and

$$u^{(-1,d)}(p,t)=d\left(\frac{g(p,t)(p+p_3d)}{4\mu p}\right)^{1/2}\times\begin{bmatrix}\dfrac{pd}{R(t)g(p,t)}\chi^{(d)}(\mathbf{p})\\ \chi^{(d)}(\mathbf{p})\end{bmatrix},\quad (C9)$$

where

$$g(p,t)=\omega(p,t)+\mu\,,\quad (C10)$$

$$\omega(p,t)=[p^2/R(t)^2+\mu^2]^{1/2},\quad (C11)$$

and

$$\chi^{(d)}(\mathbf{p})=\begin{pmatrix}1\\ (p_1+ip_2)/(pd+p_3)\end{pmatrix}.\quad (C12)$$

One can directly verify that Eq. (B4) and the equations which follow it in Appendix B [with $\mathbf{p}$ replaced by $\mathbf{p}/R(t)$] hold in this representation. Some additional equations which hold in this representation are

$$u^{(a,d)}(\mathbf{p},t)^\dagger u^{(-a,d)}(\mathbf{p},t)=p/\mu R(t)\quad (C13)$$

and

$$u^{(-a,d)}(\mathbf{p},t)=-d\gamma^5 u^{(a,d)}(\mathbf{p},t).\quad (C14)$$

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Vacuum-Polarization Contributions to the Sixth-Order Anomalous Magnetic Moment of the Muon and Electron*

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We report on the calculation of second-order vacuum-polarization contributions to the sixth-order electron and muon anomalous magnetic moments. With these results the difference of muon and electron moments has now been completely calculated through sixth order in quantum electrodynamics. The only remaining sixth-order contributions to the electron moment which have not been completely calculated are those graphs without fermion loop insertions.

I. INTRODUCTION

In view of the recent measurement by Wesley and Rich[1] of the anomalous magnetic moment of the electron, $a_e=\frac{1}{2}(g_e-2)$, to a precision of 6 ppm and the expected precision of future measurements of the muon anomalous magnetic moment $a_\mu=\frac{1}{2}(g_\mu-2)$,[2] a complete calculation of the sixth-order quantum-electrodynamic contributions to a_e and a_μ is awaited with increasing urgency.

The status of the sixth-order calculations for the electron moment is as follows: The contribution arising from the insertion of fourth-order vacuum-polarization graphs into the second-order vertex[3] [Fig. 1(a)] as well as the photon-photon scattering contribution[4] [Fig. 1(b)] has been evaluated. The graphs which have not been evaluated consist of all those obtained by insertion of the second-order vacuum-polarization graphs into the fourth-order vertices [Fig. 1(c)] and all sixth-order vertices with no electron loop insertion [Fig. 1(d)]. In addition there is a dispersion-theoretical estimate of the sixth-order electron magnetic moment.[5,6]

The quantum-electrodynamic contributions to the difference of the muon and electron moments $a_\mu-a_e$ in sixth order arise from the insertion of electron loops of the vacuum-polarization type [Figs. 1(a) and 1(c)] in the muon vertices of the second[7–10] and fourth orders[7,9]

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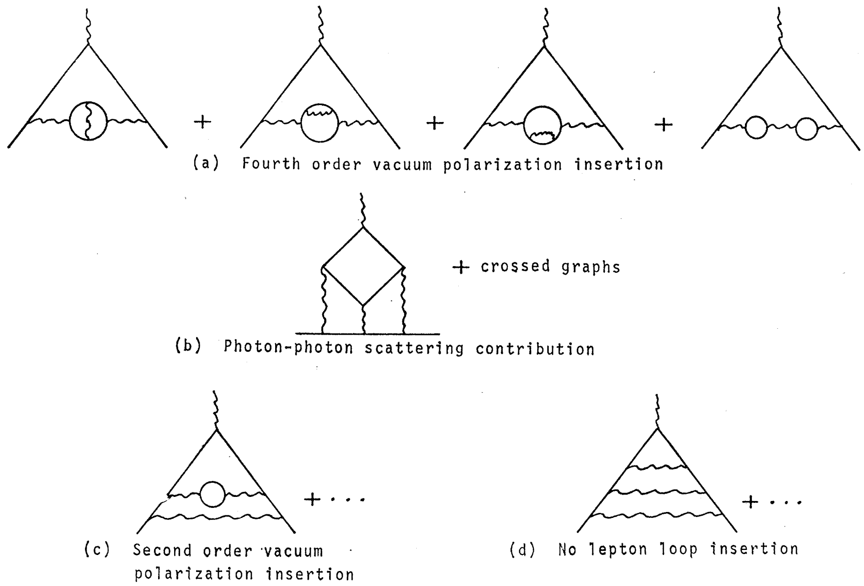

FIG. 1. Four types of sixth-order Feynman diagrams which contribute to the anomalous magnetic moment of the muon and electron.

and of the photon-photon scattering type[4] [Fig. 1(b)]. These have been completely evaluated except for certain nonlogarithmic remainders in the vacuum-polarization contributions [Fig. 1(c)] which have only been estimated.[7]

In this paper we report the results of calculations which complete the evaluation of all the vacuum-polarization contributions to a_e and a_μ through sixth order. In particular, the difference of muon and electron anomalous moments has now been completely calculated through sixth order in quantum electrodynamics. As in other calculations which have been recently reported,[4] the final results have been obtained via numerical integration, but to sufficient accuracy for comparison with experiments at a precision better than 1 ppm for the electron anomalous moment and 10 ppm for the muon moment. The final confrontation of experiment with the theoretical results of quantum electrodynamics through order α^3 will, however, require the complete evaluation of the contributions of the graphs of Fig. 1(d) (there are 28 distinct graphs of this type) to a_e, a task which, though difficult, seems technically feasible with present algebraic and numerical computation techniques.

II. SECOND-ORDER VACUUM-POLARIZATION CONTRIBUTIONS TO SIXTH-ORDER MOMENT

The Feynman diagrams contributing to the fourth-order lepton vertex in quantum electrodynamics are shown for reference in Fig. 2. Mass, charge, and wave-function renormalization counter terms are understood. The contributions of the individual diagrams to the fourth-order magnetic moment have been given by Petermann,[11] and are listed numerically in column 2 of Table I. The corner [Fig. 2(c)] and fermion self-energy [Fig. 2(d)] diagram contributions are separately logarithmically infrared divergent for photon mass $\lambda \to 0$, although the sum of their contributions to $F_2(0)$ is infrared finite.

As a check of our computational scheme, we have first reevaluated the fourth-order moment contributions from Fig. 2. The reduction of these integrals to the Feynman parametric form has been obtained using the algebraic techniques and renormalization procedure described in previous work.[4,12] Trace projections, index contractions, and loop integral replacements were obtained using the algebraic computation program REDUCE,[13] and also by hand calculation as a further check. The Feynman parametric integrals could be reduced in a trivial way to four dimensions for diagrams (a), (c), and (d) and to three dimensions for the ladder diagram (b) of Fig. 2. The numerical results, obtained using the Sheppey program described earlier,[4] agreed in each case with the analytic calculations to the required accuracy.

Once the integrals for the fourth-order diagrams (a)–(d) are written down, the integrals corresponding to the sixth-order graphs generated by the insertion of second-order vacuum-polarization electron loops in the

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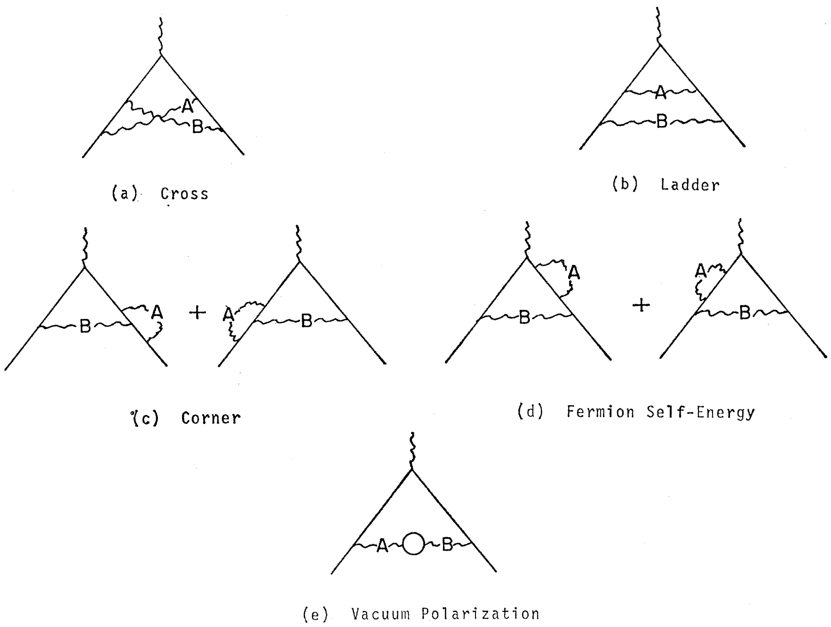

FIG. 2. Feynman diagrams contributing to the fourth-order lepton vertex in quantum electrodynamics. Sixth-order Feynman diagrams of the type shown in Fig. 1(c) are generated by insertion of a second-order vacuum-polarization electron loop in either one of the photon propagators labeled A and B.

photon propagators (labeled A and B in Fig. 2) can be easily obtained by the effective replacement[14]

$$\frac{1}{k^2-\lambda^2+i\epsilon}\rightarrow\frac{\alpha}{\pi}\int_0^1 dt\frac{t^2(1-\frac{1}{3}t^2)}{1-t^2}\times\frac{1}{k^2-[4m_e^2/(1-t^2)]+i\epsilon} \quad (2.1)$$

at the expense of one extra integration. The prescription for the double-loop vacuum polarization contribution corresponding to Fig. 2(e) is discussed in Refs. 7 and 9.

The results of our calculations for the second-order electron-pair vacuum-polarization contributions to the sixth-order anomalous moments of the electron and muon are summarized in columns 3 and 5 of Table I. After the usual self-mass, subvertex, and charge renormalizations, the contributions of the self-energy [Fig. 2(d)] and corner [Fig. 2(c)] diagrams with vacuum-polarization insertions in the main loop photon propagator (B) are individually infrared divergent.[15] The logarithmic dependence on photon mass λ of these contributions can be readily determined analytically. In the case of the corner graph, the logarithmic contribution can be isolated analytically through the use of intermediate renormalization.[16] We have also found and used an even simpler method which applies to both corner and self-energy graphs. This method is described in the Appendix. The remaining infrared-finite contributions were obtained numerically. The combined numerical result for the corner and self-energy contributions ($\lambda/m\rightarrow 0$) is

$$F_2(0)[2c(B)+2d(B)]=(\alpha^3/\pi^3)\begin{cases}-0.115(6) & \text{electron}\\ -2.06(6) & \text{muon.}\end{cases} \quad (2.2)$$

The corner and self-energy diagrams with the vacuum polarization insertions in the photon propagator (A) of the internal loop are individually infrared finite.[17] The sum of these contributions is

$$F_2(0)[2c(A)+2d(A)]=(\alpha^3/\pi^3)\begin{cases}-0.089(2) & \text{electron}\\ -1.96(7) & \text{muon.}\end{cases} \quad (2.3)$$

The error limits correspond approximately to 90% confidence limits, and could be improved if necessary. Further information on the analytic behavior of individual contributions as a function of $\rho=m_e^2/m_\mu^2$ for $\rho\ll 1$ is shown in Table I.

[14] G. Källén, Helv. Phys. Acta **25**, 417 (1952).

[15] In addition, the unrenormalized amplitude for the ladder diagram [Fig. 2(b)] and its subvertex renormalization subtraction term contain canceling infrared-divergent contributions. In the ladder diagram, the photon A cannot cause the infrared divergence since it is not attached to external lines. Thus the divergence is associated with the photon B only. On the other hand, the infrared divergence of the subtraction term arises not from the photon B, but from the photon A. Hence the infrared divergence of the ladder and subtraction graphs arise from entirely different parts of the domain of integration. The ensuing computational problem is resolved by simply symmetrizing the variables associated with the photons A and B.

[16] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965); P. K. Kuo and D. R. Yennie, Ann. Phys. (N. Y.) **51**, 496 (1969); T. Appelquist, *ibid.* **54**, 27 (1969).

[17] The contributions of vacuum-polarization insertions in the photon line of type A in the corner [Fig. 2(c)] and self-energy [Fig. 2(d)] diagrams are infrared finite but contain canceling $\ln^2\rho$ terms for $\rho=(m_e/m_\mu)^2\ll 1$.

TABLE I. Second-order electron-pair vacuum-polarization contributions to the sixth-order anomalous magnetic moments of the muon and electron. The functions used in column 4 are defined as follows: $f(\rho)=\frac{2}{3}[\ln(m_\mu/m_e)-25/12+\frac{3}{4}\pi^2 m_e/m_\mu]$, $g(\rho)=(2/9)[\ln(m_\mu/m_e)]^2-(25/27)\ln(m_\mu/m_e)+\pi^2/27+317/324$, and $h(\rho)=(119/27-4\pi^2/9)\ln(m_\mu/m_e)-61/162+\pi^2/27$, where $\rho=(m_e/m_\mu)^2$.

Graph	$F_2^{(4)}(0)$ (electron) $(\alpha^2/\pi^2)\times$	$F_2^{(6)}(0)$ (electron) $(\alpha^3/\pi^3)\times$	$F_2^{(6)}(0)$ $(\rho=m_e^2/m_\mu^2\ll1)$ $(\alpha^3/\pi^3)\times$	$F_2^{(6)}(0)$ (muon) $(\alpha^3/\pi^3)\times$
Ladder	0.778	0.0532(4)	$2f(\rho)[0.778]-0.53(6)$	2.88(6)
Cross	−0.467	−0.0032(3)	$2f(\rho)[-0.467]+0.76(1)$	−1.28(1)
Corner	$-0.564-\ln(\lambda/m_e)$	$\begin{cases}-0.051(3)\\ \quad-0.0314\ln(\lambda/m_e)\\ +0.0273(3)\end{cases}$	$\begin{cases}f(\rho)[-0.564-\ln(\lambda/m_\mu)]-0.18(3)\\ f(\rho)[-0.654]-0.53(7)\end{cases}$	
Fermion SE	$-0.090+\ln(\lambda/m_e)$	$\begin{cases}-0.064(3)\\ \quad+0.0314\ln(\lambda/m_e)\\ -0.1161(14)\end{cases}$	$\begin{cases}f(\rho)[-0.090+\ln(\lambda/m_\mu)]-0.45(3)\\ \text{(included with corner result)}\end{cases}$	−4.02(13)
Vac. pol.	0.016	0.00255(2)	$\begin{cases}g(\rho)\pm0.02\\ h(\rho)\pm0.002\end{cases}$	2.82(2)
α^2 vac. pol. (proper)	···	0.05291(6)	$\frac{1}{4}\ln(m_\mu/m_e)+\frac{1}{2}\zeta(3)-5/12-0.03(2)$	1.49(2)
γ-γ scatt.	···	0.36(4)	$6.4(1)\ln(m_\mu/m_e)-16(1)$	18.4(11)
Total	−0.3285	0.26±0.05		20.3±1.3

The combined result for second-order vacuum-polarization insertions in the fourth-order ladder, cross, corner, and self-energy diagrams is

$$F_2(0)[2a-d(A\text{ and }B)]=(\alpha^3/\pi^3)\begin{cases}-0.154(9) & \text{electron}\\ -2.42(20) & \text{muon,}\end{cases}\qquad(2.4)$$

which is the main result of this paper. The previous estimate[7] for this contribution to the sixth-order muon moment was

$$\frac{4\alpha}{3\pi}\left(\ln\frac{m_\mu}{m_e}-\frac{25}{12}+\frac{3\pi^2}{4}\frac{m_e}{m_\mu}\right)(-0.3442\cdots)\frac{\alpha^2}{\pi^2}=(-1.506)\frac{\alpha^3}{\pi^3}.\qquad(2.5)$$

We have also checked numerically all previous calculations[3,7,9,10] of fourth-order vacuum-polarization contributions to the sixth-order moments of improper double vacuum-polarization loop contributions [including the mixed electron-pair and muon-pair contribution from insertions in Fig. 2(e)] and the proper fourth-order vacuum polarization (V.P.) insertion as given by Källén and Sabry.[18] The total result is

$$F_2(0)\ (\text{fourth-order V.P.})=(\alpha^3/\pi^3)\begin{cases}0.05546(8) & \text{electron}\\ 4.31(4) & \text{muon,}\end{cases}\qquad(2.6)$$

in excellent agreement with previous analytic results:

$$F_2(0)\ (\text{fourth-order V.P.})=(\alpha^3/\pi^3)\begin{cases}0.05536 & \text{electron (Ref. 3)}\\ 4.34+O(m_e/m_\mu) & \text{muon (Refs. 9,10).}\end{cases}\qquad(2.7)$$

III. COMPARISON WITH EXPERIMENT

The total contribution to the electron moment in sixth order, calculated thus far from quantum electrodynamics, is

$$a_e=F_2{}^e(0)=(\alpha^3/\pi^3)[-0.154(9)+0.055+0.36(4)]=(\alpha^3/\pi^3)[0.26\pm0.05].\qquad(3.1)$$

The three terms correspond to (1) the second-order vacuum-polarization contributions evaluated in this work [Fig. 1(c)], (2) fourth-order vacuum-polarization contributions[3] which we have checked [Fig. 1(a)], and (3) the photon-photon scattering contribution obtained in Ref. 4 [Fig. 1(b)].

If we combine[19] these contributions with the Drell-Pagels[5]-Parsons[6] dispersion-theory estimate of the remaining sixth-order contributions, we obtain

$$a_e{}^{\text{theor}}=\alpha/2\pi-0.32848\alpha^2/\pi^2+\text{“}0.39\text{”}\alpha^3/\pi^3\qquad(3.2)$$

for the current (preliminary) theoretical estimate of the sixth-order electron moment. This can be compared with the preliminary results of the recent measurement by Wesley and Rich[1,20] (W.R.):

$$a_e{}^{\text{expt}}(\text{W.R.})=1\,159\,644(7)\times10^{-9}=\alpha/2\pi-0.32848\alpha^2/\pi^2+(0.54\pm0.58)\alpha^3/\pi^3,\qquad(3.3)$$

[18] G. Källén and A. Sabry, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 29, No. 17 (1955); see also J. A. Mignaco and E. Remiddi (Ref. 3).

[19] The contributon of second-order vacuum polarization to the dispersion estimate (Refs. 5 and 6) of the sixth-order electron moment is negligible, so there is little difficulty with double counting here.

[20] Previous measurements of the electron moment and comparisons with theory are summarized by S. J. Brodsky and S. D. Drell, Ann. Rev. Nucl. Sci. (to be published).

using $\alpha^{-1}=137.03608(26)$.[21] Of course, a complete evaluation of the non-lepton-loop contributions [Fig. 1(d)] is required for the final confrontation of experimental results with quantum electrodynamics through order α^3.

Insofar as the difference of muon and electron moments is concerned, the results of this paper for second-order vacuum-polarization contributions complete the calculation through sixth order in quantum electrodynamics. The combined quantum electrodynamics result[22] is

$$\begin{aligned}
a_\mu^{\text{theor}}-a_e^{\text{theor}}
&=\frac{1}{3}\left[\ln\frac{m_\mu}{m_e}-\frac{25}{12}+\frac{3\pi^2}{4}\frac{m_e}{m_\mu}+3\left(3-4\ln\frac{m_\mu}{m_e}\right)\frac{m_e^2}{m_\mu^2}\right.\\
&\qquad\left.+O\left(\frac{m_e^3}{m_\mu^3}\right)\right]\frac{\alpha^2}{\pi^2}-\left(\frac{1}{45}\frac{m_e^2}{m_\mu^2}\right)\frac{\alpha^2}{\pi^2}\\
&\qquad+[-2.42(20)+4.31(4)+18.4\pm1.1]\alpha^3/\pi^3\\
&=1.09426\alpha^2/\pi^2+(20.3\pm1.3)\alpha^3/\pi^3\\
&=616(1)\times10^{-8}, \qquad (3.4)
\end{aligned}$$

including the sixth-order contributions from (1) second-order vacuum polarization, (2) fourth-order vacuum polarization,[7,9,10] and (3) photon-photon scattering[4] due to the electron current. This result differs by $(-0.9\pm0.3)\alpha^3/\pi^3$ from previous compilations[4,20] which did not include the complete nonlogarithmic remainder of the second-order vacuum-polarization contributions. In addition, hadronic vacuum polarization calculated from the Orsay data for electron-positron annihilation in the ρ, ω, and ϕ regions gives the contribution[23]

$$\Delta a_\mu^{\text{theor}}(\text{hadronic})=6.5(5)\times10^{-8}. \qquad (3.5)$$

The result of the CERN measurement[24] for the anomalous moment of the muon is

$$a_\mu^{\text{expt}}=0.001\,166\,16(31). \qquad (3.6)$$

If we combine this with the result of Ref. 1, we can check the theoretical result for the difference of muon and electron moments directly:

$$\begin{aligned}
a_\mu^{\text{expt}}-a_e^{\text{expt}}&=652(32)\times10^{-8},\\
a_\mu^{\text{theor}}-a_e^{\text{theor}}&=623(2)\times10^{-8}.
\end{aligned} \qquad (3.7)$$

The theoretical uncertainty does not take into account further (positive) contributions from hadronic vacuum polarization beyond the ϕ resonance region or possible weak-interaction contributions[25] which may be of order 1×10^{-8}. The comparison of theory with experiment is presently limited by the $\pm31\times10^{-8}$ uncertainty in the determination of the muon moment.

After this work was completed, we learned that a similar calculation of the second-order vacuum-polarization contributions to the difference of a_μ and a_e has been carried out by Lautrup, Peterman, and de Rafael[26] and by Lautrup.[27] Their result is in agreement with ours. We would like to thank Dr. de Rafael for communicating their results to us.

ACKNOWLEDGMENTS

We wish to thank Dr. Thomas Appelquist for his help in the early stages of this calculation, and Professor Donald R. Yennie for helpful comments. The cooperation of the SLAC Computation Center and the assistance of Peter Baer are also greatly appreciated.

APPENDIX: NEW TECHNIQUE OF RENORMALIZATION

The ultraviolet divergence is a familiar problem in quantum electrodynamics, and there is a well-established procedure of renormalization which enables us to extract the finite physically meaningful part of the divergent integral in an unambiguous manner. However, the usual prescription of renormalization found in every textbook of quantum electrodynamics is not necessarily the most convenient for practical calculation, in particular, for computer calculation. In this Appendix we discuss variations of the renormalization method which are more appropriate for computational purposes.

Let us first consider the problem of vertex renormalization. Suppose M_r is the vertex renormalization part of a divergent Feynman integral M such that the physically meaningful part is given exactly by

$$M-M_r. \qquad (A1)$$

M_r must have the same ultraviolet structure as M. The variables of integration can be chosen so that the ultraviolet-divergent parts of M and M_r cancel out *at every point* in the domain of integration. However, M_r defined in the conventional manner may cause a computational difficulty because it frequently has an infrared divergence which may or may not be present in M itself. Thus, even though the ultraviolet divergence is eliminated by the subtraction procedure (A1), the integral $M-M_r$ may still be divergent because of the infrared divergence. In order to avoid this, we may simply choose a different M_r. One such choice is the subtraction

[21] B. N. Taylor, W. H. Parker, and D. N. Langenberg, Rev. Mod. Phys. **41**, 375 (1969).

[22] The fourth-order contribution to the muon-electron moment difference is the result obtained by H. Suura and E. H. Wichmann, Phys. Rev. **105**, 1930 (1957); A. Petermann, *ibid.* **105**, 1931 (1957); Fortschr. Physik **6**, 505 (1958); H. H. Elend, Phys. Letters **20**, 682; **21**, 720 (1966); G. W. Erickson and H. Liu (unpublished).

[23] M. Gourdin and E. de Rafael, Nucl. Phys. **10B**, 667 (1969).

[24] J. Bailey, W. Bartl, G. von Bochman, R. C. A. Brown, F. J. M. Farley, H. Jöstlein, E. Picasso, and R. W. Williams, Phys. Letters **28B**, 287 (1968).

[25] R. A. Shaffer, Phys. Rev. **135**, B187 (1967); S. J. Brodsky and J. D. Sullivan, *ibid.* **156**, 1644 (1967); T. Burnett and M. J. Levine, Phys. Letters **24B**, 467 (1967).

[26] B. E. Lautrup, A. Peterman, and E. de Rafael, CERN Report No. TH.1190 (unpublished).

[27] B. E. Lautrup, CERN Report No. TH.1191 (unpublished).

term obtained by the method of intermediate renormalization.[16] This term, which we denote by M_{ir}, is free from infrared divergence. With the help of M_{ir} we may rewrite the subtraction procedure as

$$(M-M_{\text{ir}})+(M_{\text{ir}}-M_{\text{r}})\,, \tag{A2}$$

instead of (A1). The advantages in doing this are that (1) M_{ir} removes the ultraviolet divergence of M without introducing an infrared divergence and that (2) $M_{\text{ir}}-M_{\text{r}}$ can be evaluated analytically or reduced to an integral of lower dimension which can be integrated much more easily.

A disadvantage of this approach is that an additional (though less tedious than that of M itself) trace calculation, etc., must be carried out to find M_{ir} which we throw away in the end. However, this can be easily avoided once we realize that it is not M_{ir} itself that interests us and any substitute will do if it satisfies the two conditions mentioned above. An almost trivially simple way of obtaining such a substitute is to look at M itself, locate the domain of Feynman parameters from where the ultraviolet divergence arises, expand the denominators (numerators) of the integrand of M in the neighborhood of this domain to first (zeroth) order, and use the resulting expression as the integrand of the new subtraction term M_0. This integral has precisely the same ultraviolet structure as M (at least for vertex and fermion self-energy renormalization). Furthermore, there is a considerable flexibility in the choice of first-order terms in the denominator, so that M_0 can be made both infrared-divergence free and reducible to lower-order integrals. (The intermediate renormalization can be regarded as one of several possible choices.) Once we find such an M_0, the renormalization proceeds as follows:

$$(M-M_0)+(M_0-M_{\text{r}})\,, \tag{A3}$$

where the first term is now free from any divergence and can be evaluated numerically on the computer, and the second term contains the infrared divergence but can be evaluated analytically or reduced to integrals of lower order. A further advantage of this method is that, unlike the intermediate renormalization method, it can be easily applied to both vertex and self-energy renormalizations.

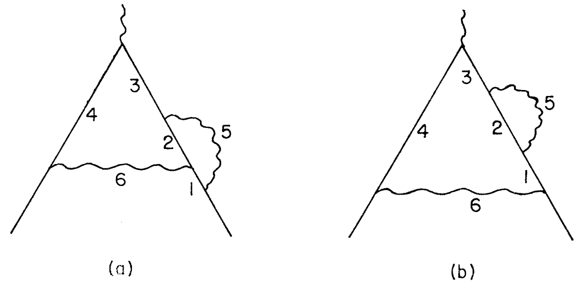

FIG. 3. Fourth-order (a) corner and (b) self-energy diagrams. Feynman parameters $z_1, z_2, \ldots, z_6$ are associated with the lines $1, 2, \ldots, 6$.

To show how this new method works, let us take the fourth-order corner graph as an example. After parametrizing [see Fig. 3(a) for assignment of Feynman parameters] and simplifying this graph according to the procedure described in Ref. 4, we can write the corresponding magnetic-moment contribution in the form[28]

$$\begin{aligned} M_{\text{corner}}=\frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2\int dz[2B_{12}(A_4-A_4{}^2) \\ -B_{14}(8A_2A_4-4A_1-4A_4-5\alpha_3) \\ +B_{34}(2A_1A_2+\alpha_1+3\alpha_3+2)]/U^3V \\ +\frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2\int dz[2(A_4-A_4{}^2)(1-A_1A_2) \\ +(1-A_4{}^2)(2\alpha_1+\alpha_3)]/U^2V^2\,, \end{aligned} \tag{A4}$$

where

$$\begin{aligned} dz&\equiv\prod_{i=1}^{6} dz_i\delta\Big(\sum_{i=1}^{6} z_i-1\Big)\,,\\ U&=U_1U_3+z_2(z_1+z_5)\,,\\ U_1&=B_{34}=z_1+z_2+z_5\,,\\ U_3&=B_{12}=z_3+z_4+z_6\,,\\ B_{14}&=-z_2\,,\\ V&=z_1+z_2+z_3+z_4+(\lambda/m)^2(z_5+z_6)\\ &\quad-z_1z_5/(z_1+z_5)-z_6(z_3+z_4)/U_3-U^{-1}(z_1+z_5)\\ &\quad\times z_2U_3[z_1/(z_1+z_5)-z_6/U_3]^2\,,\\ A_1&=z_5/(z_1+z_5)+U^{-1}z_2U_3[z_1/(z_1+z_5)-z_6/U_3]\,,\\ A_4&=z_6/U_3+U^{-1}z_2(z_1+z_5)[z_1/(z_1+z_5)-z_6/U_3]\,,\\ A_2&=A_1+A_4-1\,,\\ \alpha_1&=2z_4z_2/U\,,\\ \alpha_3&=-2z_4U_1/U\,, \end{aligned} \tag{A5}$$

λ being the photon mass. As is easily seen (A4) is divergent at the corner

$$z_1+z_2+z_5=0\,. \tag{A6}$$

In fact this is the only ultraviolet divergence of (A4). Of course, as it is, (A4) is not well defined, and we have to perform an appropriate regularization for proper definition.

[28] Of course, in order to obtain the full contribution of the corner and self-energy graphs to the magnetic moment, M_{corner} and M_{se} must be doubled to take account of the other corner and self-energy graphs.

In order to carry out the vertex renormalization according to our new prescription, we must first expand U, V, etc., to first order in z_1, z_2, and z_5. This leads us to the definition

$$\begin{aligned} U_0 &= U_1U_3, \\ V_0 &= z_1+z_2+z_3+z_4+(\lambda/m)^2(z_5+z_6) \\ &\quad -z_6(z_3+z_4)/U_3-z_5(z_1+z_2)/U_1, \\ A_{4,0} &= z_6/U_3. \end{aligned} \quad \text{(A7)}$$

Then, noting that only the first term of the first integral in (A4) is actually divergent, we define

$$M_0=\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2\int dz\, B_{12}(A_{4,0}-A_{4,0}{}^2)/U_0{}^3V_0. \quad \text{(A8)}$$

We can now write down our renormalization procedure as

$$M_{\text{corner}}-M_{\text{r}}=(M_{\text{corner}}-M_0)+(M_0-M_{\text{r}}), \quad \text{(A9)}$$

where $M_{\text{corner}}-M_0$ is free from any divergence and can be readily evaluated by computer. M_{r} is obtained immediately from the known results of the second-order calculation

$$M_r=\frac{\alpha}{2\pi}\left\{\frac{\alpha}{\pi}\left[\frac{1}{4}\ln\left(\frac{\Lambda^2}{m^2}\right)-\frac{1}{2}\ln\left(\frac{m^2}{\lambda^2}\right)+\frac{9}{8}\right]\right\}, \quad \text{(A10)}$$

where Λ is the ultraviolet cutoff mass.

Actually the last term $z_5(z_1+z_2)/U_1$ of V_0 in (A7) is not what we obtain by linearizing V in the neighborhood of $z_1+z_2+z_5=0$. This term can be chosen rather arbitrarily insofar as it does not affect the ultraviolet structure of M_0. The particular choice is made for the convenience of analytic evaluation. By simple calculation we obtain

$$M_0=(\alpha/\pi)^2[\tfrac{1}{8}\ln(\Lambda^2/m^2)-\tfrac{1}{16}], \quad \text{(A11)}$$

which leads to

$$M_0-M_{\text{r}}=(\alpha/\pi)^2[\tfrac{1}{4}\ln(m^2/\lambda^2)-\tfrac{5}{8}]. \quad \text{(A12)}$$

Our renormalization procedure for the fermion self-energy graph [Fig. 3(b)] may be written as[28]

$$\begin{aligned} M_{\text{se}}-M_{\delta m}-M_{\text{wr}} \\ =(M_{\text{se}}-M_0)+(M_0-M_{\delta m}-M_{\text{wr}}), \end{aligned} \quad \text{(A13)}$$

where $M_{\delta m}$ and M_{wr} are self-mass and wave-function-renormalization subtraction terms, given by

$$\begin{aligned} M_{\delta m} &= \left(\frac{\alpha}{2\pi}\right)\left\{\frac{3\alpha}{4\pi}\left[\ln\left(\frac{\Lambda^2}{m^2}\right)+\frac{1}{2}\right]\right\}, \\ M_{\text{wr}} &= \left(\frac{\alpha}{2\pi}\right)\left\{-\frac{\alpha}{\pi}\left[\frac{1}{4}\ln\left(\frac{\Lambda^2}{m^2}\right)-\frac{1}{2}\ln\left(\frac{m^2}{\lambda^2}\right)+\frac{9}{8}\right]\right\}. \end{aligned} \quad \text{(A14)}$$

The integrals M_{se} and M_0 can be written as

$$\begin{aligned} M_{\text{se}} &= \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2\int dz\left[\frac{B_{34}(6A_1-6A_2+4A_1A_2)}{U^3V} - \frac{A_1(A_1-1)^2(A_2+2)}{U^2V^2}\right], \\ M_0 &= \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2\int dz\left[\frac{B_{34}(6A_{1,0}-6A_{2,0}+4A_{1,0}A_{2,0})}{U_0{}^3V_0} - \frac{A_{1,0}(A_{1,0}-1)^2(A_{2,0}+2)}{U_0{}^2V_0{}^2}\right], \end{aligned} \quad \text{(A15)}$$

where

$$\begin{aligned} U &= U_1U_2+z_2z_5, \\ U_1 &= z_1+z_3+z_4+z_6, \\ U_2 &= B_{34}=z_2+z_5, \\ V &= z_1+z_2+z_3+z_4+(\lambda/m)^2(z_5+z_6) \\ &\quad +z_6(z_6/U_1-1)-z_2z_5z_6{}^2/U_1U, \\ A_1 &= z_6U_2/U, \\ A_2 &= z_5z_6/U, \\ U_0 &= U_1U_2, \\ V_0 &= z_1+z_2+z_3+z_4+(\lambda/m)^2(z_5+z_6) \\ &\quad +z_6(z_6/U_1-1)-z_2z_5/U_2, \\ A_{1,0} &= z_6/U_1, \\ A_{2,0} &= z_5z_6/U_0. \end{aligned} \quad \text{(A16)}$$

As before, the integration of M_0 can be carried out analytically, giving

$$M_0=(\alpha/\pi)^2[\tfrac{1}{4}\ln(\Lambda^2/m^2)-\tfrac{1}{8}] \quad \text{(A17)}$$

and

$$M_0-M_{\delta m}-M_{\text{wr}}=(\alpha/\pi)^2[\tfrac{1}{4}-\tfrac{1}{4}\ln(m^2/\lambda^2)]. \quad \text{(A18)}$$

Finally we note that, in case of the ladder diagram, it is most convenient to carry out the subvertex renormalization in the conventional way because of the cancellation of infrared divergences between the ladder and accompanying subtraction diagrams. The only care we have to exercise is to symmetrize the integrand with respect to the variables associated with the two virtual photons.[15]