Near $q=0$,

$$P^{(0)}\sim q^{-1/3},$$
$$P^{(k)}\sim \exp[C_k \ln^{2k} q]. \quad (46)$$

In the absence of satellites it is possible to construct a dual counterterm[12] because of the power behavior of $P^{(0)}$. For $k\geqslant 1$ the singularity of $P^{(k)}$ dominates that of $P^{(0)}$.

Although we are unable to show that counterterms do not exist, we consider their existence to be highly unlikely.

[12] G. Frye and K. Susskind, Phys. Letters **31B**, 589 (1970).

[13] P. G. O. Freund and R. J. Rivers, Phys. Letters **29B**, 510 (1969); P. G. O. Freund, Nuovo Cimento Letters **4**, 147 (1970).

Finally, we consider the single nonplanar orientable loop of Fig. 4, which gives the unrenormalized two-Reggeon cuts and may give some indication of the nature of the Pomeranchuk singularity.[13] In the absence of satellites the loop converges for $u<-\frac{4}{3}$ because of the power behavior of $P^{(0)}$. With satellites the singularity of $P^{(k)}$ [Eq. (46)] would cause the loop to diverge for all values of u. As a corollary, the $s^{1/3}$ behavior[11,12] associated with the branchpoints will be destroyed.

In summary, if the one-loop diagrams have any meaning at all, it is only within the restricted context of the N-point function without satellites. Increasing the level degeneracy as in Eq. (1) by including satellites seems to remove any chance of renormalization (in the planar loop) or convergence (in the nonplanar loop).

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Generalizations of the Dirac Representation

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Representation spaces of the relativistic symmetry are investigated, which are "infinite-dimensional" generalizations of the space of solutions of the Dirac equation. The representations are extended by the discrete operators C, P, and T. Application of these representations to the description of baryons and mesons is discussed.

I. INTRODUCTION

JUDGING from the experience of the past few years, it appears that Dirac's γ's are only some special cases of more general quantities with physical significance. As is well known, the usual γ_μ and $\sigma_{\mu\nu}$ are an irreducible matrix representation of the generators of $SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}$,[1] and the space of solutions of the Dirac equation is an irreducible representation space of the relativistic symmetry[2] $\mathfrak{S}=\mathcal{P}_{P_\mu,L_{\mu\nu}} \vdash SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$,[1,3] where $\vdash$ denotes semidirect sum. In the connection with infinite multiplets the applicability of several unitary representations of $SO(3,2)$ or $\mathfrak{S}$ has been investigated, e.g., the four Majorana representations[4] or the oscillatorlike representations of $SO(4,2)$,[5,6] which are in fact singleton representations[7] of $SO(3,2)$.[8]

The Dirac representation of $\mathfrak{S}$ has a great advantage as compared to these representations: It is not only an irreducible representation of $\mathfrak{S}$, but it is also an irreducible representation of the full quantum-mechanical Poincaré group, including charge conjugation, and also an irreducible representation of $\mathfrak{S}$ extended by the discrete operations, space inversion U_P, time inversion A_T, and charge conjugation U_C. In analogy to the notation for the Poincaré group, we want to call $\mathfrak{S}$ extended by U_P, A_T, and U_C the full relativistic symmetry $\mathfrak{S}^F$. The infinite-dimensional irreducible representations of $\mathfrak{S}$ considered so far are not irreducible representations of $\mathfrak{S}^F$; the discrete operations will transform out of an irreducible representation space. Thus the following question arises: Are there infinite-dimensional generalizations of the Dirac representation, i.e., are there infinite-dimensional representations of the full relativistic symmetry that remain irreducible when restricted to $\mathfrak{S}$? The answer to this question will be the subject of the present paper. It will turn out that there are two classes of infinite-

[1] The subscripts X_i on the symbol for the group G_{X_i} indicate that X_i are the generators of G. This notation is necessary to allow us to distinguish between mathematically isomorphic groups, which have different physical observables.

[2] P. Budini and C. Fronsdal, Phys. Rev. Letters **14**, 968 (1965).

[3] A. Böhm and G. B. Mainland, Fortschr. Physik **18** (1970).

[4] A. Böhm, in *Lectures in Theoretical Physics* (Gordon and Breach, New York, 1968), Vol. 10B, p. 483; Phys. Rev. **175**, 1767 (1968).

[5] Y. Nambu, in *Proceedings of the 1967 International Conference on Particles and Fields* (Interscience, New York, 1968), and references therein.

[6] A. O. Barut, in *Lectures in Theoretical Physics* (Gordon and Breach, New York, 1968), Vol. 10B, p. 377, and references therein.

[7] J. B. Ehrman, (a) Proc. Cambridge Phil. Soc. **53**, 290 (1957); (b) thesis, Princeton University, 1954 (unpublished).

[8] These are the singleton representations with the multiplicity pattern given in Figs. 7-5 and 7-14 of Ref. 7(b). For $j_{\min}=0$ the $SO(4,2)$ irreducible representation reduces to a sum of two inequivalent irreducible representations of $SO(3,2)$ with $n_{\min}=1$ and $n_{\min}=2$.

dimensional irreducible representations of $\mathfrak{S}$ which are very similar to each other and which we will call $\mathfrak{S}^{(R,1/2)}$ and $\mathfrak{S}^{(R,0)}$. $\mathfrak{S}^{(R,1/2)}$ contains only half-integer spins and is the infinite-dimensional generalization of $\mathfrak{S}^{(\text{Dirac})}$. $\mathfrak{S}^{(R,0)}$ contains only integer spins and is the infinite-dimensional generalization of the representation of $\mathfrak{S}$ which contains only zero spin [i.e., in which $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ is represented trivially]. The zero-spin representation has been used for the description of zero-spin mesons, and the Dirac representation has been used for the description of $\frac{1}{2}$-spin baryons, and it appears that $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ are uniquely predestined for the description of the infinite tower of mesons and baryons, respectively.

The application of $\mathfrak{S}^{(R,0)}$ to the description of the meson spectrum and the breaking of $\mathfrak{S}^{(R,0)}$ to give the mass spectrum has already been treated in a previous letter.[9] In the present paper we will describe in Sec. II the construction of the representations $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ disregarding symmetry breaking. In Sec. III we will investigate the action of the discrete operations U_C, U_P, and A_T in $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$; this will give us some insight into the physical interpretation. In Sec. IV we discuss the application of $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ to the description of hadrons.

II. REPRESENTATION OF RESTRICTED RELATIVE SYMMETRY

A. Restricted Relativistic Symmetry

The restricted relativistic symmetry $\mathfrak{S}$ is essentially the enveloping algebra of the Poincaré group $\mathcal{E}(\mathcal{P})$ in certain representations adjoint by a Lorentz-vector operator. $\mathfrak{S}$ is the associative algebra generated by

$$P_\mu,\ M=(P_\mu P^\mu)^{1/2},\ L_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu},\ S_{\mu\nu},\ \Gamma_\mu,\ \nu,\mu,=0,1,2,3$$

in which the multiplication is defined by the relations[10]

$$[P_\mu,P_\nu]=0\,, \tag{1a}$$

$$[L_{\mu\nu},P_\rho]=i(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu)\,, \tag{1b}$$

$$[L_{\mu\nu},L_{\rho\sigma}]=i(g_{\mu\rho}L_{\nu\sigma}+g_{\nu\sigma}L_{\mu\rho}-g_{\mu\sigma}L_{\nu\rho}-g_{\nu\rho}L_{\mu\sigma})\,, \tag{1c}$$

$$[M_{\mu\nu},S_{\rho\sigma}]=0\,, \tag{1d}$$

$$\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}=0\,, \tag{1e}$$

$$[P_\mu,S_{\rho\sigma}]=0\ [P_\mu,\Gamma_\nu]=0\,, \tag{1f}$$

$$[S_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma})\,, \tag{1g}$$

$$[L_{\rho\sigma},\Gamma_\mu]=[S_{\rho\sigma},\Gamma_\mu]=i(g_{\sigma\mu}\Gamma_\rho-g_{\rho\mu}\Gamma_\sigma)\,, \tag{1h}$$

$$[\Gamma_\rho,\Gamma_\sigma]=-iS_{\rho\sigma}\,, \tag{1i}$$

where $\mu,\nu,\rho,\sigma=0,1,2,3$ and $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$. P_μ and $L_{\mu\nu}$ are the generators[11] of the Poincaré group and represent, therefore, the usual physical observables momenta and angular momenta. The splitting $L_{\mu\nu}=M_{\mu\nu}+S_{\mu\nu}$ is familiar from the space of the solutions of the Dirac equation, which is a special case of the representation spaces of $\mathfrak{S}$. $M_{\mu\nu}$ is called the "orbital part" and $S_{\mu\nu}$ the "spin part" of the angular momentum.

We are not interested in all representations of $\mathfrak{S}$ but only in representations with the following properties.

(1) $\mathfrak{S}$ is an algebra of continuous[12] operators in a dense subspace Φ of the Hilbert space $\mathcal{H}$. This assures us that all the algebraic operations are defined.

(2) The subalgebra generated by P_μ and $L_{\mu\nu}$; $\mathcal{E}(\mathcal{P})$ integrates[13] to a unitary representation of the group $\mathcal{P}$ with $P_\mu P^\mu>0$.[14]

(3) The subalgebra generated by $S_{\mu\nu}$ and Γ_μ; $\mathcal{E}(SO(3,2)_{S_{\mu\nu},\Gamma_\nu})$ integrates to a (unitary) representation of the group $SO(3,2)$.

Requirement (2) is necessary for the physical interpretation; requirement (3) is for mathematical convenience only.[15]

(4) The representation is irreducible, i.e., there exists no proper closed subspace invariant under $\mathfrak{S}$, and the central elements of $\mathfrak{S}$ are multiplets of the unit operator.[16]

It is easy to see from the defining relations (1) that there is no operator in $\mathfrak{S}$ that transforms out of an irreducible representation of $SO(3,2)$. Similarly one can see easily that m^2, the eigenvalue of $P_\mu P^\mu$, and ϵ, the sign of the eigenvalue[17] of P_0, are invariants. Consequently, the irreducible representations of $\mathfrak{S}$ are char-

[9] A. Böhm, Phys. Rev. Letters **23**, 436 (1969).

[10] A consequence of (1) is

$$[L_{\mu\nu},S_{\rho\sigma}]=i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma}).$$

Because of this formal analogy with the commutation relations of the semidirect sum of the Lie algebra of the Poincaré group $\mathcal{L}(p_{L_{\mu\nu},P_\mu})$ and the Lie algebra $\mathcal{L}(SO(3,2))_{S_{\mu\nu},\Gamma_\mu}$, $\mathfrak{S}$ has been denoted (Ref. 2) as the semidirect product $[=\mathcal{P}_{L_{\mu\nu}P_\mu}\vdash SO(3,2)_{S_{\mu\nu},\Gamma_\mu}]$, which is misleading insofar as the parameters of the group generators $L_{\mu\nu}$ and $S_{\mu\nu}$ are not independent.

[11] The same word generator is used for two different things: (a) generator of a group, (b) generator of an associative algebra.

[12] Continuous means continuous with respect to the topology of Φ. We do not give here the mathematical details but just remark that the prescription for the construction of such a space Φ has been given in Appendix B of A. Böhm, J. Math. Phys. **8**, 1557 (1967).

[13] This means that the operator $\Delta_{\mathcal{P}}=P_0+\mathbf{P}^2+\mathbf{N}^2+\mathbf{M}^2$ is essentially self-adjoint in $\Phi\subset\mathcal{H}$.

[14] With regard to the future introduction of symmetry breaking, we should replace the requirement of "representations with $P_\mu P^\mu>0$" by "representations of $\mathcal{P}$ with the little group $SO(3)$."

[15] The unitarity is not fulfilled for the Dirac representation. A consequence of (2) and (3) is that all the linear symmetric elements of $\mathfrak{S}$ are essentially self-adjoint on Φ.

[16] Because of the requirement of integrability, the irreducible representations of $\mathcal{E}(SO(3,2))$ belong, of course, to irreducible unitary representations of the group $SO(3,2)$ and the same is true also for $\mathcal{P}$.

[17] Conventionally one uses positive- and negative-energy solutions in the space of solutions of the Dirac equation; this is not only unnecessary but also inconvenient as it has to be supplemented by the usual reinterpretation. As shown in Ref. 3, one can restrict oneself to positive-energy states to obtain an appropriate description of the spin-$\frac{1}{2}$ particle-antiparticle system.

acterized[18] by m^2, ϵ, and the irreducible representation of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ which it contains.

In the present work we restrict ourselves to representations of $\mathfrak{S}$ that contain irreducible representations of $SO(3,2)$ of a specific class, which we shall denote $(R,0)$ and $(R,\frac{1}{2})$. We shall, therefore, first give a brief description of these representations of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$; more details can be found in Refs. 19 and 7.

B. Some Properties of Irreducible Representations $(R,0)$ and $(R,\frac{1}{2})$ of $SO(3,2)$

The irreducible representation of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ can be reduced with respect to the following chains of subgroups:

$$SO(3,2)_{S_{\mu\nu},\Gamma_\mu} > SO(3,1)_{S_{\mu\nu}} > SO(3)_{S_{ij}} > SO(2)_{S_{12}}, \qquad (2)$$

$$SO(3,2)_{S_{\mu\nu},\Gamma_\mu} > SO(3,1)_{S_{ij},\Gamma_i} > SO(3)_{S_{ij}} > SO(2)_{S_{12}}, \qquad (3)$$

$$SO(3,2)_{S_{\mu\nu},\Gamma_\mu} > SO(3)_{S_{ij}} \otimes SO(2)_{\Gamma_0} > SO(3)_{S_{ij}} > SO(2)_{S_{12}} \quad (\mu,\nu=0,1,2,3;\, i,j=1,2,3). \qquad (4)$$

$SO(3,1)_{S_{\mu\nu}}$ is the (spin part of the) homogeneous Lorentz group, $SO(3,1)_{S_{ij}\Gamma_i}$ is algebraically equivalent to $SO(3,1)_{S_{\mu\nu}}$ but has a different physical meaning, and $SO(3)_{S_{ij}}$ is the spin-rotation group.

The irreducible representations $(R,0)$ and $(R,\frac{1}{2})$ have the following properties.

(1) They contain an irreducible representation of the maximal compact subgroup $SO(3)_{S_{ij}} \otimes SO(2)_{\Gamma_0}$ at most once (singleton representations[7]). Therefore, the basis vectors in the irreducible representation space $\mathfrak{H}^{(R,\cdot)}$ (where the dot stands for 0 or $\frac{1}{2}$) are completely characterized by the system of commuting operators

$$\mathbf{S}^2,\ S_{12},\ \Gamma_0. \qquad (4')$$

We denote these basis vectors by

$$|s,n,s_3\rangle \qquad (4'')$$

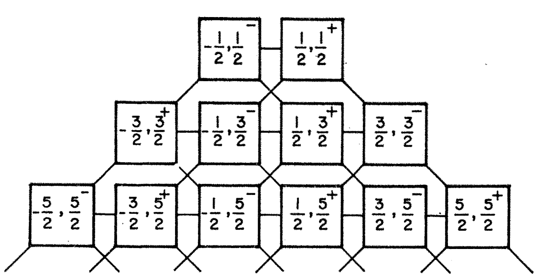

FIG. 1. Multiplicity pattern of the half-integer-spin representation $(R,\frac{1}{2})$ of $SO(3,2)$. The numbers in the boxes give the values of $[n,s^P]$.

[18] We shall see later that they are also completely characterized by these quantities.

[19] L. Jaffe, J. Math. Phys. (to be published).

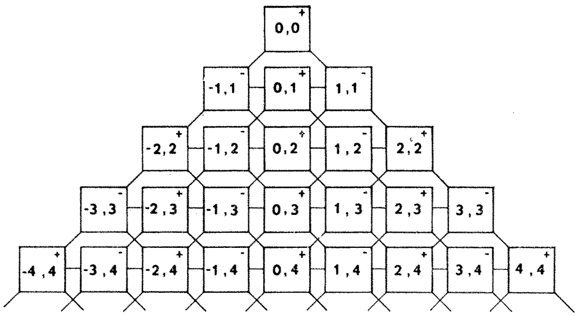

FIG. 2. Multiplicity pattern of the integer-spin representation $(R,0)$ of $SO(3,2)$.

and they have the property

$$\Gamma_0|s,n,s_3\rangle = n|s,n,s_3\rangle,$$
$$\mathbf{S}^2|s,n,s_3\rangle = s(s+1)|s,n,s_3\rangle, \qquad (4''')$$
$$S_{12}|s,n,s_3\rangle = s_3|s,n,s_3\rangle.$$

(2) They are characterized by one continuous parameter $R>2$, which is the eigenvalue of the second-order Casimir operator

$$\Gamma_\mu\Gamma^\mu + \tfrac{1}{2}S_{\mu\nu}S^{\mu\nu} = R \qquad (5)$$

and which is connected with the eigenvalue of the fourth-order Casimir operator $P_1 = -W_\mu W^\mu$, with $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}\Gamma^\sigma$, by

$$P_1 = \tfrac{1}{4}R(R-2). \qquad (6)$$

(3) The reduction of the irreducible representation $(R,\frac{1}{2})$ and $(R,0)$ with respect to $SO(3)_{S_{ij}} \times SO(2)_{\Gamma_0}$ is given by the multiplicity pattern[7,19] of Figs. 1 and 2, respectively. Each box $[n,s]$ in the figures characterizes the irreducible representation of $SO(2)_{\Gamma_0} \times SO(3)_{S_{ij}}$ which it contains, and the lines connecting these boxes indicate that there are nonzero matrix elements of Γ_i and S_{0i} between these irreducible representation spaces of $SO(2) \times SO(3)$.

(4) The reduction of the irreducible representation $(R,\frac{1}{2})$ and $(R,0)$ with respect to $SO(3,1)_{S_{\mu\nu}}$ [and also with respect to $SO(3,1)_{S_{ij},\Gamma_i}$] is given by

$$\mathfrak{H}^{(R,1/2)} \underset{SO(3,1)}{=} \sum_{k_0=\pm\frac{1}{2},\pm\frac{3}{2},\pm\ldots}^{\infty} \oplus\,\mathfrak{H}(k_0,c), \qquad (7)$$

$$\mathfrak{H}^{(R,0)} \underset{SO(3,1)}{=} \sum_{k_0=\pm 0,\pm 1,\pm 2\ldots}^{\infty} \oplus\,\mathfrak{H}(k_0,c) \qquad (8)$$

($\underset{G}{=}$ means reduction with respect to the subgroup G), where $\mathfrak{H}(k_0,c)$ are the usual irreducible representation spaces of the group $SO(3,1)$.[20] Thus c is also an in-

[20] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964). In distinction to the notation in Naimark's book we use the notation

$$c = |c^{(\text{Naimark})}|, \quad k_0 = \text{sign}(c^{\text{Naimark}})\,k_0^{(\text{Naimark})}.$$

variant of the $SO(3,2)$ representation; it is connected with R by

$$ic=[\tfrac{1}{2}(R-2)]^{1/2}. \qquad (9)$$

As the reduction with respect to $SO(3,1)$ is discrete, we can introduce into $\mathcal{H}^{(R,\cdot)}$ a basis system in which the noncompact subgroup $SO(3,1)_{S_{\mu\nu}}$ is diagonal. This basis we denote by

$$f^{k_c j}{}_{j_3}$$

and it has the property

$$-\tfrac{1}{2}S_\mu S^{\mu\nu} f^{k_0 j}{}_{j_3}=(1-c^2-k_0{}^2)f^{k_0 j}{}_{j_3},$$

$$(\sum_{i=1,2,3} S_{i0}S_i)f^{k_0 j}{}_{j_3}=ik_0 c f^{k_0 j}{}_{j_3}, \qquad (2')$$

$$\mathbf{S}^2 f^{k_0 j}{}_{j_3}=j(j+1)f^{k_0 j}{}_{j_3},\quad S_{12}f^{k_0 j}{}_{j_3}=j_3 f_{j_3}.$$

The transformation coefficients $\langle k_0|n\rangle_{(j)}$ between the two basis systems (2′) and (4″)

$$|sns_3\rangle=\sum_{k_0} f^{k_0\, j=s}{}_{j_3=s_3}\langle k_0|n\rangle_{(j)} \qquad (10)$$

have been calculated in Ref. 19 and are listed in Appendix II of Ref. 19.

The matrix elements of the generators in the two basis systems have also been calculated there and are listed in Appendix III of Ref. 19. For Γ_3 we have, e.g.,

$$\Gamma_3|sns_3\rangle=\sum_{s'n's_3'}|s',n',s_3'\rangle\langle s's_3'|1,0,s,s_3\rangle \times(2j+1)^{1/2}\Gamma_{s's}{}^{n'n}, \qquad (11)$$

where $\langle s's_3'|1,0,s,s_3\rangle$ is the $SU(2)$ Clebsch-Gordan coefficient and $\Gamma_{s's}{}^{n'n}$ is the reduced matrix element listed in Eqs. (III 15) and (III 18) of Ref. 19. $\Gamma_{s's}{}^{n'n}\neq 0$ only for $n'=n\pm1$ and $s'=s\pm1$, s.

To illustrate that the irreducible representation $(R,\frac{1}{2})$ are infinite-dimensional generalizations of the four-dimensional Dirac representation[21] $(R=-\frac{5}{2},\frac{1}{2})$, we list here its corresponding properties: The reduction of $(R=-\frac{5}{2},\frac{1}{2})$ with respect to $SO(3)_{S_{ij}}\times SO(2)_{\Gamma_0}$ is given by the multiplicity pattern

$$[n=-\tfrac{1}{2},\, s=\tfrac{1}{2}]\leftrightarrow[n=\tfrac{1}{2},\, s=\tfrac{1}{2}]. \qquad (12)$$

The reduction with respect to $SO(3,1)_{S_{\mu\nu}}$ is given by

$$\mathcal{H}^{(R=-5/2,1/2)} \underset{SO(3,1)}{=} \sum_{k_0=\pm\frac{1}{2}} \oplus\mathcal{H}(k_0,c=\tfrac{3}{2}). \qquad (12')$$

Comparing (12) with Fig. 1, we see that the "lowest" states of $(R,\frac{1}{2})$ correspond to the states of the Dirac representation. In the same way we can consider $(R,0)$ as the infinite-dimensional generalization of the one-dimensional trivial representation $(R=0,\,0)$ of $SO(3,2)$, with the multiplicity pattern (see Fig. 2)

$$[n=0,\, s=0] \qquad (13)$$

and the reduction with respect to $SO(3,1)$

$$\mathcal{H}^{(R=0,0)} \underset{SO(3,1)}{=} \mathcal{H}(k_0=0,c=1). \qquad (14)$$

C. Representations $\mathfrak{S}^{(R,1/2)}$ and $\mathfrak{S}^{(R,0)}$ of Relativistic Symmetry

To induce the representations $(R,\cdot)$ of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ to irreducible representations $\mathfrak{S}^{(R,\cdot)}$ of $\mathfrak{S}$, we start with the generalized eigenvectors

$$|\mathbf{p},\zeta\rangle \qquad (15)$$

of the system of commuting operators P_i [22] $(i=1,2,3)$,

$$P_i|\mathbf{p},\zeta\rangle=p_i|\mathbf{p},\zeta\rangle; \qquad (16)$$

here ζ is a (set of) degeneracy parameter distinguishing the different generalized vectors with the property (16). We now define as usual

$$w_\mu=\tfrac{1}{2}\epsilon_{\mu\rho\sigma\nu}L^{\rho\sigma}P^\nu=\tfrac{1}{2}\epsilon_{\mu\rho\sigma\nu}S^{\rho\sigma}P^\nu \qquad (17)$$

[for the second equality we have used one of the defining relations (1e)],

$$W=-w_\mu w^\mu=\tfrac{1}{2}S_{\rho\sigma}S^{\rho\sigma}P_\nu P^\nu-S_{\rho\mu}S_\sigma{}^\mu P^\rho P^\sigma, \qquad (18)$$

and $U(L(p))$, the representative of the rotation-free Lorentz transformation with $L(p)p=(m,0,0,0)$. Then

$$|\mathbf{p},=0,\zeta\rangle=U(L(p))|\mathbf{p},\zeta\rangle \qquad (19)$$

and a straightforward calculation gives

$$U^{-1}(L(p))w_3U(L(p))|\mathbf{p},\zeta\rangle =mU^{-1}(L(p))S_{12}|\mathbf{p}=0,\zeta\rangle, \qquad (20)$$

$$W|\mathbf{p},\zeta\rangle=m^2U^{-1}(L(p))\tfrac{1}{2}S_{ij}S^{ij}|\mathbf{p}=0,\zeta\rangle, \qquad (21)$$

$$P_\mu\Gamma^\mu|\mathbf{p},\zeta\rangle=mU^{-1}(L(p))\Gamma_0|\mathbf{p}=0,\zeta\rangle. \qquad (22)$$

From the fact that $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ commutes with P_μ, it follows that the states (19) are transformed into each other by $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ transformations. Therefore, the set $|\mathbf{p}=0,\zeta\rangle$ spans a representation space of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$.

Since we restrict ourselves to irreducible representations of $\mathfrak{S}$, this representation space of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$

[21] The Dirac representation (Ref. 3) is obtained if one requires in addition to the defining relations (1) the additional relation $\{\Gamma_\rho,\Gamma_\sigma\}=\frac{1}{2}g_{\rho\sigma}$ ("representation relation"). A representation relation also exists for the Majorana representations. Unfortunately, for the representations $(R,\cdot)$ we could not find such a simple algebraic relation that determines their properties.

[22] To make this statement rigorous we remark that from the assumed integrability of the representation of $\mathcal{E}(\mathcal{P})$ it follows that P_i are essentially self-adjoint on Φ and strongly commuting. Let Φ be the suitable constructed nuclear space (see Ref. 12) and $\Phi^\times$ its conjugate such that $\Phi\subset H\subset\Phi^\times$ is a Gelfand triplet; then it follows from the "Dirac spectral theorem" [see, e.g., K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups* (Polish Scientific Publishers, Warszawa, 1968), Ch. II, or A. Böhm, in *Boulder Lectures in Theoretical Physics* (Colorado U. P., Boulder, 1966), Vol. 9A, p. 255.] that there exists $|p,\zeta\rangle\in\Phi^\times$ such that (16) is true. P_i is here the extension of the operator P_i of Φ to a continuous operator in $\Phi^\times$. Finite group transformations $U(a,\Lambda)$ can be extended to continuous operators in $\Phi^\times$ and we call them $U(a,\Lambda)$ again. Generally, we shall use the same symbol for an operator in $\Phi^\times$ and its restrictions to any subspace.

must be irreducible. We choose it to be the representation space of $(R,\cdot)$. In this irreducible representation space of $(R,\cdot)$ we choose the basis (4″). So we have

$$|\mathbf{p}=0,\zeta\rangle=|\mathbf{p}=0,\,s,\,n,\,s_3,\,\xi\rangle\,, \tag{23}$$

where ξ is a possible further degeneracy parameter, and we find from comparison of (4‴) with (20)–(22) that

$$P_\mu\Gamma^\mu|\mathbf{p},s,n,s_3,\xi\rangle=mn|\mathbf{p},s,n,s_3,\xi\rangle\,, \tag{24}$$

$$W|\mathbf{p},s,n,s_3,\xi\rangle=m^2s(s+1)|\mathbf{p},s,n,s_3,\xi\rangle\,, \tag{25}$$

$$U^{-1}(L(p))w_3U(L(p))|\mathbf{p},s,n,s_3,\xi\rangle=ms_3|\mathbf{p},s,n,s_3,\xi\rangle\,. \tag{26}$$

From (25) and (26) we see that $|\mathbf{p},s,n,s_3,\xi\rangle$ is the usual canonical basis of the irreducible Poincaré group representation (s,m). From this it follows that no operation of $\mathcal{P}$ or $\mathcal{E}(\mathcal{P})$ can change ξ, so that ξ is redundant and

$$P_i,\ U^{-1}(L(p))w_3U(L(p)),\ W,\ P_\mu\Gamma^\mu \tag{23'}$$

constitute a complete system of commuting operators for $\mathfrak{S}^{(R,\cdot)}$. We now also see that $(m,\epsilon,(R,\cdot))$ completely specify the irreducible representations of $\mathfrak{S}$, because W, and therefore the spin s, is no longer an invariant, but has the spectrum given by the multiplicity pattern of Figs. 1 and 2. The operators that change the spin and transform between different—equivalent or inequivalent—irreducible representations of $\mathcal{P}$ are Γ_i; e.g.,

$$(L^{-1}(p))\Gamma_3U(L(p))|\mathbf{p},s,n,s_3\rangle=\sum|\mathbf{p},s',n',s_3'\rangle\times\langle s's_3'|1,0,ss_3\rangle\Gamma_{s's}{}^{n'n}\,. \tag{11'}$$

The irreducible representation space of $\mathfrak{S}_{(m)}{}^{(R,\cdot)}$ which is spanned by these generalized eigenvectors $|\mathbf{p},s,n,s_3\rangle$ is called $\mathcal{H}_{(m)}{}^{(R,\cdot)}$.

The multiplicity pattern in Figs. 1 and 2 of $SO(3,2)^{(R,1/2)}$ and $SO(3,2)^{(R,0)}$ now extends to the multiplicity pattern of $\mathfrak{S}^{(R,1/2)}$ and $\mathfrak{S}^{(R,0)}$. To each box $[n,s]$ now corresponds the set of states $\{|\mathbf{p},s,s_3,n\rangle, -s\leq s\leq s,\ p$ such that $p_\mu p^\mu=m^2\}$, so that to each box now corresponds the irreducible representation space of the Poincaré group $\mathcal{H}(m,s,n)$, where n distinguishes here between the equivalent irreducible representations of $\mathcal{P}$ with mass m and spin s. As an irreducible representation space of $\mathcal{P}$ is the mathematical image of an "elementary particle," each box in the multiplicity pattern corresponds to an elementary particle and each elementary particle is now not only characterized by mass m and spin s but in addition by the new quantum number n.[23] Since $\mathcal{H}^{(R,0)}$ contains only integer spins, and $\mathcal{H}^{(R,1/2)}$ contains only half-integer spins, $\mathfrak{S}^{(R,0)}$ describes an infinite tower of mesons and $\mathfrak{S}^{(R,1/2)}$ an infinite tower of baryons.

[23] The introduction of such a new quantum number, corresponding to the principal quantum number of the hydrogen atom, has been advocated before, in particular, by A. C. Barut (Ref. 6) and Y. Nambu (Ref. 5); however, the spectrum of their quantum number n is, owing to their use of a different representation, different from that of our n.

So far every particle of the infinite tower of particles described by $\mathcal{H}^{(R,\cdot)}(m)$ has the same mass m. To obtain a realistic mass spectrum, one has to break the relativistic symmetry $\mathfrak{S}$ by a "generalized wave equation" such as, e.g., relation (10) in Ref. 4(b) for the Majorana representation. Results of such a symmetry breaking for the meson representation $\mathfrak{S}^{(R,0)}$ have been described in Ref. 9. Then m will depend upon n and s and $\mathcal{H}(m(n,s),s,n)$ will describe an elementary particle with spin s, principal quantum number n, and mass $m=m(s,n)$.

We have here only introduced the canonical basis (23), because it is only this basis which we will need for the further investigations. Better insight into the concept of the relativistic symmetry can be obtained from the spinor basis, which is considered in the Appendix.

III. PROPERTIES OF DISCRETE OPERATORS C, P, T

We shall now study the representations of the full relativistic symmetry $\mathfrak{S}^F$ which is obtained from $\mathfrak{S}$ by adjoining to it the discrete operations of charge conjugation U_C, space inversion U_P, and time inversion A_T. This will give us further insight into the physical interpretation of the states of the representation space of $\mathfrak{S}^{(R,1/2)}$ and $\mathfrak{S}^{(R,0)}$. The main problem that remained open in Ref. 9 was the interpretation of the states with negative quantum number n for the representation $\mathfrak{S}^{(R,0)}$. From the analogy with the Dirac representation, we would expect that in $\mathfrak{S}^{(R,1/2)}$ the antibaryons can be assigned to the negative-n states. In the following we shall *derive* that the negative-n states are the U_C transforms of the positive-n states so that the negative-n states in $\mathfrak{S}^{(R,1/2)}$ not only can but must be the antibaryon states. For the meson representation we shall derive that the negative-n states are not the U_C transforms but the A_T transforms of the positive-n states.

The relations of the discrete operations with the generators of the Poincaré group follow from their physical interpretation and are well known[24,25]:

$$U_CP_\mu U_C^{-1}=P_\mu\,,\qquad U_CL_{\mu\nu}U_C^{-1}=L_{\mu\nu}\,,\qquad U_C\text{ unitary;} \tag{27}$$

$$U_PP_iU_P^{-1}=-P_i\,,\qquad U_PP_0U_P^{-1}=P_0\,,\qquad U_PL_{ij}U_P^{-1}=L_{ij}\,,\qquad U_PL_{i0}U_P^{-1}=-L_{i0}\,,\qquad U_P\text{ unitary;} \tag{28}$$

$$A_TP_iA_T^{-1}=-P_i\,,\qquad A_TP_0A_T^{-1}=P_0\,,\qquad A_TL_{ij}A_T^{-1}=-L_{ij}\,,\qquad A_TL_{i0}A_T^{-1}=L_{i0}\,,\qquad A_T\text{ antiunitary.} \tag{29}$$

[24] E. P. Wigner, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gürsey (Gordon and Breach, New York, 1964), p. 37.

[25] Haim Goldberg, Nuovo Cimento **60A**, 509 (1969).

The equivalent of these relations are relations (2) of Ref. 25 and (7.9) of Ref. 24.

The relations among the discrete operations U_P, U_C, and A_T are given in the multiplication table Table I of Ref. 25 and are derived from their physical interpretation.

We consider the two representations $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ simultaneously and assume that the integer-spin representation $\mathfrak{S}^{(R,0)}$ describes mesons and the half-integer-spin representation $\mathfrak{S}^{(R,1/2)}$ describes baryons. Therefore, the phase factors will be[25]

$$(\pi_C,(-1)^{2s}\epsilon_T,(-1)^{2s}\epsilon_I,(-1)^{2s}\epsilon_C)=(+\ +++) \quad \text{for } \mathfrak{S}^{(R,0)} \tag{30}$$

and

$$(\pi_C,(-1)^{2s}\epsilon_T,(-1)^{2s}\epsilon_I,(-1)^{2s}\epsilon_C)=(-\ +++) \quad \text{for } \mathfrak{S}^{(R,1/2)}. \tag{31}$$

It remains to determine the relations between Γ_μ and the discrete operations. We shall only utilize those relations which are a consequence of the previous relations.

If we assume that Γ_0 and U_P can be diagonalized simultaneously, as is the case for the Dirac representation, then, because of the equivalence of the two $SO(3,1)$ subgroups $SO(3,1)_{S_{\mu\nu}}$ and $SO(3,1)_{S_{ij},\Gamma_i}$ and the vector character of S_{i0} as a consequence of (28), it follows that Γ_i can be a vector:

$$U_P\Gamma_iU_P^{-1}=-\Gamma_i \tag{32}$$

and, as a consequence of these properties, it can be shown that for all singleton representations of $SO(3,2)$

$$U_P=\eta e^{i\pi\Gamma_0} \qquad \text{on the states at rest}, \tag{33a}$$

or, more generally,

$$U_P=\eta e^{i\pi P_\mu\Gamma^\mu/M}S, \tag{33b}$$

where S is the operation $(p_0p_i)\to(p_0,-p_i)$. η is an over-all phase factor. Thus the subspaces $\mathcal{H}(m,s,n)$ are parity eigenspaces, or to each box in the multiplicity pattern corresponds a definite parity. We fix the phase factor if we assign for mesons to $[n=0,s=0]$ the parity $+1$, i.e., the σ states $s^P=0^+$, and for baryons to $[n=\frac{1}{2},s=\frac{1}{2}]$ the parity $+1$, i.e., the baryon state $s^P=\frac{1}{2}^+$. Then

$$\begin{aligned}\eta&=1 \qquad \text{for } \mathfrak{S}^{(R,0)} \quad (\text{mesons})\\ &=e^{-i\pi/2} \text{ for } \mathfrak{S}^{(R,1/2)} \quad (\text{baryons}).\end{aligned} \tag{34}$$

From the relation[25]

$$U_PU_C=\pi_CU_CU_P \quad \text{with } \pi_C=\begin{cases}+ \text{ for mesons}\\ - \text{ for baryons}\end{cases} \tag{35a}$$

and from (33a), it follows that

$$\Gamma_0U_C=\pi_CU_C\Gamma_0. \tag{36}$$

From the relation[25]

$$\epsilon_I\epsilon_TU_PA_T=A_TU_P \quad [\epsilon_T(-1)^{2s}=\epsilon_I(-1)^{2s}=1] \tag{35b}$$

and (33a), it follows (because of the antilinearity of A_T) that

$$\eta^2\Gamma_0A_T=-A_T\Gamma_0 \quad \text{with } \eta^2=\begin{cases}+ \text{ for mesons}\\ - \text{ for baryons.}\end{cases} \tag{37}$$

From the relations (33), (34), (36), and (37), we obtain the actions of U_P, U_C, and A_T on the generalized basis states (and therewith on every state of the representation space).

For baryons and mesons we have, from (33) with (34),

$$U_P|\mathbf{p},s,s_3,n\rangle=(-1)^{[n]}|-\mathbf{p},s,s_3,n\rangle, \tag{38}$$

where $[n]$ is the largest integer which is smaller than or equal to n. $(-1)^{[n]}$ is given in the upper right-hand corner of the boxes of the pattern in Figs. 1 and 2.

For baryons and mesons we obtain from (36)

$$\Gamma_0(U_C|\mathbf{p}=0,s,s_3,n\rangle)=n\pi_C(U_C|\mathbf{p}=0,s,s_3,n\rangle). \tag{39}$$

Because of (27),

$$U(0,\Lambda)U_C|\mathbf{p},s,s_3,n\rangle=\sum_{s_3'}U_c|(\Lambda p)_i,s,s_3',n\rangle \times D_{s_3's_3}{}^s(R), \tag{27'}$$

$$U(a)U_C|\mathbf{p},s,s_3,n\rangle=e^{ia_\mu p^\mu}U_C|\mathbf{p},s,s_3,n\rangle,$$

we see that the state $U_C|\mathbf{p},s,s_3,n\rangle$ has the same transformation properties under the Poincaré transformations as the state $|\mathbf{p},s,s_3,n\rangle$. A consequence of (39) and (27′) is

$$(1/M)P_\mu\Gamma^\mu(U_C|\mathbf{p},s,s_3,n\rangle)=n\pi_C(U_C|\mathbf{p},s,s_3,n\rangle) \tag{40a}$$

or

$$(1/M)P_\mu\Gamma^\mu(U_C|\mathbf{p},s,s_3,\pi_Cn\rangle)=n(U_C|\mathbf{p},s,s_3,\pi_Cn\rangle). \tag{40b}$$

Comparing (27′) and (40b) with the corresponding equations for $|p,s,s_3,\pi_Cn\rangle$, we see that

$$|\mathbf{p},s,s_3,n\rangle \quad \text{and} \quad U_C|\mathbf{p},s,s_3,\pi_Cn\rangle$$

have the same transformations under all operations of $\mathfrak{S}$. If we assume that (23′) is not only a complete system of commuting operators for $\mathfrak{S}^{(R,\cdot)}$ but is also a complete system for the irreducible representation of the full relativistic symmetry, then

$$U_C|\mathbf{p},s,s_3,n\rangle=a|\mathbf{p},s,s_3,\pi_Cn\rangle, \tag{41a}$$

where $a=a(p,s,s_3,n)$ is a proportionality factor. From (27′) one sees that a must be independent of s_3 and p, so that

$$a=a(n,s). \tag{41b}$$

From the phase convention $U_C{}^2=1$ and (41), one obtains further

$$a(n,s)a(\pi_C n,s)=1. \tag{42}$$

For baryons, $\pi_C=-1$ and (41a) is

$$U_C|\mathbf{p},s,s_3,n\rangle=a(n,s)|\mathbf{p},s,s_3,-n\rangle. \tag{41c}$$

We assign the baryon (with parity convention $+1$) to the states belonging to the box $[n=\frac{1}{2}, s=\frac{1}{2}]$ of the multiplicity pattern. Then it follows from (41c) and the physical interpretation of U_C that the states corresponding to $[n=-\frac{1}{2}, s=\frac{1}{2}]$ which have parity -1 must be antibaryon states in agreement with our experience. Correspondingly for the baryon resonances, which we assign to the higher s states of the multiplicity pattern, we have that the $[n,s]$ with $n>0$ represent the baryons and with $n<0$ represent the antibaryons.

For meson $\pi_C=+1$ we conclude from (41a), (42), and the unitarity of U_C that

$$U_C|\mathbf{p},s,s_3,n\rangle=a(n,s)|\mathbf{p},s,s_3,n\rangle, \tag{41d}$$

with

$$a(n,s)=a^*(n,s) \quad\text{or}\quad a(n,s)=+1 \text{ or } -1. \tag{41e}$$

Thus the mesons assigned to this representation are eigenstates of the charge-conjugation operator and have C parity $+1$ or -1.

We remark that (41d) has been derived under the assumption that (23′) is already a complete system of commutating observables which is at best true for noncharged mesons and thus $a(n,s)$ is the usual C_n. If the system (23′) of commuting observables is incomplete, because of the presence of some additional quantum number like, e.g., charges, U_C may transform out of an irreducible representation space of $\mathfrak{S}^{(R,0)}$. (To obtain U_C eigenstates, we would have to form

$$|\mathbf{p},s,n,\pm\rangle=|\mathbf{p},s,n\rangle\pm U_C|\mathbf{p},s,n\rangle, \tag{43}$$

which are, however, unphysical because they are not charge eigenstates.)

From (37) we obtain for meson and baryon rest states

$$\Gamma_0(A_T|\mathbf{p}=0,s,s_3,n\rangle=-\eta^2(A_T|\mathbf{p}=0,s,s_3,n\rangle). \tag{44a}$$

From this, one calculates using (29) and

$$A_TU(L^{-1}(p))=U(L(p))A_T, \tag{29′}$$

that

$$(1/M)\Gamma_\mu P^\mu(A_T|\mathbf{p}s,s_3,n\rangle)=-\eta^2n(A_T|\mathbf{p},s,s_3n\rangle). \tag{44b}$$

(29′) is a consequence of the general relation (7.9b) of Ref. 24:

$$A_TU(\mathbf{a},\mathbf{B})=U(-\sigma\mathbf{a}^*\sigma,\mathbf{B}^*\sigma)A_T, \tag{29″a}$$

where

$$\begin{aligned}&\mathbf{a}=a_0+\mathbf{a}\boldsymbol{\sigma},\\ &\mathbf{B}=D^{k_0=1/2,c=3/2}(\Lambda),\\ &\boldsymbol{\sigma}=-\sigma_2=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}:.\end{aligned} \tag{29″b}$$

From (29″a) one derives [most conveniently using the notation (29″b) and proceeding along the same pattern as in Ref. 24]

$$U(\Lambda)[\sum_{s_3}(A_T|-\mathbf{p},s,s_3,n\rangle)C_{s_3\kappa}^{-1}]$$
$$=\sum_{\kappa'}[\sum_{s_3'}(A_T|-(\Lambda p)_i,s,s_3',n\rangle)C_{s_3'\kappa'}^{-1}] \times D_{\kappa'\kappa}^{(s)}(R), \tag{45}$$

$$U(a)[\sum_{s_3}(A_T|-p,s,s_3,n\rangle)C_{s_3\kappa}^{-1}]$$
$$=e^{ia_\mu p^\mu}[\sum_{s_3}(A_T|-\mathbf{p},s,s_3,n\rangle)C_{s_3\kappa}^{-1}], \tag{46}$$

where

$$C_{\kappa\tau}=(-1)^{s+\kappa}\delta_{\kappa,-\tau},\quad -s\leq\tau,\kappa\leq+s.$$

Also from (44b) one obtains

$$(1/M)\Gamma_\mu P^\mu[\sum_{s_3}(A_T|-\mathbf{p},s,s_3,-\eta^2n\rangle)C_{s_3\kappa}^{-1}]$$
$$=n[\sum_{s_3}(A_T|-\mathbf{p},s,s_3,-\eta^2n\rangle)C_{s_3\kappa}^{-1}]. \tag{47}$$

Comparing expressions (45)–(47) with the expressions for the action of these operators on the states $|\mathbf{p},s,s_3,n\rangle$, we see that the vector

$$\sum_{s_3}(A_T|-\mathbf{p},s,s_3,-\eta^2n\rangle)C_{s_3\kappa}^{-1}$$

transforms under these operations just like

$$|\mathbf{p},s,s_3,n\rangle.$$

Again under the assumption that (23′) is a complete system of commuting observables, we conclude that

$$A_T|-\mathbf{p},s,-s_3,-\eta^2n\rangle(-1)^{s+s_3}=\alpha'(s,n)|\mathbf{p},s,s_3,n\rangle$$

or, with $A_T^2=\epsilon_T$ and the new proportionality factor $=\epsilon_T\alpha'^{*-1}$,

$$A_T|\mathbf{p},s,s_3,n\rangle=\alpha(s,n)(-1)^{s+s_3}|-\mathbf{p},s,-s_3,-\eta^2n\rangle. \tag{48}$$

That the proportionality factor does not depend upon p and s_3 can be shown using, e.g., (29′) and (29).

Applying A_T to (48), one obtains

$$\epsilon_T=(-1)^{2s}\alpha^*(s,n)\alpha(s,-\eta^2n), \tag{49}$$

from which, for the case under consideration, $\epsilon_T(-1)^{2s}=1$, we obtain

$$\alpha^*(s,n)\alpha(s,-\eta^2n)=1. \tag{50}$$

For baryons, $\eta^2=-1$, we obtain from (48) and (50)

$$A_T|\mathbf{p},s,s_3,n\rangle=\alpha(s,n)(-1)^{s+s_3}|-\mathbf{p},s,-s_3,n\rangle, \tag{51}$$

with $|\alpha(s,n)|=1$. For mesons, $\eta^2=1$, we obtain from (48)

$$A_T|\mathbf{p},s,s_3,n\rangle=\alpha(s,n)(-1)^{s+s_3}|-\mathbf{p},s,-s_3,-n\rangle, \tag{52}$$

with $\alpha^*(sn)\alpha(s,-n)=1$; i.e., time inversion transforms a meson space $\mathcal{H}^{[n,s]}$ into the meson space $\mathcal{H}^{[-n,s]}$, Therewith, we have found the meaning of the particle spaces $\mathcal{H}^{[n,s]}$ with negative eigenvalue of $\Gamma_\mu P^\mu/M$. For baryons, these are the C-conjugated states of the states with positive eigenvalue of $\Gamma_\mu P^\mu/M$, i.e., with the usual interpretation, the antiparticle states. The meson states are C-conjugation eigenstates, and the states with negative n are the T-conjugated states of the states with positive n.

So we have seen that the irreducible representations $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ of the relativistic symmetry $\mathfrak{S}_{P_\mu,L_{\mu\nu},\Gamma_\nu}$ are also irreducible representations of $\mathfrak{S}^F=\mathcal{P}_{P_\mu L_{\mu\nu}TCP}{}^{\text{Full}}\dashv SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$; the discrete operations U_P, U_C, and A_T do not transform out of it. This is, e.g., not the case for the simpler Majorana representations. Since in the above derivation we have only used properties which are also true in $\mathfrak{S}^{(\text{Majorana})}$, we immediately see that for the half-integer-spin Majorana representations, U_C transforms the representation with positive spectrum of $\Gamma_\mu P^\mu/M$ into the representation with negative spectrum of $\Gamma_\mu P^\mu/M$:

$$U_C\colon \mathfrak{S}^{(M,1/2,+)}\longrightarrow\mathfrak{S}^{(M,1/2,-)}$$

and for the integer-spin Majorana representation, A_T performs this transformation:

$$A_T\colon \mathfrak{S}^{(M,0,+)}\longrightarrow\mathfrak{S}^{(M,0,-)}.$$

Thus extension of the Majorana representation by the discrete operations P, T, C will always require representation doubling. The same is true for the oscillator-like representation of $SO(4,2)$.

We calculate the action of a CPT transformation on the states $|\mathbf{p},s,s_3,n\rangle$. From (38), (41), and (48) it follows that for baryons as well as for mesons

$$U_CU_PA_T|\mathbf{p},s,s_3,n\rangle =\omega(s,n)(-1)^{s+s_3}|\mathbf{p},\,s,\,-s_3,\,-n\rangle, \quad (53)$$

with

$$\omega(s,n)=\alpha(s,n)a(s,n)(-1)^{[-n]}.$$

Thus the negative-n states are the CPT transforms of the positive-n states with opposite helicity for baryons as well as for mesons.

There remains an independent relation between the discrete operations which we have not yet utilized:

$$A_TU_C=\pi_C\epsilon_C\epsilon_IU_CA_T \quad \text{with } \epsilon_C\epsilon_I=1. \quad (35\text{c})$$

Applying both sides of (35c) to the state $|\mathbf{p},s,s_3,n\rangle$ and using (41) and (48), we obtain

$$a^*(n,s)\alpha(s,\pi_Cn)=\pi_Ca(-\eta^2n,\,s)\alpha(s,n). \quad (54)$$

For mesons, $\eta^2=+1$ and $\pi_C=+1$; we have, therefore,

$$a^*(n,s)\alpha(s,n)=\alpha(-n,\,s)\alpha(s,n),$$

so that we obtain with (41e)

$$a(n,s)=a(-n,\,s). \quad (55)$$

Thus we have derived that the charge parity of a meson and its T conjugate must be the same.

Present experimental data seem to favor for mesons a charge parity of

$$a(n,s)=(-1)^s \quad (56)$$

(there are at least three $I^G=1^+$ mesons in the R region). However, a C parity that alternates like the P parity,

$$a(n,s)=(-1)^n \quad [\text{or also } a(n,s)=-(-1)^n] \quad (57)$$

is experimentally not excluded. From the theoretical point of view, it appears to be very difficult to give a justification for (56). The C parity that could readily be obtained is $a(n,s)=\text{const}$; this, however, is definitely in disagreement with experiment. With suitable assumptions about the properties of the operators that transform between different hadron spaces (currents), (57) can be given a theoretical justification.

The assumption made at the beginning of this section, that U_P and Γ_0, or more generally U_P and $P_\mu\Gamma^\mu$, commute, is a natural one, but not the only possibility. One can easily see from the defining relations (1) and (28) that

$$U_P\Gamma_0=-\Gamma_0U_P, \quad (58\text{a})$$

$$U_P\Gamma_i=\Gamma_iU_P, \quad (58\text{b})$$

is a permissible choice for the relation between Γ_μ and U_P. With (58) we obtain instead of (38)

$$U_p|\mathbf{p},s,s_3,n\rangle=\kappa(n,s)|-\mathbf{p},\,s,\,s_3,\,-n\rangle, \quad (59)$$

where $\kappa(n,s)$ is a phase factor with $\kappa(n)\kappa(-n)=1$ (from $U_p{}^2=1$), $\kappa^*(-n)=\kappa(n)$ (from unitarity of U_P), and $\kappa=\text{const}$ [from (58b)], so that

$$\kappa=+1 \quad\text{or}\quad \kappa=-1. \quad (59')$$

For physical reasons we will choose states that span eigenspaces of U_P rather than eigenstates of $P_\mu\Gamma^\mu$, because we are used to the assumption that elementary particles have a definite parity. So we define the new states

$$\left|\mathbf{p},s,s_3,(n),{}^1_2\right\rangle =(1/\sqrt{2})(|\mathbf{p},s,s_3,n\rangle\pm\kappa|\mathbf{p},\,s,\,s_3,\,-n\rangle), \quad (60)$$

which are easily checked to have the desired property

$$U_p\left|\mathbf{p},s,s_3,(n),{}^1_2\right\rangle=\pm\left|-\mathbf{p},\,s,\,s_3,\,(n),\,{}^1_2\right\rangle \quad (61)$$

and which further obey

$$P_\mu\Gamma^\mu\left|\mathbf{p},s,s_3(n),{}^1_2\right\rangle=\kappa nm\left|\mathbf{p},s,s_3,(n),{}^2_1\right\rangle. \quad (62)$$

We choose now the parity convention such that the states with $n=0$ have parity $+1$; then we see from (60) that we have to choose $\kappa=1$.

For the representation $\mathfrak{S}^{(R,1/2)}$ the physical content has essentially not changed. For a given s and each value of $|n|$, we have again a particle-antiparticle system of opposite parity [because of (35a)] and the spectrum of s^P in an irreducible representation space $\mathcal{H}^{(R,1/2)}$ is the same as in the previous case. The only difference is that we now have Eq. (62) for the physical states instead of Eq. (24).

For the representations $\mathfrak{S}^{(R,0)}$ the physical content for the case (58) is different from that of the case (32). The parity eigenstates are again charge-conjugation eigenstates. For a given s we have now one parity equal to $+1$ $(=\kappa)$ state for $|n|=0$, and for any other value of $|n|$ we have a pair of states with opposite parity. Thus the s^P content in an irreducible representation space $\mathcal{H}^{(R,0)}$ is different from the one given by Fig. 2, whose parity assignment came from the assumption (32).

IV. DISCUSSION

The description of the baryons by $\mathfrak{S}^{(R,1/2)}$ appears as natural as the description of the electron by the Dirac representation $\mathfrak{S}^{\text{Dirac}}=\mathfrak{S}^{(R=-5/2,1/2)}$. The choice between the two cases (32) [(33)] and (58) for the parity operator is easily decided for case (32). This gives an exact extension of the Dirac case by which the particle states are eigenstates of $P_\mu\Gamma^\mu$ and therewith establishes the nice correspondence between the boxes $[n,s]$ in the pattern of Fig. 1 and the elementary-particle spaces. The pattern of Fig. 1 accommodates the baryons of higher spin with spin degeneracy, as well as the antiparticles, in a way which is in agreement with our old ideas about the baryon properties and with the new experimental data for baryon resonances. From the physical point of view, the two choices for the parity operator do not seem to lead to results that can be distinguished from each other by the experimental data.

The description of the mesons by $\mathfrak{S}^{(R,0)}$, with the choice (32) [(33)] for the parity operator, was not quite what we would have expected, due to the appearance of the negative-n states. From the investigation in the previous section, however, these negative-n states appear not only perfectly acceptable but even necessary for a complete description that includes the discrete operations C, P, and T. The question of how to distinguish experimentally between the meson states which differ only in the sign of the quantum number n remains open. Since they are the T conjugates of each other, they can only be distinguished by observables that do not commute with T and are, therefore, degenerate in all the well-known quantum numbers. The problem of the physical interpretation of these T-conjugated states need not be present, if we choose for the parity operator relation (58). However, then we will have for each value of $(|n|,s)$, except for $n=0$, a doublet of parity eigenspaces with opposite parity. After the symmetry breaking[9] has been taken into account, this will lead to the prediction of a doublet of particles with the same mass and spin but opposite parity, which seems to be strongly disfavored experimentally. It would, e.g., predict that there are two mesons of the mass of $A_2{}^H$, one with $s^P=2^+$ and the other with $s^P=2^-$, which seems to be in disagreement with latest experimental results.[26] It therefore seems that experimental results choose the parity operator (32) [(33)] for mesons also, which leads to a more beautiful scheme than (58) but also to more curious predictions.

Further evidence for the applicability of the representations $\mathfrak{S}^{(R,0)}$ and $\mathfrak{S}^{(R,1/2)}$ to the description of hadrons will evolve after the symmetry breaking has been taken into account. This will be discussed in a forthcoming work.

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APPENDIX

The clearest presentation of the algebraic relations (1) that define the relativistic symmetry can probably be given in the spinor basis. The spinor basis of the irreducible representation space of the Poincaré group is the basis in which the splitting of the Lorentz transformations generated by $L_{\mu\nu}$ into an "orbital" part generated by $M_{\mu\nu}$ and a "spin" part generated by $S_{\mu\nu}$ is made explicit.

Let $\mathcal{P}^x$ be called the Poincaré group generated by P_μ and $M_{\mu\nu}$ [from relation (1) it can be seen that P_μ and $M_{\mu\nu}$ fulfill the commutation relation of the Poincaré group], then it follows from (1) that the Lie algebra $\mathcal{L}(\mathcal{P}^x)$ and $\mathcal{L}(SO(3,2)_{\Gamma_\mu,S_{\mu\nu}})$ commute. Let $\mathcal{H}(m,\epsilon=+1)$ be the irreducible representation space of $\mathcal{P}^x$ [because of relation (1e), $s^x=0$] and $\phi(p)$ its generalized basis vectors, and let $\mathcal{H}^{(R,\cdot)}$ be the representation space of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$, then as a consequence of the "direct product,"

$$\mathcal{P}^x\otimes SO(3,2)_{S_{\mu\nu},\Gamma_\mu} = \text{set of global transformations of } \mathfrak{S}, \quad \text{(A1)}$$

we obtain the irreducible representation space of $\mathfrak{S}$ as the direct product of the spaces $\mathcal{H}(m)$ and $\mathcal{H}^{(R,\cdot)}$:

$$\mathcal{H}^{(R,\cdot)}(m)=\mathcal{H}(m)\otimes\mathcal{H}^{(R,\cdot)}. \quad \text{(A1}')$$

The spinor basis is then the corresponding product of the basis systems:

$$f^{k_0j}{}_{j_3}(p)=\Phi(p)\otimes f^{k_0j}{}_{j_3}, \quad \text{(A2)}$$

[26] W. Kienzle, Invited paper at the Washington Meeting of the American Physical Society, 1970 (unpublished); CERN report (unpublished).

where $f^{k_0 j}{}_{j_3}$ is the basis (2′) of $\mathfrak{H}^{(R,\cdot)}$. Direct product means the direct product in the usual sense but with the same parameters for the transformations generated by $M_{\mu\nu}$ and by $S_{\mu\nu}$ [i.e., $\mathcal{P}^x \otimes SO(3,2)$ equals the set of all elements of $\mathcal{P}^x \otimes SO(3,2)$ which fulfill $\alpha^{\mu\nu}=\beta^{\mu\nu}$, where $e^{i\alpha_{\mu\nu}M_{\mu\nu}}$ and $e^{i\beta_{\mu\nu}S_{\mu\nu}}$ are the global transformations of $SO(3,1)_{M_{\mu\nu}}$ and $SO(3,1)_{S_{\mu\nu}}$, respectively]. From definition (A2) follows the simple transformation property of the spinor basis under Lorentz transformations, i.e., $\Lambda \in SO(3,1)_{L_{\mu\nu}}$:

$$U(\Lambda) f^{k_0 j}{}_{j_3}(p) = \sum_{j' j_3'} f^{k_0 j'}{}_{j_3'}(\Lambda p) D^{(k_0) j' j}{}_{j_3' j_3}(\Lambda)\,, \quad \text{(A3)}$$

where $D^{(k_0) j' j}{}_{j_3', j_3}(\Lambda)$ is the representation matrix of Λ in the representation $(k_0, c=(1/i)[\frac{1}{2}(R-2)]^{1/2})$. Γ_μ and $S_{\mu\nu}$ act only on the indices of $f^{k_0 j}{}_{j_3}(p)$ without effecting p. It is clear that there is no physical transformation generated by the $S_{\mu\nu}$ alone.

In contrast to the spinor basis, the canonical basis (23) of $\mathfrak{H}^{(R,\cdot)}(m)$ is obtained from the basis (4″) by

$$|\mathbf{p},s,n,s_3\rangle = U(L^{-1}(p))(\phi(p_0, \mathbf{p}=0) \otimes |s,n,s_3\rangle)\,, \quad \text{(A4)}$$

where $L^{-1}(p)$ is the boost (19).

It is illustrative to check that the basis defined by (A4) really has the correct transformation properties of the canonical basis[23] of $\mathcal{P}$:

$$U(\Lambda)|\mathbf{p} s n s_3\rangle = \sum_{s_3'} |(\Lambda p) i,s,n,s_3'\rangle D_{s_3' s_3}(R)\,, \quad \text{(A5)}$$

with

$$R = L(\Lambda p)\Lambda L^{-1}(p)\,.$$

The calculation is as follows:

$$\begin{aligned} U(\Lambda)|\mathbf{p}, sns_3\rangle &= U(\Lambda L^{-1})(\phi(\mathbf{p}=0)\otimes |s,n,s_3\rangle) \\ &= U(L^{-1}(\Lambda p))U(R)(\phi(\mathbf{p}=0)\otimes |s,n,s_3\rangle) \\ &= U(L^{-1}(\Lambda p))(U(R)\phi(\mathbf{p}=0)\otimes U(R)|s,n,s_3\rangle) \\ &= U(L^{-1}(\Lambda p))\Big(\phi(\mathbf{p}=0)\otimes\Big(\sum_{s_3'} |s,n,s_3'\rangle D_{s_3' s_3}{}^{(s)}(R)\Big)\Big)\,, \end{aligned}$$

which gives (A5) because $L^{-1}(\Lambda p)$ is rotation free.

It is from physical considerations that the canonical basis is preferred over the simpler spinor basis.

As an elementary particle is assumed to have definite spin and not a definite j equals the "spin part of the angular momentum," it is clear that the canonical basis $|\mathbf{p},s,s_3,n\rangle$ is the physical basis and not the spinor basis, and it also appears that n is the physical quantum number and not k_0. The transformation matrix between the spinor basis and the canonical basis can be calculated along the same lines as in the Appendix of Ref. 4(a); it is given by

$$|\mathbf{p}, s, n, s_3\rangle = \sum_{j_3 j k_0} f^{k_0 j}{}_{j_3}(p) U^{(k_0) j}{}_{j_3}(p,s_3,s,n)\,, \quad \text{(A6)}$$

where

$$U^{(k_0) j}{}_{j_3}(\mathbf{p},s_3,s,n) = D^{(k_0,c) js}{}_{j_3 s_3}(L^{-1}(p))\langle k_0|n\rangle_{(s)}\,. \quad \text{(A7)}$$

The summation in (A6) goes over all $-j \le j_3 \le +j$, $j=k_0$, k_0+1, ..., and $k_0=\pm\frac{1}{2}, \pm\frac{3}{2}$, ..., for $\mathfrak{S}^{(R,1/2)}$ and $k_0=0, \pm1, \pm2$, ..., for $\mathfrak{S}^{(R,0)}$. $U^{(k)}(\mathbf{p},s_3,s,n)$ is the infinite-dimensional generalization of the Dirac spinor.[3] $\langle k_0|n\rangle_{(s)}$ is given in (10) and $D^{(k_0,c)js}{}_{s s_3}(L^{-1}(p))$ is the representation matrix [27] of $L^{-1}(p)$ in the representation $(k_0, c=-i[\frac{1}{2}(R-2)]^{1/2})$.

[27] S. Ström; Arkiv Fysik **33**, 465 (1967); R. Delbourgo, K. Koller, and P. Mahanta, Nuovo Cimento **52A**, 1254 (1967).