smlk}{}^{ab}(y)\,. \tag{2.49}$$

We observe that we have an identity

$$\begin{aligned}
\delta(x_0-y_0)[j_{4k}{}^a(x),j_l{}^b(y)]+\delta(x_0-y_0)[j_k{}^a(x),j_{4l}{}^b(y)]
&=\left(\frac{\partial}{\partial x_4}+\frac{\partial}{\partial y_4}\right)\delta(x_0-y_0)[j_k{}^a(x),j_l{}^b(y)]\\
&\quad-\frac{\partial}{\partial x_k}\delta(x_0-y_0)[j_4{}^a(x),j_l{}^b(y)]\\
&\quad-\frac{\partial}{\partial y_l}\delta(x_0-y_0)[j_k{}^a(x),j_4{}^b(y)]\,.
\end{aligned}$$

One can check, however, that this relation is consistent with Eq. (2.47) and hence will not give any new relation.

We also find another identity,

$$\delta(x_0-y_0)[j_4{}^a(x),j_{4k}{}^b(y)]$$
$$=\left(\frac{\partial}{\partial x_4}+\frac{\partial}{\partial y_4}\right)\delta(x_0-y_0)[j_4{}^a(x),j_k{}^b(y)]$$
$$-\delta(x_0-y_0)[D^a(x),j_k{}^b(y)]$$
$$+\frac{\partial}{\partial x_l}\delta(x_0-y_0)[j_l{}^a(x),j_k{}^b(y)]$$
$$-\frac{\partial}{\partial y_k}\delta(x_0-y_0)[j_4{}^a(x),j_4{}^b(y)], \quad (2.50)$$

from which we can compute $\delta(x_0-y_0)[j_4{}^a(x),j_{4k}{}^b(y)]$ completely in terms of other quantities; the result is

$$\delta(x_0-y_0)[j_4{}^a(x),j_{4k}{}^b(y)]$$
$$=\{Q_{4k}{}^{ab}(x)-\sigma_k{}^{ab}(x)\}\delta^{(4)}(x-y)$$
$$+\left\{\delta_{lk}Q_4{}^{ab}(y)-\frac{\partial}{\partial y_4}S_{lk}{}^{ab}(y)+f_{lk}{}^{ab}(y)\right\}\frac{\partial}{\partial x_l}\delta^{(4)}(x-y)$$
$$-\{f_{slk}{}^{ab}(y)+G_{slk}{}^{ab}(y)\}\frac{\partial^2}{\partial x_l\partial x_s}\delta^{(4)}(x-y), \quad (2.51)$$

where we have set

$$Q_{\mu\nu}{}^{ab}(x)=\frac{\partial}{\partial x_\mu}Q_\nu{}^{ab}(x)-\frac{\partial}{\partial x_\nu}Q_\mu{}^{ab}(x).$$

In the case of the $SU(3)$ theory Ciccariello *et al.*[18] suggest that we may have the following relation:

$$\left[\int d^3x j_4{}^a(x),j_{\mu\nu}{}^b(y)\right]=Q_{\mu\nu}{}^{ab}(y)=-f_{abc}j_{\mu\nu}{}^{ab}(y).$$

This is compatible with Eq. (2.51) only if we have $\sigma_k{}^{ab}(x)=0$.

Also, if we demand that the order of $\delta(x_0-y_0)\times[j_4{}^a(x),j_{4k}{}^b(y)]$ is at most 1, then we must have $f_{slk}{}^{ab}(y)+G_{slk}{}^{ab}(y)=0$. Then one can show, after some algebra, that we must have $H_{\mu\nu}{}^{ab}(x)=0$ and that $f_{slk}{}^{ab}$ is completely antisymmetric under the interchange of s, l, and k and of a and b.

When we set $A(x)=j_4{}^a(x)$ and $B(y)=j_{4k}{}^b(y)$, we obtain a relation which is identically satisfied, and it gives no new relation. But if we choose $A(x)=j_l{}^a(x)$ and $B(y)=j_{4k}{}^b(y)$ in Eq. (2.1), it leads to

$$[K_s,\delta(x_0-y_0)[j_l{}^a(x),j_{4k}{}^b(y)]]$$
$$=\left\{L_{s4}{}^{(x)}+L_{s4}{}^{(y)}+(x_s-y_s)\frac{\partial}{\partial x_4}\right\}$$
$$\times\delta(x_0-y_0)[j_l{}^a(x),j_{4k}{}^b(y)]$$
$$-(x_s-y_s)\delta(x_0-y_0)[j_{4l}{}^a(x),j_{4k}{}^b(y)]$$
$$-\frac{\partial}{\partial x_l}\{(x_s-y_s)\delta(x_0-y_0)[j_4{}^a(x),j_{4k}{}^b(y)]\}$$
$$+\delta(x_0-y_0)[j_l{}^a(x),j_{sk}{}^b(y)]. \quad (2.52)$$

From this, together with Eqs. (2.51) and (1.22), we find that the order of $\delta(x_0-y_0)[j_{4l}{}^a(x),j_{4k}{}^b(y)]$ is at most 3. Moreover, if we assume that its order is at most 2 (*Ansatz* VII), then Eq. (2.52) gives us

$$2\{[K_r,X_{mslk}{}^{ab}(y)]-L_{r4}{}^{(y)}X_{mslk}{}^{ab}(y)\}$$
$$=-\delta_{lm}[f_{rsk}{}^{ab}(y)+f_{srk}{}^{ab}(y)+2G_{rsk}{}^{ab}(y)]$$
$$-\delta_{ls}[f_{rmk}{}^{ab}(y)+f_{mrk}{}^{ab}(y)+2G_{rmk}{}^{ab}(y)]$$
$$+\delta_{rm}f_{slk}{}^{ab}(y)+\delta_{rs}f_{mlk}{}^{ab}(y)$$
$$-\delta_{km}f_{slr}{}^{ab}(y)-\delta_{ks}f_{mlr}{}^{ab}(y), \quad (2.53)$$

$$[K_r,X_{slk}{}^{ab}(y)]-L_{r4}{}^{(y)}X_{slk}{}^{ab}(y)$$
$$=2Y_{srlk}{}^{ab}(y)+\delta_{rk}\delta_{ls}Q_4{}^{ab}(y)-\delta_{ls}\frac{\partial}{\partial y_4}S_{rk}{}^{ab}(y)$$
$$+\delta_{ls}f_{rk}{}^{ab}(y)-\delta_{rs}f_{lk}{}^{ab}(y)+\delta_{ks}f_{lr}{}^{ab}(y)$$
$$-\frac{\partial}{\partial y_r}f_{slk}{}^{ab}(y)+\frac{\partial}{\partial y_k}f_{slr}{}^{ab}(y), \quad (2.54)$$

$$[K_r,X_{lk}{}^{ab}(y)]-L_{r4}{}^{(y)}X_{lk}{}^{ab}(y)$$
$$=Y_{rlk}{}^{ab}(y)+\frac{\partial}{\partial y_r}f_{lk}{}^{ab}(y)-\frac{\partial}{\partial y_k}f_{lr}{}^{ab}(y). \quad (2.55)$$

Notice that a set composed of $X_{mslk}{}^{ab}(x)$, $f_{lsk}{}^{ab}(x)$, $G_{lmk}{}^{ab}(x)$, and $S_{lk}{}^{ab}(x)$ is closed under commutation with the boost operator K_s, as we see from Eqs. (2.53), (2.49), (2.42), (2.44), and (2.24). Therefore, they form a basis for a representation of the Lorentz group. Indeed, its solution is given by Eqs. (1.24) and (1.25). However, its derivation is a bit involved and too cumbersome to be reproduced here; therefore, we shall give a simple sketch of the proof as follows.

In Eq. (2.53), let us successively set $m=s$, $m=l$, $m=k$, and $l=k$ and sum over respective variables. This gives four equations. Solving them, we find that one can write

$$f_{rsk}{}^{ab}(x)=[K_r,A_{sk}{}^{ab}(x)]+\text{lower-order terms},$$
$$f_{srk}{}^{ab}(x)=[K_r,B_{sk}{}^{ab}(x)]+\text{lower-order terms}, \quad (2.56)$$
$$f_{skr}{}^{ab}(x)=[K_r,C_{sk}{}^{ab}(x)]+\text{lower-order terms},$$

[18] S. Ciccariello, R. Gatto, A. Sartori, and M. Tonin, Phys. Letters **30B**, 546 (1969).

where "lower-order terms" designates those proportional to either δ_{rs}, δ_{rk}, or δ_{sk}, and where $A_{sk}{}^{ab}(x)$, $B_{sk}{}^{ab}(x)$, and $C_{sk}{}^{ab}(x)$ are some operators contracted from $X_{mslk}{}^{ab}$. Now, using the Jacobi identity for K_s, K_l, and $f_{mkr}{}^{ba}(x)$ in combination with Eqs. (2.56) and (2.49), we can then prove that $X_{smlk}{}^{ab}(y)$ must be completely symmetric under interchange of s, m, l, and k, apart from lower-order terms which are proportional to δ_{sm}, δ_{lk}, δ_{sk}, etc. Completely symmetrizing Eqs. (2.49) and (2.53) among suitable variables, we see that the completely symmetric parts of $X_{smlk}{}^{ab}$ and $f_{slk}{}^{ab}$, together with some lower-order terms, are closed under commutation with the boost operator K_s; in other words, they must form a representation space of the Lorentz group. The completely symmetric part of $X_{smlk}{}^{ab}$ must contain parts corresponding to spins 4, 2, or 0. Since a finite-dimensional representation of the Lorentz group is always fully reducible and can be written in terms of tensors, symmetric parts of both $X_{smlk}{}^{ab}$ and $f_{slk}{}^{ab}(x)$ must be parts of a fourth-rank symmetric Lorentz tensor $X_{\mu\nu\lambda\rho}{}^{ab}$, if the former contains a part corresponding to spin 4. However, unfortunately, this assignment suffers from the incorrect sign in one of the commutators of Eqs. (2.49) and (2.53). Thus, we conclude that $X_{smlk}{}^{ab}$ must not contain a spin-4 part and hence it is reducible to lower-order terms. As we can see from Eq. (2.53), $f_{slk}{}^{ab}$ must also be written as a sum of δ functions involving δ_{sl}, δ_{sk}, and δ_{kl}. If we do not want to use the representation theory of the Lorentz group, we have to make repeated use of Jacobi identities among K_s, K_t, and $X_{smlk}{}^{ab}(x)$ in order to reach the same conclusion. At any rate, expressing both $X_{smlk}{}^{ab}(x)$ and $f_{slk}{}^{ab}(x)$ as sums of terms containing δ_{sm}, δ_{lk}, δ_{sk}, etc., and inserting them into Eqs. (2.49) and (2.53), one can solve the problem with the help of Eqs. (2.24), (2.42), and (2.44) in a form

$$\begin{aligned}2X_{mslk}{}^{ab}(x)&=\delta_{ml}S_{lk}{}^{ab}(x)+\delta_{sl}S_{mk}{}^{ab}(x)\\&\quad+(\delta_{ml}\delta_{sk}+\delta_{sl}\delta_{mk})E^{ab}(x)\,,\\ f_{slk}{}^{ab}(x)&=\delta_{sl}X_k{}^{ab}(x)+\delta_{sk}X_l{}^{ba}(x)\,,\end{aligned}\tag{2.57}$$

where $X_k{}^{ab}(x)$ and $E^{ab}(x)$ satisfy

$$\begin{aligned}&[K_s,X_k{}^{ab}(x)]-L_{s4}{}^{(x)}X_k{}^{ab}(x)=S_{sk}{}^{ab}(x)+\delta_{sk}E^{ab}(x)\,,\\&[K_s,S_{lk}{}^{ab}(x)]-L_{s4}{}^{(x)}S_{lk}{}^{ab}(x)\\&\quad=-\delta_{sl}X_k{}^{ab}(x)-\delta_{sk}X_l{}^{ba}(x)\\&\qquad-2\delta_{lk}H_{4s}{}^{ab}(x)-2\delta_{sk}H_{4l}{}^{ab}(x)\,,\\&[K_s,E^{ab}(x)]=-X_s{}^{ab}(x)-X_s{}^{ba}(x)\,.\end{aligned}\tag{2.58}$$

When one sets

$$\begin{aligned}T_{sk}{}^{ab}(x)&=\tfrac{1}{2}[S_{sk}{}^{ab}(x)+S_{ks}{}^{ba}(x)]\,,\\T_{4k}{}^{ab}(x)&=X_k{}^{ab}(x)-H_{4k}{}^{ab}(x)\,,\\T_{k4}{}^{ab}(x)&=X_k{}^{ba}(x)-H_{k4}{}^{ba}(x)\,,\\T_{44}{}^{ab}(x)&=-E^{ab}(x)=-E^{ba}(x)\,,\end{aligned}\tag{2.59}$$

these equations are combined into a single equation

$$\begin{aligned}&[K_s,T_{\mu\nu}{}^{ab}(x)]-L_{s4}{}^{(x)}T_{\mu\nu}{}^{ab}(x)\\&\quad=\delta_{\mu4}T_{s\nu}{}^{ab}(x)+\delta_{\nu4}T_{\mu s}{}^{ab}(x)\\&\qquad-\delta_{\mu s}T_{4\nu}{}^{ab}(x)-\delta_{\nu s}T_{\mu4}{}^{ab}(x)\,,\end{aligned}\tag{2.60}$$

i.e., $T_{\mu\nu}{}^{ab}(x)$ is a Lorentz tensor. From Eq. (2.59), we see that $T_{\mu\nu}{}^{ab}(x)$ satisfies the symmetry condition

$$T_{\mu\nu}{}^{ab}(x)=T_{\nu\mu}{}^{ba}(x)\,.\tag{2.61}$$

Moreover, when we set

$$R_{\mu\nu}{}^{ab}(x)=T_{\mu\nu}{}^{ab}(x)+H_{\mu\nu}{}^{ab}(x)\,,\tag{2.62}$$

then $R_{\mu\nu}{}^{ab}(x)$ satisfies Eq. (1.24), as can be seen from Eqs. (2.61) and (1.19). Also, $X_{mslk}{}^{ab}(x)$, $f_{slk}{}^{ab}(x)$, and $S_{lk}{}^{ab}(x)$ are now expressed in terms of $R_{\mu\nu}{}^{ab}(x)$ as in Eq. (1.25); i.e.,

$$\begin{aligned}S_{lk}{}^{ab}(x)&=R_{lk}{}^{ab}(x)-\delta_{lk}H_{44}{}^{ab}(x)\,,\\f_{slk}{}^{ab}(x)&=\delta_{sl}R_{4k}{}^{ab}(x)+\delta_{sk}R_{4l}{}^{ba}(x)\,,\\2X_{mslk}{}^{ab}(x)&=\delta_{ml}R_{sk}{}^{ab}(x)+\delta_{sl}R_{mk}{}^{ab}(x)\\&\quad-(\delta_{ml}\delta_{sk}+\delta_{sl}\delta_{mk})R_{44}{}^{ab}(x)\,.\end{aligned}\tag{2.63}$$

When we assume that the order of

$$\delta(x_0-y_0)[j_l{}^a(x),j_{4k}{}^b(y)]$$

is at most 1, i.e., $X_{mslk}{}^{ab}(x)=0$, then Eq. (2.63) implies

$$R_{sk}{}^{ab}(x)=\delta_{sk}R_{44}{}^{ab}(x)\,.$$

Since $R_{\mu\nu}{}^{ab}(x)$ is a Lorentz tensor, this leads immediately to

$$R_{\mu\nu}{}^{ab}(x)=\delta_{\mu\nu}R^{ab}(x)\,.\tag{2.64}$$

Then Eq. (2.63) gives $f_{slk}{}^{ab}(x)=0$. Also, $R_{\mu\mu}{}^{ab}(x)=R_{\mu\mu}{}^{ba}(x)$ [see Eq. (1.24)] implies $R^{ab}(x)=R^{ba}(x)$; it follows that $H_{\mu\nu}{}^{ab}(x)=0$ [again by Eq. (1.24)], and hence $S_{lk}{}^{ab}(x)=\delta_{lk}R^{ab}(x)$.

If we assume that the order of

$$\delta(x_0-y_0)[j_{4l}{}^a(x),j_{4k}{}^b(y)]$$

is also at most 1, i.e., $Y_{mslk}{}^{ab}(x)=0$, then Eqs. (2.54) and (2.48) imply that $X_{slk}{}^{ab}(x)$ and $f_{lk}{}^{ab}(x)$ are closed under commutation with K_s. In that case, a similar calculation proves that we must have

$$\begin{aligned}f_{lk}{}^{ab}(x)&=\frac{\partial}{\partial x_k}R_{4l}{}^{ba}(x)\,,\\X_{slk}{}^{ab}(x)&=\delta_{sl}\left\{\frac{\partial}{\partial x_k}[R_{44}{}^{ba}(x)-H_{44}{}^{ba}(x)]\right.\\&\qquad\left.-\frac{\partial}{\partial x_4}R_{4k}{}^{ab}(x)-Q_k{}^{ab}(x)\right\}\,.\end{aligned}\tag{2.65}$$

So, if the order of both commutators $\delta(x_0-y_0)\times[j_l{}^a(x),j_{4k}{}^b(y)]$ and $\delta(x_0-y_0)[j_{4l}{}^a(x),j_{4k}{}^b(y)]$ is at most 1, we must have $f_{lk}{}^{ab}(x)=0$, because of Eq. (2.64). This implies that $\delta(x_0-y_0)[j_l{}^a(x),j_k{}^b(y)]=0$; i.e., the space components commute as in the field algebra. Therefore, if we do not wish this type of commutation relation, at least one of $\delta(x_0-y_0)[j_l^a(x),j_{4k}^b(y)]$ and $\delta(x_0-y_0)[j_{4l}^a(x),j_{4k}^b(y)]$ must be of order 2 [i.e., must contain nonzero second-order derivatives with respect to $\delta^{(4)}(x-y)$]. Note that the field algebra is consistent with $R_{\mu\nu}^{ab}(x)=C\delta_{\mu\nu}\delta_{ab}$.

We remark that if we repeat our procedure by taking, for instance, $A(x)=D^a(x)$ and $B(y)=j_{\mu\nu}^b(y)$, we will introduce quantities involving second-order time derivatives of $j_\mu^a(x)$. Hence our set of relations is maximal if we do not use terms containing $(\partial^2/\partial x_4\partial x_4)j_\mu(x)$.

Finally, we note that since $H_{\mu\mu}^{ab}(x)=0$, we have $\langle H_{\mu\nu}^{ab}(x)\rangle_0=0$. It follows that $\langle G_{skl}^{ab}(x)\rangle_0=\langle\Sigma_{kl}^{ab}(x)\rangle_0=\langle F_{kl}^{ab}(x)\rangle_0=0$ and that Eq. (2.63) implies

$$\langle X_{mslk}^{ab}(x)\rangle_0=0.$$

Thus the vacuum expectation values of all equal-time commutators which we have considered so far do not contain the second-order derivative of $\delta^{(4)}(x-y)$, in conformity with the Lehmann-Källén representation.

As another application of Eq. (2.1), we notice that the ordinary canonical commutation relation of scalar fields, $\delta(x_0-y_0)[\phi(x),\phi(y)]=0$, leads to the fact that $\delta(x_0-y_0)[(\partial/\partial x_0)\phi(x),\phi(y)]$ must be proportional to $\delta^{(4)}(x-y)$; this result has been noted by Kazes.[17]

III. COVARIANT TIME-ORDERED PRODUCT OF TWO CURRENTS

It is well known that time-ordered products of two or more currents are in general not Lorentz covariant. Hence it is important to investigate (i) whether it is possible to construct a covariant T^* product, (ii) how the seagull, which makes the T product covariant, should be constructed from the commutators, and (iii) what additional constraints are placed on the seagull by the requirement that the Ward-Takahashi identities, obtained by ignoring Schwinger terms and seagull terms, be valid.

These problems have been discussed by several authors[2–4] under various restrictive assumptions. However, Gross and Jackiw[5] have recently solved the problem under much weaker assumptions. Indeed, they find that the answer for (i) and (ii) is affirmative while the condition for (iii) is somewhat model dependent. Although we have perhaps nothing really new to add to this subject in view of this fact, nevertheless it may be instructive to demonstrate explicitly these points in our case, since we employ an approach entirely different from the one used by Gross and Jackiw.[5]

To obtain the Lorentz property of a time-ordered product, we start from the following identity:

$$\begin{aligned}[K_s,T(A(x)B(y))]-(L_{s4}^{(x)}+L_{s4}^{(y)})T(A(x)B(y))\\ =T(\Delta_sA(x)B(y))+T(A(x)\Delta_sB(y))\\ -i(x_s-y_s)\delta(x_0-y_0)[A(x),B(y)],\end{aligned}\tag{3.1}$$

where $\Delta_sA(x)$ and $\Delta_sB(y)$ are defined in Eq. (2.2). Since $\delta(x_0-y_0)[j_\mu^a(x),j_\nu^b(y)]$ is not zero, we find that Eq. (3.1) demonstrates the well-known fact that $T(j_\mu^a(x),j_\nu^b(y))$ is not Lorentz covariant.

To find the correct Lorentz-covariant tensor, let us set

$$\begin{aligned}M_{\mu\nu}^{ab}(x,y)&=T^*(j_\mu^a(x)j_\nu^b(y))\\ &=T(j_\mu^a(x)j_\nu^b(y))+\Delta_{\mu\nu}^{ab}(x)\delta^{(4)}(x-y)\end{aligned}\tag{3.2}$$

and demand that $M_{\mu\nu}^{ab}$ satisfies the Lorentz-tensor condition

$$\begin{aligned}[K_s,M_{\mu\nu}^{ab}(x,y)]-(L_{s4}^{(x)}+L_{s4}^{(y)})M_{\mu\nu}^{ab}(x,y)\\ =\delta_{\mu4}M_{s\nu}^{ab}(x,y)+\delta_{\nu4}M_{\mu s}^{ab}(x,y)\\ -\delta_{\mu s}M_{4\nu}^{ab}(x,y)-\delta_{\nu s}M_{\mu4}^{ab}(x,y).\end{aligned}\tag{3.3}$$

Using *Ansatz* III, we find that $\Delta_{\mu\nu}^{ab}(x,y)$ must satisfy

$$\begin{aligned}\{[K_s,\Delta_{\mu\nu}^{ab}(x)]-L_{s4}^{(x)}\Delta_{\mu\nu}^{ab}(x)\}\delta^{(4)}(x-y)\\ =\{\delta_{\mu4}\Delta_{s\nu}^{ab}(x)-\delta_{\mu s}\Delta_{4\nu}^{ab}(x)\\ +\delta_{\nu4}\Delta_{\mu s}^{ab}(x)-\delta_{\nu s}\Delta_{\mu4}^{ab}(x)\}\delta^{(4)}(x-y)\\ +i(x_s-y_s)\delta(x_0-y_0)[j_\mu^a(x),j_\nu^b(y)].\end{aligned}\tag{3.4}$$

The purpose of this section is to demonstrate the existence of a $\Delta_{\mu\nu}^{ab}(x)$ which satisfies this condition. To this end, we assume *Ansätze* I–VI, but we need not assume the validity of the last *Ansatz*, VII. Setting

$$\begin{aligned}&\Delta_{44}^{ab}(x)=0,\\ &\Delta_{k4}^{ab}(x)=-iH_{k4}^{ab}(x),\\ &\Delta_{4k}^{ab}(x)=+iH_{4k}^{ab}(x)=-iH_{4k}^{ba}(x),\\ &\Delta_{kl}^{ab}(x)=-\tfrac{1}{2}i[S_{kl}^{ab}(x)+S_{lk}^{ba}(x)],\end{aligned}\tag{3.5}$$

one can check that Eq. (3.4) is satisfied in view of Eqs. (1.4), (1.10), (1.15), and (1.19)–(1.21). Hence with the identification Eq. (3.5), we have proved the existence of a covariant time-ordered product, $T^*(j_\mu^a(x)j_\nu^b(y))$. Also, Eq. (3.5) implies that $\Delta_{\mu\nu}^{ab}(x)$ satisfies the symmetry condition

$$\Delta_{\mu\nu}^{ab}(x)=\Delta_{\nu\mu}^{ba}(x),\tag{3.6}$$

so that we have

$$T^*(j_\mu^a(x)j_\nu^b(y))=T^*(j_\nu^b(y)j_\mu^a(x)).\tag{3.7}$$

Note that, when we have $H_{\mu\nu}^{ab}(x)=0$, our definition of $\Delta_{\mu\nu}^{ab}(x)$ reduces to the one originally given by Dashen and Lee[2] [for the case $D^a(x)=\partial_\mu j_\mu^a(x)=0$] and by Yang,[3] who generalized the result without the conservation law but with the additional assumptions $H_{\mu\nu}^{ab}(x)=0$. Hence we conclude that we can find the covariant time-ordered product in a much more general case, although this may not be surprising in view of the work by Gross and Jackiw.[5]

Also, one can construct the covariant time-ordered products, $T^*(D^a(x)j_\nu{}^b(y))$, by the formula

$$\frac{\partial}{\partial x_\mu}T^*(j_\mu{}^a(x)j_\nu{}^b(y))=T^*(D^a(x)j_\nu{}^b(y))-iQ_\nu{}^{ab}(x)\delta^{(4)}(x-y)\,,\quad(3.8)$$

where $T^*(D^a(x)j_\nu{}^b(y))$ is given by

$$\begin{aligned}T^*(D^a(x)j_4{}^b(y))&=T(D^a(x)j_4{}^b(y))\\&\quad-i\left(H_{\mu4}{}^{ab}(y)\frac{\partial}{\partial x_\mu}-H_{44}{}^{ab}(y)\frac{\partial}{\partial x_4}\right)\delta^{(4)}(x-y)\,,\\T^*(D^a(x)j_k{}^b(y))&=T(D^a(x)j_k{}^b(y))\\&\quad+i\left(H_{\mu k}{}^{ab}(y)\frac{\partial}{\partial x_\mu}-H_{44}{}^{ab}(y)\frac{\partial}{\partial x_k}\right)\delta^{(4)}(x-y)\,.\end{aligned}\quad(3.9)$$

Notice that the sign difference in the two expressions in Eq. (3.9) implies that $T(D^a(x)j_\nu{}^b(y))$ is not a Lorentz vector unless we have $H_{\mu\nu}{}^{ab}(x)=0$. If $H_{\mu\nu}{}^{ab}(x)=0$, then the validity of Eqs. (3.8) and (3.9) may be interpreted to imply the cancellation of the Schwinger term and the seagull term in the usual Lagrangian terminology. However, for the case $H_{\mu\nu}{}^{ab}(x)\neq0$, the situation is more involved,[5] and its physical interpretation is less clear.

Analogously, we can define the covariant time-ordered product $T^*(D^a(x)D^b(y))$ by

$$\begin{aligned}\frac{\partial}{\partial y_\nu}T^*(D^a(x)j_\nu{}^b(y))&=T^*(D^a(x)D^b(y))\\&=T(D^a(x)D^b(y))+i\sigma^{ab}(x)\delta^{(4)}(x-y)\\&\quad+i\frac{\partial}{\partial y_\nu}H_{\mu\nu}{}^{ab}(y)\frac{\partial}{\partial x_\mu}\delta^{(4)}(x-y)\\&\quad-2i\frac{\partial}{\partial y_4}H_{\mu4}{}^{ab}(y)\frac{\partial}{\partial x_\mu}\delta^{(4)}(x-y)\\&\quad-i\frac{\partial}{\partial y_\mu}H_{44}{}^{ab}(y)\frac{\partial}{\partial x_\mu}\delta^{(4)}(x-y)\\&\quad+2i\frac{\partial}{\partial y_4}H_{44}{}^{ab}(y)\frac{\partial}{\partial x_4}\delta^{(4)}(x-y)\,.\end{aligned}\quad(3.10)$$

Notice that the second-order derivatives with respect to $\delta^{(4)}(x-y)$ cancel out when we use Eqs. (1.2) and (1.21). We may remark that $T(D^a(x)D^b(y))$ is not a Lorentz scalar unless we have $F_{kl}{}^{ab}(x)=0$ and $F_k{}^{ab}(x)=0$, as can be seen from Eqs. (3.1) and (1.8).

In concluding this section, we remark that there is no uniqueness in definining the covariant time-ordered product. Probably, a reasonable restriction on $\Delta_{\mu\nu}{}^{ab}(x)$ is that it must satisfy

$$\begin{aligned}\Delta_{\mu\nu}{}^{ab}(x)&=\Delta_{\nu\mu}{}^{ba}(x)\,,\\\Delta_{44}{}^{ab}(x)&=0\,,\end{aligned}\quad(3.11)$$

since the first relation is necessary to ensure Eq. (3.7) while $\Delta_{44}{}^{ab}(x)=0$ is necessary to have $T^*(j_4{}^a(x)j_4{}^b(y))=T(j_4{}^a(x)j_4{}^b(y))$. Then the general solution for $\Delta_{\mu\nu}{}^{ab}(x)$ is constructed by adding an arbitrary Lorentz tensor $\tilde{\Delta}_{\mu\nu}{}^{ab}(x)$ to the right-hand side of Eq. (3.5); here $\tilde{\Delta}_{\mu\nu}{}^{ab}(x)$ must satisfy the condition

$$\tilde{\Delta}_{\mu\nu}{}^{ab}(x)=-\tilde{\Delta}_{\mu\nu}{}^{ba}(x)=-\tilde{\Delta}_{\nu\mu}{}^{ab}(x)\,,\quad(3.12)$$

i.e., it must be antisymmetric under exchanges of (a,b) and (μ,ν). For example, we could have $\tilde{\Delta}_{\mu\nu}{}^{ab}(x)=cQ_{\mu\nu}{}^{ab}(x)$, where c is an arbitrary constant. However, for the diagonal term $a=b$, we have no ambiguity of this kind since $\tilde{\Delta}_{\mu\nu}{}^{aa}(x)=0$ (no summation over a). In particular, we have no ambiguity for the definition of the covariant time-ordered product of two electromagnetic currents, since we have only one component $n=1$ with $a=b=1$ to start with. Also, for the case when we have $H_{\mu\nu}{}^{ab}(x)=0$ identically as in the $SU(2)$ case, it is natural to add the additional condition

$$\begin{aligned}T^*(D^a(x)j_\nu{}^b(y))&=T(D^a(x)j_\nu{}^b(y))\\&=\frac{\partial}{\partial x_\mu}T^*(j_\mu{}^a(x)j_\nu{}^b(y))+iQ_\nu{}^{ab}(x)\delta^{(4)}(x-y)\,.\end{aligned}$$

Then this implies $\tilde{\Delta}_{\mu\nu}{}^{ab}(x)=0$ again.

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