Symmetry Breaking in Representations of the Relativistic Symmetry

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We discuss the symmetry breaking in representations of the relativistic symmetry which are generalizations of the Dirac representation. The symmetry-breaking relation is based on a generalization of the de Sitter model and leads to fine structure in the mass spectrum.

IN the preceding paper[1] we have discussed irreducible representations $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ of the relativistic symmetry $\mathfrak{S}$ which are infinite-dimensional generalizations of the Dirac representation, i.e., whose representation spaces are "infinite-dimensional" generalizations of the space of solutions of the Dirac equation. In an irreducible representation space of the "unbroken" relativistic symmetry $\mathfrak{S}$, all states have the same mass; to obtain a realistic mass spectrum, one has to break $\mathfrak{S}$ by requiring in addition to the defining relations of $\mathfrak{S}$ a symmetry-breaking relation among the generators of $\mathfrak{S}$. An example of a symmetry-breaking relation is the infinite-component wave equation[2] for the Majorana representation. Other procedures of symmetry breaking for the Majorana representation have been discussed in Ref. 3. Good agreement with experimental data was obtained by an algebraic symmetry-breaking relation which is based on the de Sitter model.[4]

For the representations $\mathfrak{S}^{(R,\cdot)}$ (where the dot denotes 0 or $\frac{1}{2}$) there is, of course, also a vast number of possible choices for symmetry-breaking relations. We discuss here the symmetry breaking in the frame of a generalization of the de Sitter model of Ref. 4. This will then lead to an algebraic structure which we want to call $\mathcal{A}_2$ and which is a generalization of the algebra $\mathcal{A}_1$ in Ref. 4.

Instead of $\mathfrak{S}^{(\text{Majorana})}$ which was used in Ref. 4, we have now the representations $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ of $\mathfrak{S}$. The symmetry-breaking relation in Ref. 4 [Eq. (10) of Ref. 4] ensured that the second-order Casimir operator

$$Q=(1/\lambda^2)B_\mu B^\mu-\tfrac{1}{2}L_{\mu\nu}L^{\mu\nu}$$

of $SO(4,1)$, generated by $L_{\mu\nu}$ and

$$B_\mu=P_\mu+\tfrac{1}{2}\lambda(P_\rho P^\rho)^{-1/2}\{P^\rho,L_{\rho\mu}\},$$

was an invariant operator. λ was a new constant of dimension MeV which determined the strength of the symmetry breaking.

The generalization $\mathcal{A}_2$ of this model consists of replacing the symmetry-breaking constant λ by a symmetry-breaking operator Λ which is an element of the algebra generated by P_μ, Γ_μ, $L_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu}$. This still leaves a great variety of possibilities and we will have to make some assumptions on Λ.

Thus we consider the representations $\mathfrak{S}^{(R,1/2)}$ and $\mathfrak{S}^{(R,0)}$ derived in I with arbitrary m ($m\neq 0$) and $p_0/|p_0|=\epsilon>0$, and define

$$B_\mu=P_\mu+\tfrac{1}{2}\Lambda M^{-1}\{P^\rho,L_{\rho\mu}\},\quad M^2=P_\mu P^\mu. \tag{1}$$

We assume that Λ is Lorentz invariant,

$$[\Lambda,L_{\mu\nu}]=0 \tag{2}$$

and

$$[P_\nu,\Lambda]=0, \tag{3}$$

but that in general $[\Gamma_\nu,\Lambda]\neq 0$.[5] Then, using (2), (3), and the c.r. (commutation relation) of $\mathcal{P}$, we calculate

$$[B_\mu,B_\nu]=i\Lambda^2 L_{\mu\nu}, \tag{4}$$

$$[L_{\mu\nu},B_\rho]=i(g_{\nu\rho}B_\mu-g_{\mu\rho}B_\nu). \tag{5}$$

From (4) and (5) we see that for every $0<\lambda^2\epsilon$ of the spectrum of Λ^2 (1) connects an irreducible representation of the Poincaré group with an irreducible representation of a de Sitter group $SO(4,1)_{B_\mu L_{\mu\nu}}{}^{(\lambda)}$,[6] which is the group of motion in a de Sitter space of radius $1/\lambda$. The difference from the original de Sitter model is that now we have not only one de Sitter group and one de Sitter space but rather as many as there are elements in the spectrum of Λ. (We remark that such a connection between an irreducible representation of $\mathcal{P}$ and of $SO(4,1)^{(\lambda)}$ does not exist if $[M,\Lambda]\neq 0$.)

For $\lambda^2<0$, B_μ and $L_{\mu\nu}$ generate an $SO(3.2)_{B_\mu L_{\mu\nu}}{}^{(\lambda)}$ and, for $\lambda=0$, B_μ and $L_{\mu\nu}$ generate the original Poincaré group $\mathcal{P}_{P_\mu,L_{\mu\nu}}$.

[1] A. Böhm, preceding paper, Phys. Rev. D **2**, 367 (1970), hereafter referred to as I.

[2] Y. Nambu, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience, New York, 1968), and references therein.

[3] A. Böhm, in *Lectures in Theoretical Physics* (Gordon and Breach, New York, 1968), Vol. 10B, p. 483.

[4] A. Böhm, Phys. Rev. **175**, 1767 (1968); **145**, 1212 (1966); A. O. Barut and A. Böhm, *ibid.* **139**, B1107 (1965).

[5] We remark that if we relax (3) and require only

$$[\hat{P}_\mu,\Lambda]=0 \quad\text{with}\quad \hat{P}_\mu=P_\mu M^{-1}, \tag{3'}$$

we would obtain instead of (4)

$$[B_\mu,B_\nu]=i\Lambda^2 L_{\mu\nu}+[M,\Lambda]\hat{P}^\rho(\hat{P}_\mu L_{\rho\nu}-\hat{P}_\nu L_{\rho\mu}). \tag{4'}$$

[6] This is, in fact, only a connection between the irreducible representation (s,m) of the Poincaré group and an irreducible representation of the Lie algebra of $SO(4,1)$. The representation given by $B_\mu=P_\mu+(\lambda/2m)\{P^\rho,L_{\rho\mu}\}$ and $L_{\mu\nu}$ of the algebra of $SO(4,1)$ on $\mathcal{H}(s,m)$ does not integrate to a representation of the group $SO(4,1)$, because, roughly speaking, the irreducible representation space of a principal-series representation of $SO(4,1)$ contains twice as many states as $\mathcal{H}(s,m)$. Whether this doubling has a physical counterpart or whether only the Lie-algebra representation of $SO(4,1)$ is of physical significance is not known at present.

Using (1), one derives for the Casimir operator

$$\Lambda^2 Q = B_\mu B^\mu - \tfrac{1}{2}\Lambda^2 L_{\mu\nu} L^{\mu\nu} \tag{6}$$

of $SO(4,1)^{(\Lambda)}$ or $SO(3,2)^{(\Lambda)}$ (or $\mathcal{P}$) after some lengthy but straightforward calculation

$$\Lambda^2 Q = M^2 + (9/4)\Lambda^2 - \Lambda^2 \hat{W}, \tag{7}$$

where

$$\hat{W} = -\hat{w}^\mu \hat{w}_\mu, \quad \hat{w}^\mu = \epsilon^{\mu\nu\rho\sigma} \hat{P}_\nu L_{\rho\sigma}, \quad \hat{P}_\mu = M^{-1} P_\mu.$$

The assumption of the breaking of the symmetry $\mathfrak{S}$ in the de Sitter model of Ref. 4 was to require that Q is an invariant, i.e., $[Q,\Gamma_\lambda]=0$.[7] Accordingly, we formulate our symmetry-breaking relation[8]:

$$[Q,\Gamma_\lambda]=0. \tag{8}$$

As in Ref. 4, we shall also require here that

$$[\hat{P}_\rho,\Gamma_\lambda]=0 \text{ with } \hat{P}_\rho = M^{-1}P_\rho. \tag{9}$$

The final problem lies in the determination of the symmetry-breaking operator Λ^2. From (1), (4), (6), and (7), we see that the easiest situation will arise if Λ^2 is positive definite, and has discrete spectrum $\lambda^{(1)}$, $\lambda^{(2)}$, $\ldots$, $\lambda^{(k)}$, $\ldots$. Then to each $\lambda^{(i)}$ corresponds a $SO(4,1)_{B_\mu L_{\mu\nu}}{}^{(\lambda^i)}$ representation with the same eigenvalue α^2 of Q and each $SO(4,1)_{B_\mu L_{\mu\nu}}{}^{(\lambda^i)}$ representation space $\mathcal{H}^{SO(4,1)\lambda^i}(\alpha,s)$ is also an irreducible representation space $\mathcal{H}^{\lambda^{(i)}}(m,s)$ of the Poincaré group $\mathcal{P}_{P_\mu L_{\mu\nu}}$ by (1).[6] Here m is connected with the invariant α and with $\lambda^{(i)}$ by (7). Thus the irreducible representation spaces of $\mathcal{P}$ and therewith the elementary particles are characterized not only by m and s but in addition by a new label $\lambda^{(i)}$ (except for the case that the spectrum of Λ^2 consists of one point). It is clear that in the frame of the representations $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$, $\lambda^{(i)}$ must be connected with the new quantum number n.

We therewith come to the question of the possible forms of the operator Λ. Since we want the symmetry breaking to be a generalization of the de Sitter model, we have to require that in the representation $\mathfrak{S}^{(\mathrm{Majorana})}$ of the de Sitter model Λ^2 be a constant. That restricts the possibilities for Λ to

$$\Lambda^2 = \lambda_1^2 - \lambda_2^2[W/M^2 - (P_\mu\Gamma^\mu/M)^2], \tag{10}$$

where λ_1 and λ_2 are two constants of dimensions MeV because in $\mathfrak{S}^{(\mathrm{Majorana})}$

$$\Lambda^2 = \lambda_1^2 + \tfrac{1}{4}\lambda_2^2 = \text{const}.$$

Therewith we can summarize the defining relations of $\mathcal{A}_2$:

(1) The relations of the relativistic symmetry $\mathfrak{S}^{(R,1/2)}$ and $\mathfrak{S}^{(R,0)}$ extended by the discrete operations U_C, A_T, U_P (that is, the commutation relations of $\mathfrak{S}$ extended by CPT and the additional relations that specify $\mathfrak{S}^{(R,1/2)}$ and $\mathfrak{S}^{(R,0)}$) as discussed in I.

(2) The symmetry-breaking relation (8), where Q is defined by (6), (1), and (10), and the relation (9).

We now investigate the representation of this algebra. From (3) it follows that Λ^2 and M^2 can be simultaneously diagonalized. We can, therefore, divide the representation space into two subspaces $\mathcal{H}^+ \oplus \mathcal{H}^-$ such that spectrum $M^2>0$ on $\mathcal{H}^+$ and spectrum $M^2 \leq 0$ on $\mathcal{H}^-$ and investigate the spectrum of Λ^2 on each subspace separately. We start with spectrum $M^2>0$.

From (10), we obtain for the spectrum of Λ^2

$$(\lambda^{(i)})^2 = \lambda_{(s,n)}{}^2 = \lambda_1^2 - \lambda_2^2(s^2+s-n^2), \tag{11}$$

where the spectrum of (s,n) is given by the multiplicity pattern in Figs. 1 and 2 of I. From these multiplicity patterns we know that $n \leq s$, so that for sufficiently high s (depending upon the value of the empirical constants λ_1^2 and λ_2^2, which will turn out to be positive), $\lambda_{(s,n)}{}^2$ can be negative. From (4) we see that in this case B_μ and $L_{\mu\nu}$ generate a representation of $SO(3,2)^{(\lambda(s,n))}$.

In an irreducible representation of $\mathcal{A}_2$, the eigenvalue of the invariant operator Q is a constant α^2. By taking the expectation value of (7) in the basis $|p,m,s,s_3,n\rangle$ of I, we obtain

$$m^2 = \lambda_{(s,n)}{}^2(\alpha^2 - 9/4) + \lambda_{(s,n)}{}^2 s(s+1). \tag{7'}$$

From this we see that for sufficiently high s when $\lambda_{(s,n)}{}^2$ becomes negative, m^2 will also become negative.

Before investigating this situation further, let us recall the general principle of our approach: An irreducible representation of the algebra describes a physical system, which consists of a tower of particles or resonances. The irreducible representations are characterized by the value α^2 and the irreducible representation of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ which they contain, i.e., by $(0,R)$ or $(\tfrac{1}{2},R)$. Thus for a particular physical system (tower), α^2 has a definite value which is determined empirically. Particular physical systems that we consider are the meson tower for which $\alpha^2 \approx 9/4$ and the baryon tower for which $\alpha^2 = 4.46$. For the sake of definiteness, we will consider in the following discussion the case $\alpha^2 \approx 9/4$ which is of particular interest for us (more precisely we consider the case $\alpha^2 - 9/4 = \epsilon^2$, where $\epsilon^2 > 0$ is arbitrary small but $\neq 0$).

In the representation space $\mathcal{H}(\alpha^2 \approx 9/4, (R,0))$, (7′) becomes

$$m^2 = \lambda_{(s,n)}{}^2 s(s+1) + \lambda_{(s,n)}{}^2 \epsilon^2. \tag{7''}$$

{From (7″) we see the reason for our choice $\alpha^2 = 9/4 + \epsilon^2$. Then the lowest state $\sigma = [s=0, n=0]$ has the mass $m_\sigma{}^2 = \lambda_1^2 \epsilon^2 \neq 0$ and, as can be seen, no state with $m^2=0$ appears for which (1) would have been not defined. From physical considerations $m_\sigma{}^2$ should be chosen of the order of electromagnetic mass differences, which cannot be accounted for in this model.}

[7] One can prove that in Ref. 4 $[Q,\Gamma_\mu]=0$ not only follows from (10) but also vice versa; (10) follows from $[Q,\Gamma_\mu]=0$ if relations (1)–(11) hold.

[8] We remark that (8) is a non-Lie-algebraic relation so that $\mathcal{A}_2$ is not the enveloping algebra of a Lie group and the O'Raifeartaigh theorem does not apply. (The other possibility for the generalization of the symmetry-breaking relation in Ref. 4, $[\Lambda^2 Q,\Gamma_\lambda]=0$, will lead to difficulties.)

Therefore we have the following situation: In the representation space $\mathcal{H}(\alpha^2\approx 9/4, (R,0))$ we have an infinite set of states with s and n given by the multiplicity pattern in Fig. 2 of I and the states with a given $[s,n]$ have the mass $m_{(s,n)}{}^2$ given by (7″) and (11). Thus the operators Γ_i that transform between different (s,n) states (at rest) also change the mass such that (7″) is always fulfilled. This is a situation with which we are well acquainted,[3,4] except that now, owing to the indefiniteness of the symmetry-breaking operator Λ, m^2 can become negative, so that for a certain ciritical value for $[s,n]$, Γ_i transforms from positive mass-squared states into negative mass-squared states. It is this point that needs further detailed investigation.

At first one would surmise that these $m^2<0$ states correspond to the usual unitary representations of the Poincaré group with $m^2<0$ and describe tachyons.[9] In that case, however, the little group would be $SO(2,1)_{S_{01},S_{02},S_{12}}$ and the additional quantum number would not be n but the eigenvalue of Γ_3, which is continuous. We know, however, that Γ_i or in general $U(L^{-1}(p))\Gamma_i U(L(p))$, where $L^{-1}(p)$ is the boost and $U(L)$ is its representative in $\mathcal{H}(\alpha^2,(R,0))$, changes the representation s of the little group $SO(3)_{S_i}$ but not the little group. Therefore, $U(L^{-1}(p))\Gamma_i U(L(p))$transforms between irreducible representation spaces of the Poincaré group which have $SO(3)_{S_i}$ as the little group. Thus the $m^2<0$ states in the space $\mathcal{H}(\alpha^2,(R,0))$ must belong to nonunitary representations $(m^2<0,\, s)$ of the Poincaré group.

Because of the unitarity of the representation $(R,0)$ of $SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}$ the generators $L^{\mu\nu}$ must be Hermitian also (see Appendix B):

$$L_{\mu\nu}{}^\dagger = L_{\mu\nu}\,, \tag{12}$$

so that in the representations $(m^2<0,\, s)$ of $\mathcal{P}$ the $SO(3,1)_{L_{\mu\nu}}$ subgroup is represented unitarily. In the representation $(m^2<0,\, s)$ the translation group is represented nonunitarily and $(m^2<0,\, s)$ are the imaginary momenta representations[10] in which P_μ is anti-Hermitian:

$$P_\mu{}^\dagger = -P_\mu\,. \tag{13}$$

To see that this conclusion is in accord with our postulate that $SO(4,1)_{B_\mu,L_{\mu\nu}}$ plays the fundamental role in our scheme, we investigate the representation of B_μ.

On the subspace $\mathcal{H}^{(\lambda^2<0)}$ with $\lambda_{(s,n)}{}^2=\lambda^2<0$, and consequently from (7″) $m^2<0$, (1) gives[11]

$$B_\mu = P_\mu + \tfrac{1}{2}(\lambda/m)\{P^\rho, L_{\rho\mu}\}\,, \tag{1'}$$

so that from (12) and (13) we obtain $B_\mu{}^\dagger = -B_\mu$. Because of (4),

$$[B_\mu,B_\nu] = -|\lambda^2|\, iL_{\mu\nu}\,. \tag{4'}$$

Thus $L_{\mu\nu}$ and B_μ generate an $SO(3,2)_{B_\mu,L_{\mu\nu}}{}^{(\lambda)}$, in which the B_μ are anti-Hermitian. We define

$$\bar{B}_\mu = (1/\lambda)B_\mu\,,$$

which obeys

$$[\bar{B}_\mu,\bar{B}_\nu] = iL_{\mu\nu}\,, \quad \bar{B}_\mu{}^\dagger = \bar{B}_\mu\,,$$

so that $\bar{B}_\mu$, $L_{\mu\nu}$ generate a unitary representation of $SO(4,1)_{\bar{B}_\mu,L_{\mu\nu}}$ on the subspace $\mathcal{H}^{(\lambda^2<0)}$ of $\mathcal{H}(\alpha^2=9/4, (R,0))$. The eigenvalue of the Casimir operator

$$Q = \bar{B}_\mu \bar{B}^\mu - \tfrac{1}{2}L_{\mu\nu}L^{\mu\nu}$$

is α^2, so that this representation of $SO(4,1)_{\bar{B}L}$ must be a class-I or -III representation[12] characterized by (α^2,s), where s is the character of the $SO(3)_{Si}$ little-group representation.[13] Therefore the space $\mathcal{H}(\alpha^2,(R,0))$ contains only Hermitian class-I and -III representations of $SO(4,1)_{\bar{B}_\mu,L_{\mu\nu}}$ independently of sign λ^2 and the reduction is

$$\mathcal{H}(\alpha^2, (R,0)) \underset{SO(4,1)_{\bar{B}_\mu L_{\mu\nu}}}{=} \sum_{s=0,1,\ldots}^{\infty} \oplus \mu(s)(\alpha^2,s)\,,$$

where the multiplicity $\mu(s)$ of the representation (α^2,s) is $\mu(s)=2s+1$ (which follows from the multiplicity pattern Fig. 2 of I).

The above representation of the broken relativistic symmetry was induced from the representation $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}{}^{(R,0)}$ in which $SO(2)_{\Gamma_0}\otimes SO(3)_{S_i}$ is diagonal and contains, therefore, only those representations of $\mathcal{P}$ that have $SO(3)_{S_i}$ as the little group. If one wants to obtain representations of the above algebra that contains Poincaré group representations with $SO(2,1)_{S_{01},S_{02},S_{12}}$ as the little group, one has to induce from the representation $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}{}^{(R,0)}$ in which $SO(1,1)_{\Gamma_3}\otimes SO(2,1)_{S_{01},S_{02},S_{12}}$ is diagonal. This has been discussed in Appendix A and one obtains in both cases quite distinct representations and not only the same representation in a different basis. Thus, e.g., the first representation contains only unitary continuous series representations of $SO(4,1)_{\bar{B}_\mu,L_{\mu\nu}}$, whereas the second representation contains only nonunitary representations of $SO(4,1)_{\bar{B}_\mu,L_{\mu\nu}}$. Only the representations with $SO(3)_{S_i}$ as the little group contain the usual particle representations of the Poincaré group, and we restrict ourselves, therefore, to these representations and shall call them $\mathcal{H}(+, \alpha^2, (R,0))$.

In the above considerations we have only discussed the representation space with $\alpha^2-9/4=\epsilon^2\sim 0$ and $(R,0)$ that describe mesons. But it is clear that for the baryon case $(R,\frac{1}{2})$ and for $\alpha^2-9/4>0$ all the above arguments remain valid. We have thus obtained the following result.

[9] E. C. G. Sudarshan, Arkiv Fysik **39**, 585 (1969).

[10] This is a subclass of the irreducible representations of the complex Poincaré group. See, e.g., A. O. Barut, in *Lectures in Theoretical Physics* (Colorado U. P., Boulder, 1964), Vol. 7a; E. H. Roffman, Commun. Math. Phys. **4**, 237 (1967).

[11] We remark that the choice of the sign of the square root of λ^2 and m^2 is irrelevant in this consideration.

[12] Or part of it; cf. Ref. 6.

[13] See, e.g., T. D. Newton, Ann. Math. **51**, 730 (1950); A. Böhm, in *Lectures in Theoretical Physics* (Gordon and Breach, New York, 1967), Vol. 9B, p. 327.

The physical system (particle tower) that is described by the representation space $\mathcal{H}(+, \alpha^2, (R,\cdot))$ of $\mathcal{A}_2$ (where $\cdot$ stands for 0 or $\frac{1}{2}$) has the mass spectrum

$$m_{(s,n)}^2 = \{\lambda_1^2 - \lambda_2^2[s(s+1)-n^2]\}(\alpha^2 - 9/4) + \{\lambda_1^2 - \lambda_2^2[s(s+1)-n^2]\}s(s+1). \quad (14)$$

The reduction of $\mathcal{H}(+, \alpha^2, (R\cdot))$ with respect to the Poincaré group representations $\mathcal{H}(m,s)$ ($\operatorname{sgn} p_0 > 0$) is given by

$$\mathcal{H}(+, \alpha^2, (R\cdot)) \underset{\mathcal{P}}{=} \sum_{[n,s]} \oplus \mathcal{H}(m_{(s,n)}, s), \quad (15)$$

where the summation runs over all $[n,s]$ of the multiplicity pattern of Fig. 1 (for $\cdot = \frac{1}{2}$) and Fig. 2 (for $\cdot = 0$) of I. For the subspaces that correspond to $[n,s]$ with

$$\lambda_1^2/\lambda_2^2 > s(s+1) - n^2 \quad (\lambda_2^2 > 0),$$

the representation spaces $\mathcal{H}(m_{(s,n)}, s)$ are unitary representation spaces of $\mathcal{P}$ and describe, therefore, elementary particles. The other irreducible representation spaces of $\mathcal{P}$ in (15) describe nonunitary representations with imaginary momenta. The states with quantum number n and $-n$ have the same mass, as should be the case according to our considerations in I: To every particle there exists an "antiparticle" of the same mass.

λ_1^2 and λ_2^2 are empirical constants which must be the same for every physical system, i.e., in every $\mathcal{H}(+, \alpha^2, (R\cdot))$. We see that the appearance of imaginary-momenta representations of $\mathcal{P}$ depends upon the value of these empirical constants; if $\lambda_2^2 < 0$, only particle representations of $\mathcal{P}$ appear and m^2 is unbounded. It appears, however, that the mass spectrum obtained with any $\lambda_2^2 < 0$ is inconsistent with the experimental mass spectrum of particles and resonances. Therefore, our model predicts that there exists a highest spin and a highest mass of resonances.[14] This unexpected result is, however, strongly dependent upon the model assumptions about the form of the symmetry-breaking operator (10). For example, if we drop the requirement that $\Lambda^2 \to$ const for the Majorana representation, we may find a form of Λ^2 that has a positive-definite spectrum and leads to a mass spectrum not in contradiction with the presently known experimental spectrum. Adjoining of the intrinsic symmetries could also change this condition.

The physical interpretation of the imaginary-momenta representations of $\mathcal{P}$ is not clear, and we only assumed that they do not represent resonances.

For the particular case $(\alpha^2 - 9/4) \approx 0$, we obtain the mass spectrum

$$m_{(s,n)}^2 = m_0^2 + \lambda_1^2 s(s+1) - \lambda_2^2 s(s+1) \times [s(s+1) - n^2], \quad (16)$$

with $m_0^2 \approx 0$.

The comparison of the mass formula (16) with the experimental data has been done in Ref. 15 for the meson tower that consists of the ρ, A_2, R, and STU band. The empirical constants have the following values:

$$\lambda_1^2 = 0.30 \pm 0.01 \text{ BeV}^2, \quad \lambda_2^2 = 0.0061 \pm 0.0002 \text{ BeV}^2, \quad \lambda_1/\lambda_2 \approx 7. \quad (17)$$

It is in the spirit of our approach that the symmetry-breaking constants are universal and that the different physical systems are described by the different representations $(\alpha^2(R,\cdot))$ and, therefore, characterized by different values of α and $(R,\cdot)$. We therefore use the values (17), which were determined from the property of the ρ tower, for all meson and baryon towers and have as the only adjustable parameter for the mass spectrum of one tower the constant α^2, which we determine from the mass of the leading member of the tower.

We shall now compare the predictions of our model with the experimental particle spectrum.

A candidate for the representation $\mathcal{H}^+(\alpha^2(R,0))$ with $\alpha^2 \approx 9/4$ is the tower of $I=0$ mesons that starts with

$$\begin{matrix} \sigma & & \\ \omega & \omega & \\ f^l & D & f^h. \end{matrix}$$

The experimental masses of these mesons are in agreement with the predictions of the mass formula (16). This would then predict that $S^P = 2^-$ for the D and the broad f bump consists of two 2^+ resonances; i.e., f is split like $\mathcal{A}_2$.

Another meson tower fits the representation with $\alpha^2 = 3.79$. This starts with the mesons

$$\begin{matrix} \epsilon_0 & (\text{mass} \approx 681 \text{ MeV}) & \\ \phi & \phi & \\ f^{*l} & E & f^{*h} \end{matrix}$$

and predicts that E has $S^P = 2^-$ and that the broad f^* consists of two resonances with $S^P = 2^+$. ϵ_0 is the $J^{PG} = 0^{++}$ state. The higher masses calculated by (16) with the value (17) and $\alpha^2 = 3.79$ agree with the experimental masses of some charge-zero resonances.

To compare the predictions of our model with the baryon spectrum, we first calculate the masses of the N tower. From the experimental value of the nucleon mass, $m_{s=1/2,n=1/2}^2 = 0.880$ BeV^2, and the values (17) for λ_1^2 and λ_2^2, we calculate from (14) the value of α^2 which characterizes the N tower:

$$\alpha_N^2 - 9/4 = 2.21 \pm 0.01. \quad (18)$$

The values of $m_{(s,n)}^2$ that are calculated by the mass

[14] The existence of an "ionization point" has also been conjectured by completely different arguments from the new Serpukhov data: D. Horn, Phys. Letters **31B**, 30 (1970). Unfortunately, the value for the ionization point obtained from (17) and (13) is higher (approx. 11 BeV) than the one estimated from the πp total cross-section data for the baryon resonances (7.5 BeV) and is approximately the same for baryon and meson resonances.

[15] A. Böhm, Phys. Rev. Letters **23**, 436 (1969).

TABLE I. Calculated values for the mass squared of the nucleon tower with $I=0$, $Y=1$. The symbol beneath the m^2 value gives the partial wave in which a resonance that can account for this mass has been found.

	$n=\frac{1}{2}$, $p=+$ (BeV²)	$n=\frac{3}{2}$, $p=-$ (BeV²)	$n=\frac{5}{2}$, $p=+$ (BeV²)	$n=\frac{7}{2}$, $p=-$ (BeV²)	$U=\frac{9}{2}$, $p=+$ (BeV²)
$s=\frac{1}{2}$	0.88 (input)				
$s=\frac{3}{2}$	1.67	1.77 D_{13}			
$s=\frac{5}{2}$	2.69	2.84 D_{15}	3.12 F_{15}		
$s=\frac{7}{2}$	3.68 F_{17}(?)	3.92 G_{17}	4.35 F_{17}(?)	5.02 G_{17}	
$s=\frac{9}{2}$	4.07	4.39	5.04	6.04	7.36

formula (14) with the values (17) and (18) are given in Table I.

Except for the discrepancy between the predicted and measured mass for the $s=\frac{3}{2}$ states, the agreement of our predictions in Table I, with the experimental data is very good: The D_{15} resonance is the $n=\frac{3}{2}$, $s=\frac{5}{2}$ state. The F_{15} resonance consists of two different states, $n=\frac{5}{2}$, $s=\frac{5}{2}$ and $n=\frac{1}{2}$, $s=\frac{5}{2}$; phase-shift analysis cannot resolve two F_{15} states with mass square of 2.69 and 3.12 BeV². The G_{17} resonance consists of the two states $n=\frac{7}{2}$, $s=\frac{7}{2}$ and $n=\frac{3}{2}$, $s=\frac{7}{2}$, and there is an indication of an $F_{17}(1980)$ resonance, which might consists of the $n=\frac{5}{2}$, $s=\frac{7}{2}$ and $n=\frac{1}{2}$, $s=\frac{7}{2}$ states.

The situation is similar for the other baryon towers. We shall give in Tables II and III the $m_{(s,n)}{}^2$ for the Λ tower and the Σ tower, respectively. The values of α^2 for the Λ tower and Σ tower, calculated from $m_\Lambda{}^2$ and $m_\Sigma{}^2$, respectively, are

$$\alpha_\Lambda{}^2-9/4=3.41\,, \tag{19}$$

$$\alpha_\Sigma{}^2-9/4=4.00\,. \tag{20}$$

The mass formula (14) gives with (19) and (17) the values of $m_{(s,n)}{}^2$ in BeV² for the Λ tower listed in Table II.

There is again no disagreement between the calculated values in Table II and the experimental spectrum

TABLE II. Calculated values for the mass squared in BeV² of the Λ tower. The symbol beneath the m^2 value gives the partial wave in which a resonance that can account for this mass has been found.

	$n=\frac{1}{2}$, $p=+$	$n=\frac{3}{2}$, $p=-$	$n=\frac{5}{2}$, $p=+$	$n=\frac{7}{2}$, $p=-$
$s=\frac{1}{2}$	1.24 (input)			
$s=\frac{3}{2}$	2.05	2.13 D_{03}		
$s=\frac{5}{2}$	2.98 F_{05}	3.16 D_{05}	3.46 F_{05}	
$s=\frac{7}{2}$	4.04 F_{07}	4.28 G_{07}	4.71 F_{07}	5.38 G_{07}
$s=\frac{9}{2}$	4.43	4.75	5.40 $\Lambda(2350)\frac{9}{2}^+$	6.40

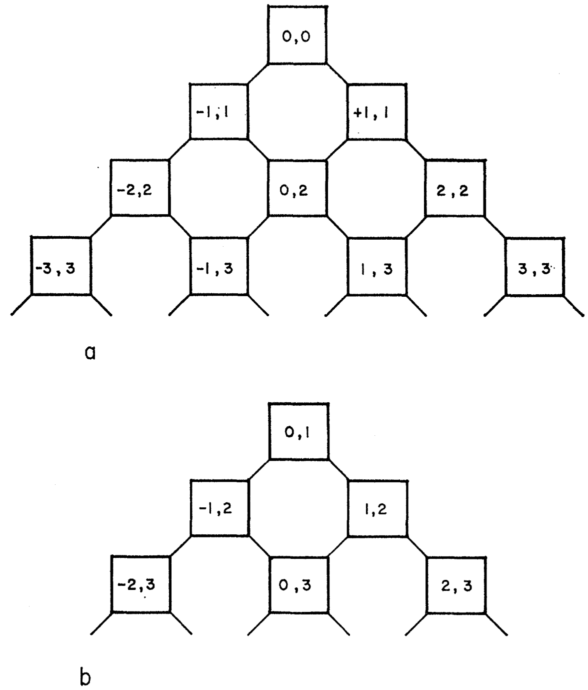

FIG. 1(a). Multiplicity pattern of the representation $\mathfrak{S}^{(R=2,0)}$. (b) Multiplicity pattern of the representation $\mathfrak{S}^{(R=2,1)}$. The numbers in the box give the values of $[n,s]$.

for the $s=\frac{5}{2}$ and $s=\frac{7}{2}$ states. Our $n=\frac{3}{2}$, $s=\frac{3}{2}$ state might be identified with the D_{03} resonance. The F_{05} resonance and the G_{07} resonance consist again of two states.

With the values (20) and (17), one calculates for the Σ tower the $m_{(s,n)}{}^2$ (in BeV²) of Table III.

The error in the calculated mass values of the above tables, originating from the error in the value (17) of λ_1 and λ_2, is approximately 10% (9–12%). Within these errors the agreement is very good.

The only disturbing feature for all three towers lies with the $s=\frac{3}{2}$ states. For the higher-spin states, only the mass of the $n=\frac{3}{2}$, $s=\frac{5}{2}$ state can exactly agree with the mass determined from the phase-shift analysis for the D_{15} or D_{05} wave, because all the other waves contain more than one state with different masses.

TABLE III. Calculated values for the mass squared in BeV² of the Σ tower. The symbol beneath the m^2 value gives the partial wave in which a resonance that can account for this mass has been found.

	$n=\frac{1}{2}$, $p=+$	$n=\frac{3}{2}$, $p=-$	$n=\frac{5}{2}$, $p=+$	$n=\frac{7}{2}$, $p=-$
$s=\frac{1}{2}$	1.41 (input)			
$s=\frac{3}{2}$	2.20 P_{13}	2.30 D_{13}		
$s=\frac{5}{2}$	3.12 F_{15}	3.31 D_{15}	3.63 F_{15}	
$s=\frac{7}{2}$	4.21 F_{17}	4.45 G_{17}	4.88	5.55

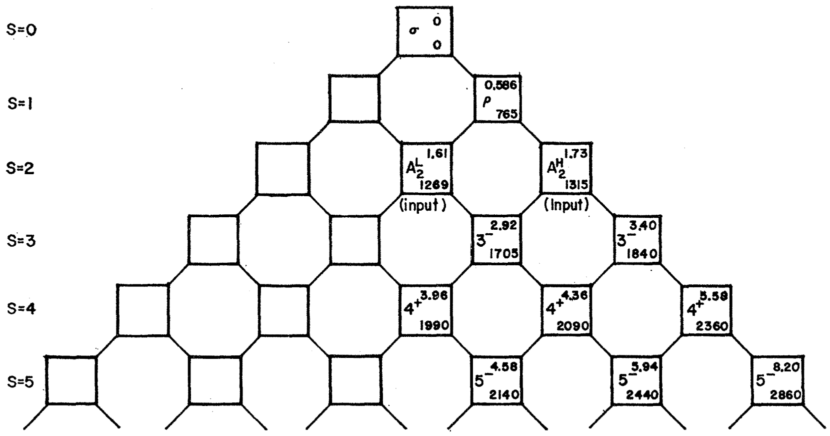

FIG. 2. Predicted particle spectrum for the representation $\mathfrak{S}^{(R=2,0)}$. The left corner of each box gives the spin parity, the number in the right upper corner is the mass squared in BeV^2, and the number in the right lower corner is the mass in MeV.

For all the meson towers, the experimentally most uncertain predictions are the existence of the $s^P=1^+$ and $s^P=2^-$ resonances. It is, therefore, reassuring to know that there exists a meson representation of $\mathfrak{S}$ which does not contain these states. This representation is obtained as the limiting case of the representations $\mathfrak{S}^{(R,0)}$ for $R\to 2$. For $R=2$ the representation space $\mathcal{H}^{(R,0)}$ of $SO(3,2)$ becomes reducible and decomposes into the two irreducible representation (irrep) spaces $\mathcal{H}^{(R=2,0)}$ and $\mathcal{H}^{(R=2,1)}$ whose multiplicity pattern is given in Figs. 1(a) and 1(b), respectively.[16,17] This happens because for $R\to 2$, $c_\pm \to 0$ [cf. Eq. (10) of I] and, therefore, the matrix elements of Γ_i between states with the same s go to zero, so that the multiplicity pattern in Fig. 2 of I decomposes into the two multiplicity patterns of Figs. 1(a) and 1(b).

From the irrep space $\mathcal{H}^{(R=2,0)}$ of $SO(3,2)$ we induce an irrep space of $\mathcal{A}_2$, which we want to call $\mathcal{H}(\alpha,(R=2,0))$. If we choose again $\eta=+1$ [cf. (23) of I], then $\mathcal{H}(\alpha=9/4,(R=2,0))$ does not contain Poincaré group representations with $s^P=1^+$ and $s^P=2^-$. Using the values (24) for the symmetry-breaking constants λ_1 and λ_2, we obtain the same mass spectrum as in Fig. 3 of Ref. 15, only that the abnormal spin-parity states (called there ρ', $A_2{}^L$, R_1, R_3, S, T) are not present.

We can, however, proceed differently and use the experimental values of the two A_2 masses as input. Then we obtain for the symmetry-breaking constants

$$\lambda_1{}^2=0.298\ \mathrm{BeV}^2,\quad \lambda_2{}^2=0.005\ \mathrm{BeV}^2, \tag{17'}$$

and the predicted mass spectrum is given in Fig. 2. Though it might appear that experimental data—especially in the lower-mass region—favor the particle spectrum of Fig. 2 over that of Fig. 3 of Ref. 15, later results—in particular in the higher-mass region for the $I=1$ resonances—appear to be in better agreement with the spectrum of Fig. 3 of Ref. 15.[18] Thus, e.g., the representation of Fig. 2 cannot account for the "anomalous" fine structure of the S bump or the existence of more than two $I=1$ mesons in the R region.[15]

It could, of course, very well be that the representation space $\mathcal{H}(\alpha^2=9/4,(R,0))$ with $R>2$ describes the $I=1$ mesons, whereas the $I=0$ mesons are described by the representation $\mathcal{H}(\alpha^2=9/4,(R=2,0))$ of Fig. 1(a). In fact, at present this would be the experimentally most favored situation. The predicted particle spectrum for the $I=0$ meson tower would then be

0^+, σ

1^-, ω

2^+, f^l $\qquad$ 2^+, f^h

3^-, $m\approx 1660$ MeV $\qquad$ 3^-, $m\approx 1840$ MeV.

There is some good evidence for an $I=0$ resonance around 1660 MeV, the $\phi(1650)\to\rho^0\pi^0$.

Note added in manuscript. Recently [P. H. Stuntebeck *et al.*, Phys. Letters **32B**, 391 (1970)], a 4-standard-deviation dip in the f peak has been reported, which indicates that there is a structure in the f similar to that in the $\mathcal{A}_2$.

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[16] J. B. Ehrman, Proc. Cambridge Phil. Soc. **53**, 290 (1957); thesis, Princeton University, 1954 (unpublished).
[17] L. Jaffe, J. Math. Phys. (to be published).
[18] Bogdan Maglić, Rapporteur's Talk, in *Proceedings of the Lund International Conference on Elementary Particles, 1969*, edited by G. von Dardel (Berlingska, Lund, Sweden, 1969).

APPENDIX A

We consider here briefly (the possibility of) representations which contain Poincaré group representations with the little group $SO(2,1)_{S_{01},S_{02},S_{12}}$. Then for unitary representations of the spectrum of $\mathcal{P}$ $M^2<0$.

Instead of the reduction $SO(3,2)\supset SO(2)\otimes SO(3)$, we need in this case the reduction

$$SO(3,2)\supset SO(1,1)_{\Gamma_3}\otimes SO(2,1)_{S_{01},S_{02},S_{12}}. \quad \text{(A1)}$$

The spectrum of the generator Γ_3 of the noncompact group $SO(1,1)$ is γ, with

$$-\infty<\gamma<+\infty. \quad \text{(A2)}$$

To find the spectrum of the Casimir operator

$$\mathcal{Q}(SO(2,1))=S_{20}^2+S_{10}^2-S_{12}^2,$$

we have to reduce the $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}{}^{(R,0)}$ representation with respect to $SO(2,1)_{S_{20},S_{10},S_{12}}$. To do this, we use the reduction of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}{}^{(R,0)}$ with respect to $SO(3,1)_{S_{\mu\nu}}$ given in Eq. (12), of I and the reduction of the $(k_0,c_\pm)$ representations of $SO(3,1)_{S_{\mu\nu}}$ with respect to $SO(2,1)$ given in Ref. 19.

Let us denote by q the eigenvalue of the operator $\mathcal{Q}(SO(2,1))$. From the result of Ref. 19, it follows that

$$(k_0,c) \underset{SO(2,1)}{=} 2\int_{q=1/4}^{\infty}\oplus dq \times D^q\oplus \sum_{k=1,2,3,\ldots,k_0}(D_k{}^{(+)}\oplus D_k{}^{(-)}), \quad \text{(A3)}$$

where D^q are the continuous-class integral-type nonexceptional representations of $SO(2,1)$ with

$$\tfrac{1}{4}\leq q<\infty. \quad \text{(A4)}$$

$D_k{}^+$ $(D_k{}^-)$ are the discrete class representations with eigenvalue $S_{12}>0$ (<0) and

$$\begin{aligned}\mathcal{Q}|D_k{}^\pm=q_{(k)}&=k(1-k), \qquad k=1,\,2,\,3,\,\ldots\\ &=-\sigma(\sigma+1), \quad \sigma=0,\,1,\,2,\,\ldots. \end{aligned} \quad \text{(A5)}$$

This, together with Eq. (12) of I,

$$(R,0) \underset{SO(1,3)}{=} \sum_{k_0=0,1,2,\ldots}^{\infty}\oplus(k_0,c_+)\oplus(k_0,c_-), \quad \text{(A6)}$$

shows that the spectrum of q in $SO(3,2)^{(R,0)}$ is given by (A4) and (A5).

By taking the expectation value between states with $p_0=0$ $p_1=p_2=0$, $p_3^2=-m^2$, we obtain from this the spectrum of $\hat{W}=M^{-2}W$:

$$\begin{aligned}\hat{W}=-\mathcal{Q}=&\ -q, \qquad \tfrac{1}{4}\leq q<\infty\\ =&\ \sigma(\sigma+1), \quad \sigma=0,1,2,\ldots\end{aligned} \quad \text{(A7)}$$

as compared to

$$\hat{W}=\mathbf{S}^2=s(s+1), \quad s=0,\,1,\,2,\,\ldots$$

[19] N. Mukunda, J. Math. Phys. **9**, 50 (1968); S. Ström, Arkiv Fysik **34**, 215 (1967).

for the case where $SO(3)_{S_i}$ was the little group.

For the spectrum of Λ^2, we obtain therewith from (10)

$$\lambda^2=\lambda_1^2-\lambda_2^2(-q+\gamma^2)=\lambda^2(q,\gamma). \quad \text{(A8)}$$

Therefore we conclude that the spectrum of Λ^2 is indefinite and continuous, with discrete points in the continuous spectra.

From (7) we obtain for the representation with $\alpha^2\approx 9/4$

$$\begin{aligned}m^2=-\lambda^2(q,\gamma)q=&\ +\{\lambda_1^2-\lambda_2^2[\sigma(\sigma+1)+\gamma]\}\sigma(\sigma+1)\\ =&\ -[\lambda_1^2-\lambda_2^2(-q+\gamma)]q.\end{aligned} \quad \text{(A9)}$$

Thus we see that the spectrum of M^2 is continuous and indefinite. For

$$\lambda^2(q,\gamma)=\lambda_1^2-\lambda_2^2(-q+\gamma)>0$$

and

$$\lambda^2(\sigma,\gamma)=\lambda_1^2-\lambda_2^2(\sigma(\sigma+1)+\gamma)<0,$$

we have unitary representations of the Poincaré group (tachyon representations) and for the other values of (q,γ) or (σ,γ) we have nonunitary representations of $\mathcal{P}$ with $m^2>0$ and $SO(2,1)_{S_{01},S_{02},S_{12}}$ as the little group.

If we choose again $\alpha^2-9/4=\epsilon$ with $|\epsilon|$ arbitrarily small but $|\epsilon|\neq 0$, we see again that $m^2\neq 0$ except when $\lambda=0$, so that (1) is always well defined. For $\lambda=0$, $B_\mu=P_\mu$ and B_μ and $L_{\mu\nu}$ generate the mass-zero representation of the Poincaré group.

For $\lambda\neq 0$, we obtain from (1)

$$\bar{B}_\mu=\lambda^{-1}B_\mu=\lambda^{-1}P_\mu+\tfrac{1}{2}m^{-1}\{P^\rho,L_{\rho\mu}\}. \quad \text{(A10)}$$

For $m^2<0$ the representation of $\mathcal{P}$ is unitary, $P_\mu{}^\dagger=P_\mu$, and one obtains from (A10)

$$\bar{B}_\mu{}^\dagger=-\bar{B}_\mu \quad \text{for } \lambda^2<0, \quad \text{(A11)}$$

i.e., if the little-group representation is of the discrete series $D_k{}^+$ or $D_k{}^-$, and

$$\bar{B}_\mu{}^\dagger=\lambda^{-1}P_\mu-\tfrac{1}{2}m^{-1}\{P^\rho,L_{\rho\mu}\} \quad \text{for } \lambda^2>0, \quad \text{(A12)}$$

i.e., if the little-group representation is of the continuous series D^q. For $m^2>0$, the representation of $\mathcal{P}$ is nonunitary, $P_\mu{}^\dagger=-P_\mu$, and one obtains from (A10)

$$\bar{B}_\mu{}^\dagger=-\bar{B}_\mu \quad \text{for } \lambda^2>0, \quad \text{(A13)}$$

i.e., if the little-group representation is of the discrete series $D_k{}^+$ or $D_k{}^-$, and

$$\bar{B}_\mu{}^\dagger=\lambda^{-1}P_\mu-\tfrac{1}{2}m^{-1}\{P^\rho,L_{\rho\mu}\} \quad \text{for } \lambda^2<0, \quad \text{(A14)}$$

i.e., if the little-group representations is of the continuous series D^q.

APPENDIX B

We show here that in a representation of $\mathfrak{S}=\mathcal{P}_{P_\mu L_{\mu\nu}} \vdash SO(3,2)_{S_{\mu\nu},\Gamma_\nu}$ in which $SO(3,2)_{S_{\mu\nu},\Gamma_\nu}$ is unitary, the generators of the physical Lorentz group $SO(3,1)_{L_{\mu\nu}}$ must be Hermitian.

From the c.r. (commutation relation),

$$[L_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma}),$$

which we abbreviate as

$$[L,S]=iS,$$

it follows by taking the adjoint and using

$$S^\dagger=S$$

that

$$[L^\dagger,S]=iS$$

and consequently,

$$[L-L^\dagger,S]=0.$$

Thus $L-L^\dagger$ is a tensor in the $SO(3,1)_{L_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu}}$ representation which commutes with $S_{\rho\sigma}$ and, therefore,

$$L-L^\dagger=\alpha M,$$

where

$$M=M_{\mu\nu}=L_{\mu\nu}-S_{\mu\nu}.$$

A consequence of this is

$$M^\dagger=(1-\alpha)M.$$

Inserting this into the c.r.,

$$[M^\dagger,M^\dagger]=iM^\dagger,$$

we obtain

$$(1-\alpha)^2[M,M]=i(1-\alpha)M,$$

which, when compared with

$$[M,M]=iM,$$

gives

$$1-\alpha=1 \quad \text{or} \quad \alpha=0,$$

so that

$$L^\dagger=L.$$

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Factorization of the Balachandran Dual-Resonance Model*

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The level structure of the Balachandran generalization of the N-point Veneziano model is considered. The model is shown to exhibit consistent factorization and level structure. Using the harmonic oscillator formalism, expressions are obtained for the vertices coupling one or two excited states to any number of ground-state particles. The form of the propagator is also obtained. Both vertices and propagators are seen to reduce to the Veneziano form when an appropriate limit is taken. Asymptotically, the degeneracy of the nth level is shown to behave like exp(const $n^{2/3}$) where the constant is given explicitly. The Gross model of N-point functions is seen to exhibit a similar asymptotic degeneracy, in contradiction to other results reported in the recent literature.

I. INTRODUCTION

RECENTLY, much work has been done on dual-resonance models,[1] in particular on those which form the simplest N-point extension of the Veneziano (beta function) model.[2] The aspects on which we concentrate are the factorization and level structure of dual-resonance models. These properties have been considered for the N-point beta function model, both directly from the integral representation[3] and also using a harmonic oscillator formulation.[4]

Balachandran[5] has recently considered a set of dual N-point functions which are generalizations of the N-point Veneziano (beta) function, nontrivial in the sense that they cannot be expressed as sums of beta functions with constant coefficients.

We demonstrate that the Balachandran model can be factorized and exhibits a consistent factorization and level structure. Introducing harmonic oscillator notation, we effectively repeat the Fubini-Gordon-Veneziano[4] procedure for this model. The asymptotic degeneracy of levels is found to behave as exp(const $n^{2/3}$) for the Balachandran model.

Before proceeding with the factorization, we will briefly define the Balachandran N-point function.[5] One first defines a homeomorphism ω of the interval $[0,1]$ onto itself, such that

(i) $\omega(0)=1$, $\omega(1)=0$,

(ii) $\omega(\omega(x))=x$,

(iii) $\omega(x)$ is holomorphic on the disks $|x|\leq 1$, $|1-x|\leq 1$. An example of such an ω is

$$\omega(x)=\frac{1-x}{1-\lambda x}, \quad -1<\lambda<\tfrac{1}{2}. \tag{1.1}$$

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[1] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); H. M. Chan and T. S. Tsun, *ibid.* **28B**, 485 (1969); M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969); C. L. Goebel and B. Sakita, *ibid.* **22**, 257 (1969).

[2] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[3] K. Bardakci and S. Mandelstam, Phys. Rev. **184**, 1640 (1969).

[4] S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969); G. Frye, Phys. Rev. D **1**, 1194 (1970); L. Susskind, *ibid.* **1**, 1182 (1970); D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters **2**, 399 (1969).

[5] A. P. Balachandran, Phys. Rev. D **1**, 2770 (1970).