Quantized Fields and Particle Creation in Expanding Universes. II

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We consider the spin-$\frac{1}{2}$ field which satisfies the fully covariant generalization of the Dirac equation. The metric, which is not quantized, is that of an expanding universe with Euclidean 3-space. The field is quantized in a manner consistent with the time development dictated by the equation of motion. Consideration of the special-relativistic limit then provides a new proof of the connection between spin and statistics. In general, there will be production of spin-$\frac{1}{2}$ particles as a result of the expansion of the universe. However, we show that in the limits of zero and infinite mass there is no spin-$\frac{1}{2}$ particle production. For arbitrary mass, we obtain an upper bound on the creation of particles of given momentum. We treat the case of an instantaneous expansion exactly (but not taking into account the reaction of the particle creation back on the gravitational field). For such an expansion, the created particle density, when integrated over all momenta, diverges as a result of the high-momentum behavior. We also consider the Friedmann expansion of a radiation-filled universe, emphasizing the effect of the initial stage of the expansion. We obtain the asymptotic form of the created particle density for large momenta, and thus show that the particle density, integrated over all momenta, is finite, in contrast to the previous case.

I. INTRODUCTION

IN this paper, we consider the quantized spin-$\frac{1}{2}$ field. As in our previous work,[1–3] the space-time interval is given by the expression

$$ds^2=-dt^2+R(t)^2(dx^2+dy^2+dz^2)\,, \qquad (1)$$

with $R(t)$ an unspecified positive function of t. The equation governing the spin-$\frac{1}{2}$ field is the totally covariant generalization of the free Dirac equation. The gravitational metric is treated as an unquantized external field, and no additional interactions are included.

II. QUANTIZATION OF SPIN-$\frac{1}{2}$ FIELD

The totally covariant generalization of the Dirac equation has the following form for the metric of (1):

$$\left(\gamma^0\frac{\partial}{\partial t}+\tfrac{3}{2}R(t)^{-1}\dot{R}(t)\gamma^0+R(t)^{-1}\boldsymbol{\gamma}\cdot\nabla+\mu\right)\psi=0\,, \qquad (2)$$

where ∇ is the ordinary gradient, and the 4×4 matrices γ^1, γ^2, γ^3, and $\gamma^4=i\gamma^0$ are the constant, Hermitian, Pauli γ matrices, which satisfy the relations

$$\gamma^j\gamma^k+\gamma^k\gamma^j=2\delta^{jk} \quad (j,k=1,2,3,4)\,. \qquad (3)$$

In Appendix A, we arrive at Eq. (2) using the Bargmann-Schrödinger method.[4]

Substituting $R(t)^{-3/2}\,e^{ia\mathbf{p}\cdot\mathbf{x}}E(\mathbf{p},t)$ for ψ in Eq. (2), where a can take on the values ±1, we find that

$$\left(\gamma^0\frac{d}{dt}+iaR(t)^{-1}\boldsymbol{\gamma}\cdot\mathbf{p}+\mu\right)E(\mathbf{p},t)=0\,. \qquad (4)$$

When $R(t)\equiv1$, then Eq. (4) is equivalent to the special-relativistic Dirac equation in momentum space. A set of independent solutions of Eq. (4) with $R(t)\equiv1$ will be denoted by

$$u^{(a,d)}(\mathbf{p})\,, \qquad (5)$$

where d, as well as a, can take on the values ±1. The index a labels the positive- and negative-energy solutions, while the index d labels the helicity, or spin component along the $\mathbf{p}$ direction. It is convenient for our purposes to use two indices rather than one. Our notation, together with the relevant properties of the $u^{(a,d)}(\mathbf{p})$, is given in Appendix B.

Since the four-component spinors of Eq. (5) form a complete set, we can write any solution of Eq. (2) in the form

$$\psi(\mathbf{x},t)=[LR(t)]^{-3/2}\sum_{\mathbf{p}}\left(\frac{\mu}{\omega(p,t)}\right)^{-1/2}\sum_{a,d}a_{(a,d)}(\mathbf{p},t)\times u^{(a,d)}(\mathbf{p},t)\exp\left[ia\left(\mathbf{p}\cdot\mathbf{x}-\int_{t_0}^{t}\omega(p,t')dt'\right)\right]. \qquad (6)$$

In Eq. (6), we have, as in Ref. 2, imposed the periodic boundary condition $\psi(\mathbf{x}+\mathbf{n}L,t)=\psi(\mathbf{x},t)$, where $\mathbf{n}$ is a vector with integer Cartesian components. The time dependence of the operators $a_{(a,d)}(\mathbf{p},t)$ is determined by Eq. (2). Further, the quantities $\omega(p,t)$ and $u^{(a,d)}(\mathbf{p},t)$ are defined by

$$\omega(p,t)=(p^2/R(t)^2+\mu^2)^{1/2} \quad (p=|\mathbf{p}|) \qquad (7)$$

and

$$u^{(a,d)}(\mathbf{p},t)=u^{(a,d)}(\mathbf{p}/R(t))\,, \qquad (8)$$

where $u^{(a,d)}(\mathbf{p})$ was defined in (5). The factors

$$(LR)^{-3/2}(\mu/\omega)^{-1/2}\exp\left[-ia\int_{t_0}^{t}\omega dt'\right]$$

are included explicitly, so that in any interval during which $R(t)$ is constant, (6) will have the usual special-

[1] L. Parker, Phys. Rev. Letters **21**, 562 (1968).
[2] L. Parker, Phys. Rev. **183**, 1057 (1969).
[3] This paper is partially based on L. Parker, Ph.D. thesis, Harvard University, 1966 (unpublished).
[4] V. Bargmann, Sitzber. Deut. Akad. Wiss. Berlin, Math.-Naturw. Kl. **1932**, 346 (1932).

relativistic form,[5] with the $a_{(a,d)}$ then being time-independent creation and annihilation operators. In accordance with their interpretation in special relativity, $a_{(1,d)}(\mathbf{p},t)$ is an annihilation operator for a particle with spin in the $\mathbf{p}d$ direction, and $a_{(-1,d)}(\mathbf{p},t)$ is a creation operator for an antiparticle with spin in the $-\mathbf{p}d$ direction. Similarly, $a_{(1,d)}^\dagger$ creates the particles which $a_{(1,d)}$ annihilates, and $a_{(-1,d)}^\dagger$ annihilates the antiparticles which $a_{(-1,d)}$ creates. We impose the anticommutation relation[6]

$$\{a_{(a,d)}(\mathbf{p},t),a_{(a',d')}^\dagger(\mathbf{p}',t)\}=\delta_{a,a'}\delta_{d,d'}\delta_{\mathbf{p},\mathbf{p}'}. \quad (9)$$

We will show that Eq. (9) is consistent with the time development dictated by the equation of motion (2). Our procedure is analogous to the one used in Ref. 2 to show that the boson commutation rules are consistent.

We assume that we are given

$$a_{(a,d)}(\mathbf{p},t_1)=A_{(a,d)}(\mathbf{p}), \quad (10)$$

with

$$\{A_{(a,d)}(\mathbf{p}),A_{(a',d')}^\dagger(\mathbf{p}')\}=\delta_{a,a'}\delta_{d,d'}\delta_{\mathbf{p},\mathbf{p}'}. \quad (11)$$

We make the ansatz analogous to that used in Ref. 2:

$$A_{(a,d)}(\mathbf{p},t)=\sum_{a'=\pm a} D_{(a)}^{(a')}(p,t)A_{(a',aa'd)}(aa'\mathbf{p}), \quad (12)$$

where the $D_{(a)}^{(a')}(p,t)$ are complex functions of the momentum magnitude $p=|\mathbf{p}|$, and the time.[7] Equation (12) will imply, for example, that the annihilation operator at time t of a particle of momentum $\mathbf{p}/R(t)$, with spin parallel to $\mathbf{p}$, is a linear combination of the annihilation operator at t_1 of a particle of momentum $\mathbf{p}/R(t_1)$, with spin parallel to $\mathbf{p}$, and of the creation operator of an antiparticle of momentum $-\mathbf{p}/R(t_1)$, with spin parallel to $-\mathbf{p}$. We now derive the integral equation satisfied by $D_{(a)}^{(a')}(p,t)$.

Substituting Eq. (12) into (6), and regrouping, we obtain

$$\psi(\mathbf{x},t)=[LR(t)]^{-3/2}\sum_{\mathbf{p}}\sum_{a,d} A_{(a,d)}(\mathbf{p})E^{(a,d)}(\mathbf{p},t)e^{ia\mathbf{p}\cdot\mathbf{x}}, \quad (13)$$

where

$$E^{(a,d)}(\mathbf{p},t)=\left(\frac{\mu}{\omega(p,t)}\right)^{1/2}\sum_{a'} D_{(a')}^{(a)}(p,t)u^{(a',aa'd)}(aa'\mathbf{p},t)\times\exp\left(-ia'\int_{t_0}^{t}\omega(p,t')dt'\right). \quad (14)$$

The summations involving a, a', and d are over ±1. Since $\{A_{(a,d)}^\dagger(\mathbf{p}),\psi(\mathbf{x},t)\}$ must satisfy Eq. (2), it follows that $E^{(a,d)}$ satisfies the equation

$$\left(\gamma^0\frac{d}{dt}+iaR(t)^{-1}\boldsymbol{\gamma}\cdot\mathbf{p}+\mu\right)E^{(a,d)}(\mathbf{p},t)=0$$

or

$$i\frac{d}{dt}E^{(a,d)}(\mathbf{p},t)=[aR(t)^{-1}\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta]E^{(a,d)}(\mathbf{p},t), \quad (15)$$

where α and β are the usual Dirac matrices, $\beta=\gamma^4=i\gamma^0$, and $\boldsymbol{\alpha}=i\gamma^4\boldsymbol{\gamma}$. The boundary condition is clearly

$$E^{(a,d)}(\mathbf{p},t_1)=\left(\frac{\mu}{\omega(p,t_1)}\right)^{1/2}u^{(a,d)}(\mathbf{p},t_1)\times\exp\left(-ia\int_{t_0}^{t_1}\omega(p,t')dt'\right). \quad (16)$$

Note that Eq. (14) indicates, according to the properties of the $u^{(a,d)}$ given in Appendix B, that $E^{(a,d)}(\mathbf{p},t)$ remains at all times an eigenvector of $\sigma_{\mathbf{p}}$ (defined in Appendix B) with eigenvalue d. That is consistent with Eq. (15) because, as is readily confirmed, $\sigma_{\mathbf{p}}$ commutes with the operator $aR(t)^{-1}\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta$, which generates the time displacement of $E^{(a,d)}$.

Alternatively, instead of starting with (12), we could have started with (13), then used (15) and (16) to show that $E^{(a,d)}$ is always an eigenvector of $\sigma_{\mathbf{p}}$, and thus arrived at (14) as the most general form of $E^{(a,d)}$. Substitution of (14) into (13) would then yield (12) and (6).

We obtain an integral equation for $E^{(a,d)}(\mathbf{p},t)$ by comparing Eq. (15) with the first-order equation satisfied exactly by the column vector $E_0^{(a,d)}$, which we define by

$$E_0^{(a,d)}(\mathbf{p},t)\equiv\left(\frac{\mu}{\omega(p,t)}\right)^{1/2}\sum_{a'} D_{(a')}^{(a)}(p,1)u^{(a',aa'd)}(aa'\mathbf{p},t)\times\exp\left(-ia'\int_{t_0}^{t}\omega(p,t')dt'\right), \quad (17)$$

where the $D_{(a')}^{(a)}(p,1)$ are time-independent coefficients. We will later want to set $D_{(a')}^{(a)}=\delta_{a'}^{a}$, so that $E_0^{(a,d)}(\mathbf{p},t_1)=E^{(a,d)}(\mathbf{p},t_1)$, but for the present it is advantageous to let the $D_{(a')}^{(a)}(p,1)$ be arbitrary.[8] The first-order equation satisfied by $E_0^{(a,d)}$ is obtained as follows. Using (17), and the equation [obtained from Eq. (B4) by replacing $\mathbf{p}$ with $\mathbf{p}/R(t)$]

$$a'\omega(p,t)u^{(a',aa'd)}(aa'\mathbf{p},t)=[aR(t)^{-1}\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta]u^{(a',aa'd)}(aa'\mathbf{p},t), \quad (18)$$

we obtain

$$i\frac{d}{dt}E_0^{(a,d)}(\mathbf{p},t)=(aR(t)^{-1}\boldsymbol{\alpha}\cdot\mathbf{p}+\mu\beta)E_0^{(a,d)}(\mathbf{p},t)+i\sum_{a'} D_{(a')}^{(a)}(p,1)\frac{d}{dt}\left[\left(\frac{\mu}{\omega(p,t)}\right)^{1/2}u^{(a',aa'd)}(aa'\mathbf{p},t)\right]\times\exp\left(-ia'\int_{t_0}^{t}\omega(p,t')dt'\right). \quad (19)$$

[5] See, e.g., F. Mandl, *Introduction to Quantum Field Theory* (Interscience, New York, 1959), p. 52.

[6] Equation (9) implies, as in special relativity, that $\{\psi(\mathbf{x},t),\psi^\dagger(\mathbf{x}',t)\}=\delta^{(3)}(\mathbf{x}-\mathbf{x}')$.

[7] We do not include an index d in the $D_{(a)}^{(a')}$ because it turns out to be unnecessary. Note that, as before, the indices a and a' take the values ±1, so that a^2 is always unity.

[8] For example, it will be easier to discuss the matrix G defined later.

According to Eq. (B17), if we define the row vector

$$\bar{W}^{(a,d)}(\mathbf{p},t)=\frac{\mu}{\omega(p,t)}\bar{u}^{(a,-d)}(-\mathbf{p},t), \qquad (20)$$

then

$$\bar{W}^{(b,abd)}(ab\mathbf{p},t)u^{(a',aa'd)}(aa'\mathbf{p},t)=\delta_{a',b}. \qquad (21)$$

Therefore, Eq. (19) can be written in the form

$$\left[i\frac{d}{dt}-\frac{a}{R(t)}\boldsymbol{\alpha}\cdot\mathbf{p}-\mu\beta+iM^{(a,d)}(\mathbf{p},t)\right]E_0^{(a,d)}(\mathbf{p},t)=0, \quad (22)$$

where $M^{(a,d)}(\mathbf{p},t)$ is the 4×4 matrix[9] defined by

$$M^{(a,d)}(\mathbf{p},t)=-\left(\frac{\omega(p,t)}{\mu}\right)^{1/2}\times\sum_b\frac{d}{dt}\left[\left(\frac{\mu}{\omega(p,t)}\right)^{1/2}u^{(b,abd)}(ab\mathbf{p},t)\right]\times\bar{W}^{(b,abd)}(ab\mathbf{p},t). \quad (23)$$

To obtain an integral equation satisfied by $E^{(a,d)}$, we will compare Eq. (22) with (15).

For that purpose, we define the 4×4 matrix

$$G^{(a,d)}(\mathbf{p},t,t')=\left(\frac{\omega(p,t')}{\omega(p,t)}\right)^{1/2}\times\sum_{a'}u^{(a',aa'd)}(aa'\mathbf{p},t)\bar{W}^{(a',aa'd)}(aa'\mathbf{p},t')\times\exp\left(-ia'\int_{t'}^{t}\omega(p,s)ds\right). \quad (24)$$

For fixed t', each column of the matrix $G^{(a,d)}$ is of the same general form as $E_0^{(a,d)}$ of Eq. (17). Therefore, each column of G satisfies Eq. (22), and we can write

$$\left[i\frac{\partial}{\partial t}-\frac{a}{R(t)}\boldsymbol{\alpha}\cdot\mathbf{p}-\mu\beta+iM^{(a,d)}(\mathbf{p},t)\right]\times G^{(a,d)}(\mathbf{p},t,t')=0. \quad (25)$$

When $t=t'$,

$$G^{(a,d)}(\mathbf{p},t,t)=\sum_{a'}u^{(a',aa'd)}(aa'\mathbf{p},t)\bar{W}^{(a',aa'd)}(aa'\mathbf{p},t) \quad (26)$$

is the projection operator onto the manifold of eigenvector of $\sigma_{\mathbf{p}}$ with eigenvalue d. This can be verified by making use of (21), and applying (26) to the two spinors $u^{(a',aa'd)}(aa'\mathbf{p},t)$ with $a'=\pm1$, which span the manifold of eigenvectors of $\sigma_{\mathbf{p}}$ with eigenvalue d, and to the two spinors $u^{(a',-aa'd)}(aa'\mathbf{p},t)$ with $a'=\pm1$, which span the manifold of eigenvectors of $\sigma_{\mathbf{p}}$ with eigenvalue $-d$.

[9] Matrix multiplication of a column vector by the row vector $\bar{W}$ from the right yields a 4×4 matrix.

Clearly,

$$\sigma_{\mathbf{p}}\frac{d}{dt}\left[\left(\frac{\mu}{\omega(p,t)}\right)^{1/2}u^{(b,abd)}(ab\mathbf{p},t)\right]=(d)\frac{d}{dt}\left[\left(\frac{\mu}{\omega(p,t)}\right)^{1/2}u^{(b,abd)}(ab\mathbf{p},t)\right].$$

Therefore,

$$G^{(a,d)}(\mathbf{p},t,t)M^{(a,d)}(\mathbf{p},t)=M^{(a,d)}(\mathbf{p},t). \quad (27)$$

The integral equation for $E^{(a,d)}(\mathbf{p},t)$, which satisfies the boundary condition

$$E^{(a,d)}(\mathbf{p},t_1)=E_0^{(a,d)}(\mathbf{p},t_1), \quad (28)$$

can now be written in the form

$$E^{(a,d)}(\mathbf{p},t)=E_0^{(a,d)}(\mathbf{p},t)+\int_{t_1}^{t}dt'G^{(a,d)}(\mathbf{p},t,t')M^{(a,d)}(\mathbf{p},t')E^{(a,d)}(\mathbf{p},t'). \quad (29)$$

We verify that (29) satisfies Eq. (4) by applying the matrix operator $[id/dt-aR(t)^{-1}\boldsymbol{\alpha}\cdot\mathbf{p}-\mu\beta+iM^{(a,d)}(\mathbf{p},t)]$ to (29), and using Eqs. (22), (25), and (27). Thus,

$$\begin{aligned}\left[i\frac{d}{dt}-\frac{a}{R(t)}\boldsymbol{\alpha}\cdot\mathbf{p}-\mu\beta+iM^{(a,d)}(\mathbf{p},t)\right]E^{(a,d)}(\mathbf{p},t)&=iG^{(a,d)}(\mathbf{p},t,t)M^{(a,d)}(\mathbf{p},t)E^{(a,d)}(\mathbf{p},t)\\&\quad+\int_{t_1}^{t}dt'\left[i\frac{\partial}{\partial t}-\frac{a}{R(t)}\boldsymbol{\alpha}\cdot\mathbf{p}-\mu\beta+iM^{(a,d)}(\mathbf{p},t)\right]\\&\quad\times G^{(a,d)}(\mathbf{p},t,t')M^{(a,d)}(\mathbf{p},t')E^{(a,d)}(\mathbf{p},t')\\&=iM^{(a,d)}(\mathbf{p},t)E^{(a,d)}(\mathbf{p},t),\end{aligned}$$

which yields Eq. (4).

Substituting Eq. (14) for $E^{(a,d)}(\mathbf{p},t)$ on both sides of (29), we find that we can write

$$D_{(a')}^{(a)}(p,t)=D_{(a')}^{(a)}(p,1)+\sum_b\int_{t_1}^{t}dt'S_{(a')}^{(b)}(p,t')D_{(b)}^{(a)}(p,t')\times\exp\left[i(a'-b)\int_{t_0}^{t}\omega(p,t')dt'\right], \quad (30)$$

with[10]

$$S_{(a')}^{(b)}(p,t)\equiv-\left(\frac{\omega(p,t)}{\mu}\right)^{1/2}\bar{W}^{(a',aa'd)}(aa'\mathbf{p},t)\times\frac{d}{dt}\left[\left(\frac{\mu}{\omega(p,t)}\right)^{1/2}u^{(b,abd)}(ab\mathbf{p},t)\right]. \quad (31)$$

[10] As will become evident later, $S_{(a')}^{(b)}$ is independent of the direction of $\mathbf{p}$, and the values of the indices a and d in (31).

In obtaining Eq. (30), we have made use of (17), (21), (23), and (24).

In proceeding further, it is convenient to work in a specific matrix representation. The representation we will use, including the matrices representing the $u^{(a,d)}(\mathbf{p},t)$, are given in Appendix B. After some calculations, one finds that in our representation

$$\frac{d}{dt}u^{(a,d)}(\mathbf{p},t)=-\dot{R}(t)R(t)^{-2}[2\omega(p,t)]^{-1}pu^{(-a,d)}(\mathbf{p},t)$$

and

$$S_{(a')}{}^{(b)}(p,t)=a'\delta_{a'}{}^{-b}S(p,t)\,, \tag{32}$$

where

$$S(p,t)\equiv\dot{R}(t)R(t)^{-2}[2\omega(p,t)^2]^{-1}\mu p\,. \tag{33}$$

Equation (30) now takes the form

$$D_{(a')}{}^{(a)}(p,t)=D_{(a')}{}^{(a)}(p,1)+a'\int_{t_1}^{t}dt'S(p,t')\times\exp\left(2ia'\int_{t_0}^{t}\omega(p,s)ds\right)D_{(-a')}{}^{(a)}(p,t')\,. \tag{34}$$

It follows that (for $a'=\pm1$ and $a=\pm1$)

$$\frac{d}{dt}D_{(a')}{}^{(a)}(p,t)=a'S(p,t)\times\exp\left(2ia'\int_{t_0}^{t}\omega dt'\right)D_{(-a')}{}^{(a)}(p,t)\,. \tag{35}$$

In accordance with Eq. (10), we impose the boundary condition

$$D_{(a')}{}^{(a)}(p,1)=\delta_{a'}{}^{a}\,. \tag{36}$$

It then follows by considering (35) and its complex conjugate, or by iteration of (34), that

$$D_{(a')}{}^{(a)}(p,t)=a'aD_{(-a')}{}^{(-a)}(p,t)^*\,. \tag{37}$$

Using (37), we find that

$$\sum_b D_{(a)}{}^{(b)}(p,t)D_{(-a)}{}^{(b)}(p,t)^*=0\,. \tag{38}$$

Then, making use of (35) and (38), we obtain

$$\frac{d}{dt}\left[\sum_b D_{(a)}{}^{(b)}(p,t)D_{(a)}{}^{(b)}(p,t)^*\right]=aS\exp\left(2ia\int_{t_0}^{t}\omega dt'\right)\sum_b D_{(-a)}{}^{(b)}D_{(a)}{}^{(b)*}+a'S\exp\left(-2ia'\int_{t_0}^{t}\omega dt'\right)\times\sum_b D_{(a)}{}^{(b)}D_{(-a)}{}^{(b)*}=0\,. \tag{39}$$

It follows from (36), (38), and (39) that

$$\sum_b D_{(a)}{}^{(b)}(p,t)D_{(a')}{}^{(b)}(p,t)^*=\delta_{aa'}\,. \tag{40}$$

Substituting (12) into (9) gives

$$\{a_{(a,d)}(\mathbf{p},t),a_{(a',d')}{}^{\dagger}(\mathbf{p}',t)\}=\sum_b\sum_c D_{(a)}{}^{(b)}D_{(a')}{}^{(c)*}\times\{A_{(b,abd)}(ab\mathbf{p}),A_{(c,a'cd')}{}^{\dagger}(a'c\mathbf{p}')\}=\sum_b D_{(a)}{}^{(b)}D_{(a')}{}^{(b)*}\delta_{ad,a'd'}\delta_{\mathbf{p},\mathbf{p}'}=\delta_{aa'}\delta_{d,d'}\delta_{\mathbf{p},\mathbf{p}'}\,, \tag{41}$$

where we have made use of (11) and (40). Therefore, the anticommutation relation (9) is consistent with the time development dictated by the equation of motion.

In fact, if Eq. (9) holds at any particular time, it will be propagated unchanged by the equation of motion. In particular, the familiar special-relativistic anticommutation relations imposed during any period in which $R(t)$ is constant will imply that Eq. (9) must hold for all t, earlier as well as later than the period of constant $R(t)$. Consequently, Eq. (9) must be the correct quantization condition when $R(t)$ is not constant.

III. SPIN AND STATISTICS

If one were to attempt to impose, at time t_1, the boson commutation relation

$$[A_{(a,d)}(\mathbf{p}),A_{(a',d')}{}^{\dagger}(\mathbf{p}')]=a\delta_{aa'}\delta_{d,d'}\delta_{\mathbf{p},\mathbf{p}'}\,,$$

which is the generalization, in our notation, of the special-relativistic boson commutation relation, then (12) would yield

$$[a_{(a,d)}(\mathbf{p},t),a_{(a',d')}{}^{\dagger}(\mathbf{p}',t)]=\sum_b\sum_c D_{(a)}{}^{(b)}D_{(a')}{}^{(c)*}\times[A_{(b,abd)}(ab\mathbf{p}),A_{(c,a'cd')}{}^{\dagger}(a'c\mathbf{p}')]=\sum_b D_{(a)}{}^{(b)}D_{(a')}{}^{(b)*}b\delta_{ad,a'd'}\delta_{\mathbf{p},\mathbf{p}'}\,,$$

which does not reduce to a $\delta_{a,a'}\delta_{d,d'}\delta_{\mathbf{p},\mathbf{p}'}$. Therefore, the boson commutation relations are generally not propagated in time by the equation of motion and cannot consistently be required to hold at all times. As we showed in Sec. II, the fermion anticommutation relations are consistent with the equation of motion (2).

This is analogous to the situation in Ref. 2, in which we showed that the boson, but not the fermion, relations were consistent with the general-relativistic equation of motion satisfied by the spin-0 field. By the same argument as was used in Ref. 2 to prove that the boson, rather than the fermion, relations should hold for the spin-0 field in special relativity, our present considerations yield a new and independent proof that the fermion, rather than the boson, relations should hold for the spin-$\frac{1}{2}$ field in special relativity.

Suppose that $R(t)$ is a function which is constant only during a limited period of time. As we showed above, only the fermion anticommutators are consistent with the generally covariant equation of motion when $R(t)$ is not constant. The continuity requirement, that the quantization relations should not suddenly jump from the fermion relations when $R(t)$ is slightly time dependent to the boson relations when $R(t)$ is constant, therefore demands that the fermion anticommutation relations should continue to hold during the period when $R(t)$ is constant, even though neither quantization scheme is ruled out solely by the special-relativistic equation of motion, which holds when $R(t)$ is constant. Alternatively, one could consider a sequence of functions $R_n(t)=1+\epsilon_n(t)$, where $\epsilon_n(t)\to 0$ as $n\to\infty$, as in Ref. 2. Our method of obtaining the connection between spin and statistics in special relativity depends only on the conditions of consistency with the generally covariant equation of motion, and continuity.

IV. PARTICLE PRODUCTION, ZERO- AND INFINITE-MASS LIMITS

During any period in which $R(t)$ is constant, the creation and annihilation operators $a_{(a,d)}(\mathbf{p},t)$ are also time independent, and unambiguously correspond to the physical or observable spin-$\frac{1}{2}$ particles. When $R(t)$ is not constant, the $a_{(a,d)}(\mathbf{p},t)$ still are creation and annihilation operators because they satisfy Eq. (9). However, it will be recalled that in Ref. 2 there existed many sets of creation and annihilation operators when $R(t)$ was not constant. It is not clear whether a similar situation exists in the spin-$\frac{1}{2}$ case, or whether the $a_{(a,d)}(\mathbf{p},t)$ are unique. Therefore, we will only assume that the particles corresponding to the $a_{(a,d)}(\mathbf{p},t)$ and the physical particles are identical during periods in which the $a_{(a,d)}(\mathbf{p},t)$ are time independent. They may possibly not be identical when the $a_{(a,d)}(\mathbf{p},t)$ are time dependent.[11]

Working in the Heisenberg picture, as before, we define the state $|0\rangle$, which contains no particles or antiparticles at t_1, by

$$A_{(1,d)}(\mathbf{p})|0\rangle=0$$

and (42)

$$A_{(-1,d)}{}^{\dagger}(\mathbf{p})|0\rangle=0\,,\quad \text{for all } \mathbf{p} \text{ and } d.$$

The average number of particles present in the state $|0\rangle$ in the mode p and volume $[LR(t)]^3$ at time t is

$$\begin{aligned}\langle N_{\mathbf{p}}(t)\rangle_0&=\sum_d \langle 0|a_{(1,d)}{}^{\dagger}(\mathbf{p},t)a_{(1,d)}(\mathbf{p},t)|0\rangle\\&=\sum_d |D_{(1)}{}^{(-1)}(p,t)|^2\\&\qquad\times\langle 0|A_{(-1,-d)}{}^{\dagger}(\mathbf{p})A_{(-1,-d)}(\mathbf{p})|0\rangle\\&=2|D_{(1)}{}^{(-1)}(p,t)|^2,\end{aligned}\tag{43}$$

where we have used Eqs. (12), (42), and the conjugate of (42). Similarly, the average number of antiparticles present in the state $|0\rangle$ in the mode $\mathbf{p}$ and volume $[LR(t)]^3$ at time t is

$$\begin{aligned}\langle \bar{N}_{\mathbf{p}}(t)\rangle_0&=\sum_d \langle 0|a_{(-1,d)}(\mathbf{p},t)a_{(-1,d)}{}^{\dagger}(\mathbf{p},t)|0\rangle\\&=2|D_{(-1)}{}^{(1)}(p,t)|^2.\end{aligned}\tag{44}$$

It follows from (37), (43), and (44) that the average number of particles in each mode equals the average number of antiparticles.

The average total number of particles present in the state $|0\rangle$ in the volume $[LR(t)]^3$ at time t is

$$\langle N(t)\rangle_0=2\sum_p |D_{(1)}{}^{(-1)}(p,t)|^2.\tag{45}$$

In the limit $L\to\infty$, when $\sum_{\mathbf{k}}\to(L/2\pi)^3\int d^3p$, the average partcile density is

$$\lim_{L\to\infty}[LR(t)]^{-3}\langle N(t)\rangle_0=[\pi^2R(t)^3]^{-1}\times\int_0^\infty dp\, p^2|D_{(1)}{}^{(-1)}(p,t)|^2.\tag{46}$$

Suppose the state of the universe is described by a statistical mixture of pure states, each of which contains definite numbers of particles and antiparticles at t_1. Then the statistical density matrix ρ is diagonal in the representation whose basis consists of the simultaneous eigenstates of the operators $A_{(1,d)}{}^{\dagger}(\mathbf{p})A_{(1,d)}(\mathbf{p})$ and $A_{(-1,d)}(\mathbf{p})A_{(-1,d)}{}^{\dagger}(\mathbf{p})$ (the t_1 representation), and ρ must contain equal numbers of corresponding creation and annihilation operators. The average number of particles in mode $\mathbf{p}$ and volume $[LR(t)]^3$ present at time t is

$$\begin{aligned}\langle N_{\mathbf{p}}(t)\rangle&=\sum_d \mathrm{Tr}[\rho a_{(1,d)}{}^{\dagger}(\mathbf{p},t)a_{(1,d)}(\mathbf{p},t)]\\&=\sum_d\sum_b\sum_c D_{(1)}{}^{(b)}(p,t)^*D_{(1)}{}^{(c)}(p,t)\\&\qquad\times\mathrm{Tr}[\rho A_{(b,bd)}{}^{\dagger}(b\mathbf{p})A_{(c,cd)}(c\mathbf{p})].\end{aligned}\tag{47}$$

By taking the trace over the basis of the t_1 representation, and noting that ρ has equal numbers of creation and annihilation operators, it becomes evident that only those terms in which $b=c$ do not vanish in (47).

[11] We call the excitations annihilated by $a_{(1,d)}$ particles, and those annihilated by $a_{(-1,d)}{}^{\dagger}$, antiparticles. When the $a_{(a,d)}$ are time dependent those excitations may not necessarily be the physically observable particles. When the $a_{(a,d)}$ are time independent, we assume that they correspond to physical particles, even when $R(t)$ is not constant.

Then using (11), (37), and (40), we find that

$$\langle N_{\mathbf{p}}(t)\rangle=\langle N_{\mathbf{p}}(t_1)\rangle+|D_{(1)}{}^{(-1)}(p,t)|^2[2-\langle N_{\mathbf{p}}(t_1)\rangle-\langle\bar{N}_{-\mathbf{p}}(t_1)\rangle], \quad (48)$$

where

$$\langle N_{\mathbf{p}}(t_1)\rangle=\sum_d \mathrm{Tr}[\rho A_{(1,d)}{}^{\dagger}(\mathbf{p})A_{(1,d)}(\mathbf{p})]$$

and

$$\langle\bar{N}_{-\mathbf{p}}(t_1)\rangle=\sum_d \mathrm{Tr}[\rho A_{(-1,d)}(-\mathbf{p})A_{(-1,d)}{}^{\dagger}(-\mathbf{p})]$$

are, respectively, the average number of particles in the mode **p** and antiparticles in the mode $-\mathbf{p}$ at time t_1. Comparison of (48) with (43) shows that the initial presence of fermions tends to decrease the number of fermions created by the expansion of the universe between times t_1 and t. For bosons, as we found in Ref. 2, the situation is reversed.

In the spin-0 case, we found that the annihilation and creation operators were time independent in the zero- and infinite-mass limits only for certain forms of $R(t)$ corresponding to particular Friedmann expansions of the universe. In the present spin-$\frac{1}{2}$ case, the $a_{(a,d)}$ are time independent in the zero- and infinite-mass limits, regardless of the forms of $R(t)$. This conclusion follows from consideration of the expression (33) for $S(p,t)$, which clearly vanishes in each of the limits when $\mu\to 0$ and when $\mu\to\infty$. It then follows from (34) and (12) that the $a_{(a,d)}$ are time independent in those limits. Thus, there is no creation of physical spin-$\frac{1}{2}$ particles in the zero- and infinite-mass limits, for any form of $R(t)$. In Sec. H of Ref. 2, the two-component, rather than four-component, spin-$\frac{1}{2}$ equation was considered, and the same result was obtained for zero mass. The present result for the infinite-mass limit was mentioned in Secs. F and G of Ref. 2.

V. SERIES EXPANSION AND UPPER BOUND

The solution of Eq. (35) satisfying the boundary condition (36) can be written in series form as follows (suppressing the label p for convenience):

$$D_{(-a)}{}^{(a)}(t)=\sum_{n=0}^{\infty}(-1)^{n+1}[2n+1,\,a,\,t] \quad (49)$$

and

$$D_{(a)}{}^{(a)}(t)=\sum_{n=0}^{\infty}(-1)^{n}[2n,a,t]^*, \quad (50)$$

where

$$[0,a,t]=1$$

and

$$[n,a,t]=\int_{t_1}^{t}dt'aS(t')\times\exp\left(-2ia\int_{t_0}^{t'}\omega ds\right)[n-1,\,a,\,t']^*. \quad (51)$$

The above series are the iterative solutions of (34), and can be confirmed by direct substitution in (35).

To obtain an upper bound on $|D_{(-a)}{}^{(a)}|$, which appears in (43) and (48), we note that

$$|[n,a,t]|\le\int_{t_1}^{t}dt'|S(t')|\times\int_{t_1}^{t'}dt''|S(t'')|\cdots\int_{t_1}^{t^{(n-1)}}dt^{(n)}|S(t^{(n)})|=\frac{1}{n!}\left(\int_{t_1}^{t}dt'|S(t')|\right)^n. \quad (52)$$

Therefore,

$$|D_{(-a)}{}^{(a)}(t)|\le\sum_{n=0}^{\infty}|[2n+1,\,a,\,t]|\le\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\int_{t_1}^{t}dt'|S(t')|\right)^{2n+1},$$

which gives the upper bound

$$|D_{(-a)}{}^{(a)}(p,t)|\le\sinh\left(\int_{t_1}^{t}dt'|S(p,t')|\right). \quad (53)$$

Using (33), one finds that for a monotonic expansion with $\dot{R}(t)>0$

$$\int_{t_1}^{t}dt'|S(p,t)|=\frac{1}{2}\left[\tan^{-1}\left(\frac{p}{\mu R(t_1)}\right)-\tan^{-1}\left(\frac{p}{\mu R(t_2)}\right)\right]. \quad (54)$$

Since $p/[\mu R(t)]$ is positive, it follows that the right-hand side of (54) is always less than or equal to $\frac{1}{4}\pi$ (this holds even if the expansion is not monotonic). Therefore, using (53), we obtain

$$|D_{(-a)}{}^{(a)}(p,t)|^2\le\sinh^2(\tfrac{1}{4}\pi)\approx 0.4. \quad (55)$$

This is consistent with the exclusion principle, which follows from the anticommutation relation (9), and requires $|D_{(-a)}{}^{(a)}|^2$ to be smaller than unity.

As p approaches ∞, (54) and the right-hand side of (53) approach zero as p^{-1}. Therefore, the integral which results from substitution of the right-hand side of (53) into (46) diverges. Consequently, although (53) provides an upper bound on the particle production in each mode, it does not provide a finite upper bound on the particle production integrated over all modes. As we shall show in Sec. VI, the integrated particle density resulting from an instantaneous expansion of the universe diverges. Since (54) is independent of the form of the monotonic $R(t)$ between times t_1 and t_2, one would therefore expect (45) to lead to a divergent upper bound on the integrated particle density.

VI. INSTANTANEOUS EXPANSION OF THE UNIVERSE

A relatively simple case, in which we can obtain the exact solution for $D_{(-1)}{}^{(1)}$, is the instantaneous expansion

$$R(t)=\begin{cases} R_1 & \text{for} \quad t<t^* \\ R_2 & \text{for} \quad t>t^* . \end{cases} \qquad (56)$$

This problem is not realistic, since $R(t)$ is not a solution of Einstein's field equation, and since the large amount of particle production which occurs in this type of expansion would have a significant effect, which we do not take into account, on the form of the expansion. Nevertheless, the problem is interesting as an exactly soluble example.

For $t<t^*$, we put

$$D_{(a)}{}^{(a')}(p,1)=\delta_a{}^{a'} . \qquad (57)$$

The 1 in the argument of $D_{(a)}{}^{(-a)}(p,1)$, or in that of any similar quantity, indicates the value of that quantity for $t<t^*$. Similarly, a 2 will indicate the value for $t>t^*$. By considering (56) as the limiting case of rapid but not instantaneous expansions, it is clear that $E(\mathbf{p},t)$ of Eq. (4) remains continuous in the limit of an instantaneous expansion, whereas ψ in Eq. (2) will jump discontinuously because of the appearance of a δ function in Eq. (2) (i.e., the time derivative of ψ becomes infinite at the time t^*, indicating a discontinuity in ψ).

Continuity of $E^{(1,1)}$ of Eq. (14) at time t^* yields the following equation (we have put $t_0=t^*$ for convenience):

$$\begin{aligned}[]&[\omega(p,1)]^{-1/2}u^{(1,1)}(\mathbf{p},1) \\ &=[\omega(p,2)]^{-1/2}[D_{(1)}{}^{(1)}(p,2)u^{(1,1)}(\mathbf{p},2) \\ &\qquad +D_{(-1)}{}^{(1)}(p,2)u^{(-1,-1)}(\mathbf{p},2)] . \end{aligned} \qquad (58)$$

Using the matrices for the $u^{(a,d)}(\mathbf{p},t)$ given in Appendix B, and solving the resulting pair of equations for $D_{(-1)}{}^{(1)}(p,2)$, yields the result

$$D_{(-1)}{}^{(1)}(p,2) =\left(\frac{g(p,1)g(p,2)}{\omega(p,1)\omega(p,2)}\right)^{1/2}\left(\frac{p/R_2}{g(p,2)}-\frac{p/R_1}{g(p,1)}\right), \qquad (59)$$

where $g(p,t)=\omega(p,t)+\mu$, and we have made use of the identity

$$\left(1+\frac{p^2/R_2{}^2}{g(p,2)^2}\right)^{-1}=\frac{g(p,2)}{2\omega(p,2)} .$$

As p approaches ∞, (59) approaches zero as p^{-1}. Therefore, as for the upper bound discussed in Sec. V, the integrated particle density obtained by substitution of (59) into (46) diverges. It can be checked by direct calculation that (59) is smaller in absolute value than the upper bound (54) for various values of $p/\mu R_1$ and $p/\mu R_2$. We now turn to an example in which the integrated particle density is finite.

VII. FRIEDMANN EXPANSION

The Friedmann universe in which radiation is predominant, and in which the 3-space is flat, is described by

$$R(t)=Kt^{1/2} \quad (t\geq 0) , \qquad (60)$$

where K is a positive constant. We assume that at $t=0$ there are no spin-$\frac{1}{2}$ particles or antiparticles present. Therefore, we let $t_1=0$ in Eq. (10) and the subsequent equations involving t_1, and we let the state of the universe be the state $|0\rangle$ of Eqs. (42).[12] Our main object will be to show that the total integrated particle density which results from the expansion is finite.

This example is more realistic than the previous one. However, we do not take into account the reaction of the particle production back on the gravitational field. That should not affect our result as to the finiteness of the particle density, since the reaction will probably tend to reduce the particle production.

It will eventually be important to take that reaction into account because only in that way will it be possible to introduce the Newtonian gravitational constant into the various quantities pertaining to the particle production. That constant appears in the Einstein field equations which determine the interaction between the gravitational metric and the particle production, but it does not appear in the original linear equations governing the quantized fields. One would expect the gravitational constant to be of importance in determining such quantities as the period in the early expansion during which significant particle production will occur, and therefore in determining the final particle density.

According to Eq. (46), the average integrated particle density at time t will be finite if and only if $|D_{(1)}{}^{(-1)}(p,t)|^2$ approaches zero faster than p^{-3}, as p approaches ∞. Only the behavior for large p is important because $|D_{(-a)}{}^{(a)}|^2$ is bounded by Eq. (55). Since $S(p,t)$ and $|D_{(-a)}{}^{(a)}(p,t)|$ are very small for large p, and we are only interested in the limiting behavior as $p\to\infty$, we approximate $|D_{(-a)}{}^{(a)}(p,t)|^2$ by means of the first term in the iterative series of Eq. (49). Thus, we obtain the approximation

$$|D_{(-a)}{}^{(a)}(p,t)|^2 =\left|\int_0^t dt' S(p,t') \exp\left[-2i\int^t \omega(p,s)ds\right]\right|^2 . \qquad (61)$$

We are interested in the particle density given by Eq. (46) for a time $t=t_2$ equal to the present age of the expansion. Clearly, most of the particle production will occur in the very early stages of the expansion (60). Therefore, the particle density at t_2, when $R(t)=R_2$, should not depend strongly on the form of the ex-

[12] If one wishes to ensure that the universe is statically bounded, so that the particles before and after the expansion are physical, then one can imagine the quantity $R(t)$ to have a very small constant value for $t\leq 0$.

pansion, except near $t=0$. In particular, we can modify $R(t)$ so that it approaches a constant value R_2 for $t\geq t_2$, without greatly modifying the result. The particles present at t_2 are then clearly physical particles. The main effect of such a modification of $R(t)$ in the integrand appearing in Eq. (61) is that $S(p,t)$ will approach zero for $t\geq t_2$.

Rather than modify the form of $R(t)$ in (60), we introduce a factor $\exp[-R(t)/R_2]$ in the integrand of (61). Such a factor ensures that the integrand will approach zero for $t\geq t_2$, but does not affect the integrand significantly for small values of t.[13] Thus, we replace (61) for times greater than or equal to t_2 by

$$|D_{(-a)}{}^{(a)}(p,2)|^2=|I|^2, \qquad (62)$$

where

$$I=\int_0^\infty dt\, S(p,t)\exp[-R(t)/R_2] \times\exp\left[-2i\int^t \omega(p,t')dt'\right]. \qquad (63)$$

To find the asymptotic form of I for large p, we first define the dimensionless quantities

$$P=p/\mu R_2, \quad u=\mu R(t)/p, \quad B=\mu(R_2/K)^2. \qquad (64)$$

The quantity P is the present physical momentum in units of the mass, for particles in the mode p. It can be shown, using the current value of Hubble's constant, that B is of the order 10^{38}, even for μ as small as the electron mass. We can write

$$\int^t \omega(p,t')dt'=\mu\int^t\left[P^2\left(\frac{R_2}{R(t')}\right)^2+1\right]^{1/2}dt'. \qquad (65)$$

Because of the factor $\exp[-R(t)/R_2]$ in (63), the contributions to (63) from t such that $R(t)\gg R_2$ can be neglected. Therefore, for $P^2\gg1$ the integral in (65), which appears in the oscillating exponential of (63), can be replaced by

$$\mu P\int^t [R_2/R(t')]dt'=2BP^2u, \qquad (66)$$

where we have made use of (60) and (64). Then, changing the variable of integration from t to u, and using (33), we can write (63) in the following form for $P^2\gg1$:

$$I=\frac{1}{2}\int_0^\infty du(1+u^2)^{-1}\exp[-(P+4iBP^2)u]. \qquad (67)$$

Watson's lemma[14] can be applied to (67), and yields the result that for $P^2\gg1$

$$I\sim\tfrac{1}{2}(P+4iBP^2)^{-1}. \qquad (68)$$

Hence, according to Eq. (62), for $P^2\gg1$

$$|D_{(-a)}{}^{(a)}(p,2)|^2\sim(8BP^2)^{-2}. \qquad (69)$$

Therefore, $|D_{(-a)}{}^{(a)}(p,2)|^2$ approaches zero as p^{-4} when p approaches ∞. Consequently, the integrated particle density given by Eq. (46) is finite. Note also that

$$BP^2=p^2/\mu K^2 \qquad (70)$$

is independent of R_2, which indicates that the result (69) is a consequence of the particle production in the early stages of the expansion. Thus, the highest-energy particles are mainly produced in the earliest stages of the expansion.

Equations (69), (70), and (46) indicate that for $p/\mu R_2\gg1$, the particle density increases as μ, the mass of the particle, increases.[15] Thus, if very massive elementary particles, or particle resonances, are physically possible, they might perhaps be produced in the earliest stages of the expansion of the universe. Elementary particles, or particle resonances, with masses even of the order of galactic masses might conceivably be produced in the initial expansion.[16] It is interesting to speculate on such a possibility. These massive "archeons" would rapidly decay into more stable elementary particles, which might eventually "recondense" into galaxies.[17] Clearly, the reaction of the particle creation back on the gravitational field would put a limit on the masses of the particles that could be produced in significant quantities in the initial stages of the expansion. Therefore, to subject such speculations to quantitative test, one must consider the nonlinear relation between the particle production and the gravitational field.

Two interesting problems involving particle production in expanding universes are (a) to study the particle production in the initial expansion, taking into account the reaction of the particle production back on the gravitational field, and (b) to attempt to generalize and find the deeper implications of the connection found in Ref. 2 between the spin-0 particle production in the later stages of an expansion and the Einstein field equations.

[13] The quantity R_2 will, in fact, not appear in the result we obtain for large p, indicating that the main contribution to the integral comes from the early stages of the expansion.

[14] E. T. Copson, *Theory of Functions of a Complex Variable* (Oxford U. P., London, 1935), p. 218.

[15] For fixed p, the particle density must vanish as $\mu\to\infty$. The result (69) is not relevant to the limit $\mu\to\infty$ with p fixed because (69) requires that $p/\mu R_2$ be large with respect to unity.

[16] E. R. Harrison (private communication) has suggested the name "archeons" for such massive elementary particles or particle resonances. The work of Harrison points toward early large-scale structure in the universe, as in E. R. Harrison, Monthly Notices Roy. Astron. Soc. **141**, 397 (1968); Phys. Rev. D **1**, 2726 (1970).

[17] The "archeons" would presumably have very large spins and magnetic moments (and might even have large boson or fermion numbers). The spins and dipole fields might eventually develop into the angular momenta and magnetic fields of the final galaxies. If one naively extends the ρ Regge trajectory as a straight line on a Chew-Frautschi plot, then one finds that an archeon resonance lying on any nearby parallel trajectory, and having a mass of about 10^{40} g, would have a spin angular momentum of roughly $10^{128}\hbar$. The angular momentum of a typical galaxy is roughly $10^{101}\hbar=10^{74}$ g cm^2 sec^{-1}.

APPENDIX A: GENERALLY COVARIANT DIRAC EQUATION

In this appendix we will use the notation of Ref. 4 (except that we set $\hbar=c=1$). According to Ref. 4, the generally covariant Dirac equation can be written as

$$\gamma^k\nabla_k\psi=\mu\psi\,, \tag{A1}$$

where k is summed from 0 to 3, and ∇_k denotes the covariant derivative. [In Eq. (2), $\boldsymbol{\nabla}$ denotes the ordinary gradient, but in this appendix we follow Bargmann's notation.] The γ^k (which are not the same as the γ^k in the text of this paper) are coordinate-dependent 4×4 matrices satisfying

$$\gamma_k\gamma_j+\gamma_j\gamma_k=2g_{kj}\,, \tag{A2}$$

where $\gamma_k=g_{kj}\gamma^j$. The covariant derivative of ψ is

$$\nabla_k\psi=\partial\psi/\partial x^k-\Gamma_k\psi\,, \tag{A3}$$

where the Γ_k are the spinor affinities, which are 4×4 matrices determined by

$$\partial\gamma_i/\partial x^k-\Gamma_{ik}{}^j\gamma_j+\gamma_i\Gamma_k-\Gamma_k\gamma_i=0\,, \tag{A4}$$

with

$$\Gamma_{ki}{}^j=\tfrac{1}{2}g^{js}(\partial_k g_{si}+\partial_i g_{sk}-\partial_s g_{ki})\,. \tag{A5}$$

The metric of Eq. (1) is given by

$$g_{00}=-1\,,\quad g_{11}=g_{22}=g_{33}=R(t)^2\,,\quad g_{ij}=0 \quad \text{for } i\neq j. \tag{A6}$$

For that metric, a solution of Eq. (A2) is

$$\begin{aligned}\gamma_0&=-\bar\gamma_0\,,\quad \gamma_1=-R(t)\bar\gamma_1\,,\\ \gamma_2&=-R(t)\bar\gamma_2\,,\quad \gamma_3=-R(t)\bar\gamma_3\,,\end{aligned} \tag{A7}$$

where the $\bar\gamma_k$ are the special-relativistic γ matrices (denoted by γ_k in the text) which satisfy the equations

$$\bar\gamma_i\bar\gamma_k+\bar\gamma_k\bar\gamma_i=2\bar g_{ik}\,, \tag{A8}$$

and

$$\bar\gamma_0{}^\dagger=-\bar\gamma_0\,,\quad \bar\gamma_1{}^\dagger=\bar\gamma_1\,,\quad \bar\gamma_2{}^\dagger=\bar\gamma_2\,,\quad \bar\gamma_3{}^\dagger=\bar\gamma_3\,.$$

In Eq. (A8), the nonvanishing components of $\bar g_{ij}$ are $\bar g_{00}=-1$, and $\bar g_{11}=\bar g_{22}=\bar g_{33}=1$.

For the metric of (A6), the only nonvanishing $\Gamma_{kl}{}^j$ are

$$\Gamma_{ii}{}^0=R(t)\dot R(t)$$

and

$$\Gamma_{0i}{}^i=\Gamma_{i0}{}^i=R(t)^{-1}\dot R(t)\quad (i=1,\,2,\,3;\text{ no sum})\,. \tag{A9}$$

By writing the Γ_i as linear combinations of the independent products of the γ matrices, and substituting into Eq. (A4), one determines the Γ_i to within an additive c-number function, which we set equal to zero. (The c number pertains to the electromagnetic field, and is zero when no electromagnetic field is present.) The resulting Γ_l are given by

$$\Gamma_0=0\,,\quad \Gamma_i=\tfrac{1}{2}\dot R(t)\bar\gamma^0\bar\gamma^i\quad (i=1,2,3)\,, \tag{A10}$$

where $\bar\gamma^0=-\bar\gamma_0$ and $\bar\gamma^i=\bar\gamma_i$ for $i=1$, 2, 3. Then, using (A3), (A10), $\gamma^0=-\bar\gamma^0$, and

$$\gamma^i=g^{ij}\gamma_j=R(t)^{-2}\gamma_i=-R(t)^{-1}\bar\gamma^i$$

for $i=1$, 2, 3, we can write Eq. (A1) in the form

$$-\bar\gamma^0\frac{\partial}{\partial t}\psi-R(t)^{-1}\sum_{i=1}^{3}\bar\gamma^i\frac{\partial}{\partial x^i}\psi +\tfrac{1}{2}R(t)^{-1}\dot R(t)\sum_{i=1}^{3}\bar\gamma^i\bar\gamma^0\bar\gamma^i\psi=\mu\psi\,,$$

which becomes, with the aid of $\bar\gamma^i\bar\gamma^0=-\bar\gamma^0\bar\gamma^i$ and $(\bar\gamma^i)^2=1$ $(i=1,\,2,\,3)$,

$$\bar\gamma^0\frac{\partial}{\partial t}\psi+\tfrac{3}{2}R(t)^{-1}\dot R(t)\bar\gamma^0\psi +R(t)^{-1}\sum_{i=1}^{3}\bar\gamma^i\frac{\partial}{\partial x^i}\psi+\mu\psi=0\,. \tag{A11}$$

This is identical to Eq. (2), if we note that in the text the $\bar\gamma^k$ are written as γ^k.

APPENDIX B: PLANE-WAVE SOLUTIONS OF THE DIRAC EQUATION

When $R(t)=1$, Eq. (4) becomes

$$\left(\gamma^0\frac{d}{dt}+ia\boldsymbol{\gamma}\cdot\mathbf{p}+\mu\right)E(\mathbf{p},t)=0\,. \tag{B1}$$

The 4×4 spin matrices are defined by

$$\sigma_k=i\gamma^4\gamma^5\gamma^k\,, \tag{B2}$$

where

$$\gamma^5=\gamma^1\gamma^2\gamma^3\gamma^4\,.$$

It is well known[18] that a set of four independent solutions of (B1) are given by

$$u^{(a,d)}(\mathbf{p})e^{-ia\omega(p)t}\,, \tag{B3}$$

where $a=\pm1$, $d=\pm1$, $\omega(p)=(p^2+\mu^2)^{1/2}$, and the $u^{(a,d)}(\mathbf{p})$ satisfy the equations

$$[-a\omega(p)\gamma^4+ia\boldsymbol{\gamma}\cdot\mathbf{p}+\mu]u^{(a,d)}(\mathbf{p})=0 \tag{B4}$$

and

$$\sigma_{\mathbf{p}}u^{(a,d)}(\mathbf{p})=du^{(a,d)}(\mathbf{p})\,, \tag{B5}$$

where

$$\sigma_{\mathbf{p}}=\boldsymbol{\sigma}\cdot\mathbf{p}/p\,. \tag{B6}$$

We normalize the $u^{(a,d)}(\mathbf{p})$ such that

$$u^{(a,d)}(\mathbf{p})^\dagger u^{(a,d)}(\mathbf{p})=\omega(p)/\mu\,. \tag{B7}$$

It can be proved that

$$u^{(a,d)}(\mathbf{p})^\dagger u^{(a',d')}(\mathbf{p})=\delta_{a,a'}\delta_{d,d'}\omega(p)/\mu\,. \tag{B8}$$

The $u^{(a,d)}(\mathbf{p})$ which we use are related to the $u^{(r)}(\mathbf{p})$

[18] See, e.g., Ref. 5, pp. 190–194.

($r=1, 2, 3, 4$) of Ref. 5 (p. 192) as follows:

$$u^{(1,1)}=u^{(1)}, \quad u^{(1,-1)}=u^{(2)},$$

and

$$u^{(-1,-1)}=u^{(3)}, \quad u^{(-1,1)}=u^{(4)}. \tag{B9}$$

It can be shown that

$$I=\sum_{a,d} au^{(a,d)}(\mathbf{p})\bar{u}^{(a,d)}(\mathbf{p}), \tag{B10}$$

where I is the 4×4 identity matrix and $\bar{u}^{(a,d)}(\mathbf{p})$ is the row matrix

$$\bar{u}^{(a,d)}(\mathbf{p})=u^{(a,d)}(\mathbf{p})^\dagger\gamma^4. \tag{B11}$$

Also, one can show that

$$\bar{u}^{(a,d)}(\mathbf{p})u^{(a',d')}(\mathbf{p})=a\delta_{a,a'}\delta_{d,d'}. \tag{B12}$$

We will also require that the following equation holds:

$$u^{(a,d)}(-\mathbf{p})=\gamma^4u^{(a,-d)}(\mathbf{p}). \tag{B13}$$

It can be directly verified that (B13) holds in the representation given in Appendix C.

Let us define

$$\bar{W}^{(a,d)}(\mathbf{p})=\frac{\mu}{\omega(p)}\bar{u}^{(a,-d)}(-\mathbf{p}). \tag{B14}$$

We now prove the following equation:

$$\bar{W}^{(a,d)}(a\mathbf{p})u^{(a',d')}(a'\mathbf{p})=\delta_{a,a'}\delta_{d,d'}. \tag{B15}$$

As a consequence of (B14) and (B12), it follows that the left-hand side of (B15) vanishes when $a'=-a$. For $a'=a$, the left-hand side of (B15) can be written as

$$\begin{aligned}\frac{\mu}{\omega(p)}&\bar{u}^{(a,-d)}(-a\mathbf{p})u^{(a,d')}(a\mathbf{p})\\ &=\frac{\mu}{\omega(p)}u^{(a,d)}(-a\mathbf{p})^\dagger\gamma^4u^{(a,d')}(a\mathbf{p})\\ &=\frac{\mu}{\omega(p)}u^{(a,-d)}(-a\mathbf{p})^\dagger u^{(a,-d')}(-a\mathbf{p})=\delta_{d,d'},\end{aligned}$$

where we have used (B13) and (B8). This completes the proof of Eq. (B15).

If follows from Eq. (B15) that

$$I=\sum_{a,d} u^{(a,d)}(a\mathbf{p})\bar{W}^{(a,d)}(a\mathbf{p}). \tag{B16}$$

One can prove that the right-hand side of (B16) is the identity matrix by applying it from the left to the complete set of linearly independent spinors $u^{(a',d')}(a'\mathbf{p})$, and using (B15).

A simple consequence of Eq. (B15), which leads to Eq. (21), is

$$\bar{W}^{(b,abd)}(ab\mathbf{p})u^{(a',aa'd)}(aa'\mathbf{p})=\delta_{a',b}. \tag{B17}$$

We define $u^{(a,d)}(\mathbf{p},t)$ by simply replacing $\mathbf{p}$ in $u^{(a,d)}(\mathbf{p})$ by $\mathbf{p}/R(t)$. Thus

$$u^{(a,d)}(\mathbf{p},t)=u^{(a,d)}(\mathbf{p}/R(t)). \tag{B18}$$

Clearly, Eq. (B4) and all the equations which follow it in this appendix remain valid when $\mathbf{p}$ is replaced by $\mathbf{p}/R(t)$ wherever it appears, since those equations do not involve time derivatives. In the text, we generally use those equations with $\mathbf{p}/R(t)$ replacing $\mathbf{p}$.

APPENDIX C: MATRIX REPRESENTATION

We use the standard representation[19] in which

$$\gamma^k=\begin{pmatrix}0 & -i\hat{\sigma}_k\\ i\hat{\sigma}_k & 0\end{pmatrix} \quad (k=1, 2, 3) \tag{C1}$$

and

$$\gamma^4=\begin{pmatrix}\hat{I} & 0\\ 0 & -\hat{I}\end{pmatrix}, \tag{C2}$$

where $\hat{I}$ is the 2×2 unit matrix and the $\hat{\sigma}_k$ are the 2×2 Pauli matrices:

$$\hat{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \hat{\sigma}_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \hat{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{C3}$$

Then

$$\gamma^5=\begin{pmatrix}0 & -\hat{I}\\ -\hat{I} & 0\end{pmatrix}, \tag{C4}$$

and the matrix $\sigma^k=i\gamma^4\gamma^5\gamma^k$ has the form

$$\sigma^k=\begin{pmatrix}\hat{\sigma}_k & 0\\ 0 & \hat{\sigma}_k\end{pmatrix} \quad (k=1, 2, 3). \tag{C5}$$

The matrix $\sigma_{\mathbf{p}}=\boldsymbol{\sigma}\cdot\mathbf{p}/p$ is then

$$\sigma_{\mathbf{p}}=\begin{pmatrix}\hat{\sigma}_p & 0\\ 0 & \hat{\sigma}_p\end{pmatrix}, \tag{C6}$$

where

$$\hat{\sigma}_{\mathbf{p}}=p^{-1}\begin{pmatrix}p_3 & p_1-ip_2\\ p_1+ip_2 & -p_3\end{pmatrix}. \tag{C7}$$

In this representation, the $u^{(a,d)}(\mathbf{p},t)=u^{(a,d)}(\mathbf{p}/R(t))$ take the form (where $d=\pm1$)

$$u^{(1,d)}(\mathbf{p},t)=\left(\frac{g(p,t)\,(p+p_3d)}{4\mu p}\right)^{1/2}\begin{bmatrix}\chi^{(d)}(\mathbf{p})\\ \dfrac{pd}{R(t)g(p,t)}\chi^{(d)}(\mathbf{p})\end{bmatrix} \tag{C8}$$

[19] M. E. Rose, *Relativistic Electron Theory* (Wiley, New York, 1961), p. 46.

and

$$u^{(-1,d)}(p,t)=d\left(\frac{g(p,t)(p+p_3d)}{4\mu p}\right)^{1/2}\times\begin{bmatrix}\dfrac{pd}{R(t)g(p,t)}\chi^{(d)}(\mathbf{p})\\ \chi^{(d)}(\mathbf{p})\end{bmatrix}, \quad (C9)$$

where

$$g(p,t)=\omega(p,t)+\mu, \quad (C10)$$

$$\omega(p,t)=[p^2/R(t)^2+\mu^2]^{1/2}, \quad (C11)$$

and

$$\chi^{(d)}(\mathbf{p})=\begin{pmatrix}1\\ (p_1+ip_2)/(pd+p_3)\end{pmatrix}. \quad (C12)$$

One can directly verify that Eq. (B4) and the equations which follow it in Appendix B [with $\mathbf{p}$ replaced by $\mathbf{p}/R(t)$] hold in this representation. Some additional equations which hold in this representation are

$$u^{(a,d)}(\mathbf{p},t)^\dagger u^{(-a,d)}(\mathbf{p},t)=p/\mu R(t) \quad (C13)$$

and

$$u^{(-a,d)}(\mathbf{p},t)=-d\gamma^5 u^{(a,d)}(\mathbf{p},t). \quad (C14)$$

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Vacuum-Polarization Contributions to the Sixth-Order Anomalous Magnetic Moment of the Muon and Electron*

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We report on the calculation of second-order vacuum-polarization contributions to the sixth-order electron and muon anomalous magnetic moments. With these results the difference of muon and electron moments has now been completely calculated through sixth order in quantum electrodynamics. The only remaining sixth-order contributions to the electron moment which have not been completely calculated are those graphs without fermion loop insertions.

I. INTRODUCTION

In view of the recent measurement by Wesley and Rich[1] of the anomalous magnetic moment of the electron, $a_e=\frac{1}{2}(g_e-2)$, to a precision of 6 ppm and the expected precision of future measurements of the muon anomalous magnetic moment $a_\mu=\frac{1}{2}(g_\mu-2)$,[2] a complete calculation of the sixth-order quantum-electrodynamic contributions to a_e and a_μ is awaited with increasing urgency.

The status of the sixth-order calculations for the electron moment is as follows: The contribution arising from the insertion of fourth-order vacuum-polarization graphs into the second-order vertex[3] [Fig. 1(a)] as well as the photon-photon scattering contribution[4] [Fig. 1(b)] has been evaluated. The graphs which have not been evaluated consist of all those obtained by insertion of the second-order vacuum-polarization graphs into the fourth-order vertices [Fig. 1(c)] and all sixth-order vertices with no electron loop insertion [Fig. 1(d)]. In addition there is a dispersion-theoretical estimate of the sixth-order electron magnetic moment.[5,6]

The quantum-electrodynamic contributions to the difference of the muon and electron moments $a_\mu-a_e$ in sixth order arise from the insertion of electron loops of the vacuum-polarization type [Figs. 1(a) and 1(c)] in the muon vertices of the second[7–10] and fourth orders[7,9]

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