here.[13] We hope to come back to this problem elsewhere.

We would like to thank Professor T. Kinoshita for some helpful comments.

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Ghost-Eliminating Modifications in Multipion Amplitudes: Factorization on Leading Trajectories

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The modification of B_n to eliminate the ghost at $\alpha_\rho = 0$ is investigated. The $J_1J_2J_3$ leading three-particle vertex in B_n is calculated. Using this form, it is shown that *no* finite number of term modification of B_n without trajectory depression satisfies consistent factorization in all multipion amplitudes. Allowing trajectory depression, although daughter levels still presumably do not factorize, a solution is found in which (a) all leading trajectories factorize, (b) are nondegenerate, and (c) the $\rho\rho\rho$ vertex need not be zero. We believe this to be the suitable generalization of the Lovelace 4-π amplitude.

I. INTRODUCTION

Since the introduction of the Veneziano 4-point[1] and n-point[2] amplitudes, a great deal of work has been done deriving the properties of planar dual amplitudes. From the work of Hopkinson and Chan[3] and others, and with the factorization results of Fubini and Veneziano,[4] and Bardakci and Mandelstam,[5] the function B_n appears to be a suitable approximation for the n-σ amplitude with all identical internal trajectories (σ: $J^{PIG} = 0^{+0+}$). There are a number of modifications which must be made to B_n to obtain an appropriate form for, say, the n-π amplitude. Among these are (a) the ghost at $\alpha_\rho = 0$, which would be a tachyon due to the positive intercept of the real ρ trajectory, must be eliminated, (b) positivity constraints arising from the requirement that all three-particle couplings be real must be imposed, and (c) the so-called abnormally coupling leading trajectories (ω-A_2 trajectory in odd number of pion channels) must be included.

The ω-A_2 trajectory inclusion has been discussed in the literature by Dorren *et al.*[6] and by Gabarró and González Mestres[7] for the 6-π amplitude, and by Canning and Jacobs[8] for the 8-π and n-π amplitudes. The ρ ghost has been discussed by Lovelace[9] for the 4-π amplitude. For $\sigma\pi\pi\pi\pi$, ghosts have been discussed by Savoy,[10] Waltz,[11] and Dethlefsen.[12] In the case of 6-π and higher amplitudes, discussions have been given by Dorren *et al.*[13] and by Olive and Zakrzewski.[14] In all of these, the properties of particles on leading trajectories are considered. The modifications of B_n are accomplished by multiplication by a simple polynomial in the external momenta and depressing the starting values of internal trajectories to restore the correct Regge behavior (or spin content). In the cases of 4-π and $\sigma\pi\pi\pi\pi$, a unique ghost-eliminating factor, α_ρ of one of the ρ trajectories present, can be found which removes all ρ ghosts and which also satisfies soft-pion Adler constraints. For six-pion and higher (even-number) pion amplitudes, the choice of α factors is ambiguous. In attempting to write an n-π amplitude which would satisfy factorization constraints, Dorren *et al.*[13] and Olive and Zakrzewski[14] found that they were forced to depress certain internal trajectories which would normally have had leading particles of well-defined parity. With the remaining smaller number of leading trajectories (not depressed) they find sets of ghost-eliminating and trajectory-modifying factors which require degeneracies on the ρ-f trajectory and possibly on the π-H,A_1 trajectory. In both models the ρ particle is at least doubly degenerate; this does not seem to be borne out by experiment. Also, in both models, leading particles are required to factorize in their couplings both to other leading particles and to daughter particles.

The purpose of this paper is twofold: First we assume that the modification of B_n to eliminate ρ ghosts is accomplished by a finite sum of terms each of which is a finite polynomial in the internal ρ trajectories times B_n with the correct Regge behavior in all channels. Using factorization only among leading trajectories, we then show that simultaneous factorization among all n-π amplitudes cannot occur. This result holds independent of the degeneracy structure and assumes only the existence of at least one nonzero $\pi\pi\rho$ coupling. Thus, if we wish to use only a finite number of terms for fixed n, we must be permitted to depress certain otherwise good internal trajectories and not have them enter into factorization requirements. A corresponding proof could be given to show that daughter levels do not satisfy factorization. Second, we show that the added flexibility of depressing trajectories is sufficient to allow a factorizing solution for leading particles. Since nonleading spins do not factorize themselves, we do not require their couplings to leading particles to factorize. We show a solution having the π-H,A_1 and ρ-f trajectories nondegenerate at all spins.

Throughout, the main tool we use is the form of the three leading particle couplings implicit in the B_n reduction on tree graphs. We derive this coupling by straightforward algebraic manipulation in the Appendix.

II. FACTORIZATION ON LEADING TRAJECTORIES IN B_n

From the previous analyses of B_n, we know that the leading particles on internal trajectories are nondegenerate. This means that B_n has the following property: If we take the residue of the $n-3$ pole in B_n on any tree graph, the portion of this residue which corresponds to particles of leading spin on each internal line is given by the propagators for particles of appropriate spin on each line, contracted with a *unique* three-particle vertex at each three-line junction.

The propagator $p^{\nu_1\cdots\nu_J\mu_1\cdots\mu_J}$ of the spin-J portion of a tensor $T^{\mu_1\cdots\mu_J}$ is given by a form

$$\prod_{i=1}^{J} g^{\nu_i\mu_{p(i)}},$$

where $p(i)$ is a permutation of the integers $1,\ldots,J$. This form must be summed over all distinct permutations, made traceless, and normalized to be idempotent. If we are not interested in the spin-less-than-J portion of the product

$$p^{\nu_1\cdots\nu_J\mu_1\cdots\mu_J}T^{\mu_1\cdots\mu_J},$$

we can neglect the permutation summation and tracelessness and equivalently use

$$p^{\nu_1\cdots\nu_J\mu_1\cdots\mu_J}=\sum_h |T_h^{\nu_1\cdots\nu_J}\rangle\langle T_h^{\mu_1\cdots\mu_J}|$$

$$\sim\prod_{i=1}^{J} g^{\nu_i\mu_i}. \qquad (1)$$

In Eq. (1) the index h is summed over the $2J+1$ independent T's.

The three-particle vertex of particles with spins J_1, J_2, and J_3 can be given in terms of the expansion coefficients over their well-defined independent couplings. When the tensors describing the spin states are as above, we can define these couplings as follows: There are β_{12} indices of J_1 contracted with β_{12} indices of J_2 (because of the implicit symmetry in the tensor wave function of each particle, we need give only the number of contractions); there are β_{23} contractions of J_3 with J_1; there are β_{31} contractions of J_3 with J_1; all remaining indices of J_1, J_2, and J_3 are contracted with the four-momenta $(P_2-P_3)^\mu$, $(P_3-P_1)^\mu$, or $(P_1-P_2)^\mu$, respectively. In the Appendix we derive the expansion coefficients prescribed by B_n to be

$$C^{J_1 J_2 J_3}_{\beta_{12}\beta_{23}\beta_{31}} = \frac{(-1/2)^{(J_1+J_2+J_3)/2-\beta_{12}-\beta_{23}-\beta_{31}}(J_1!J_2!J_3!)^{1/2}}{\beta_{12}!\beta_{23}!\beta_{31}!(J_1-\beta_{31}-\beta_{12})!(J_2-\beta_{12}-\beta_{23})!(J_3-\beta_{31}-\beta_{23})!}. \quad (2)$$

The possible imaginary character of this form does not bother us since the actual couplings for states of given isospin turn out to be real [see Ref. 16 after Eq. (5)].

For an amplitude constructed from B_n with α factors to eliminate ρ ghosts, there will be a degenerate set of leading particles on the π and the ρ trajectories with corresponding expansions of the three-particle vertices. For conciseness, we will always remove from vertices or tree-graph residues appropriate factors of Eq. (2) which, by definition, always factor out. The reduced vertices we call Γ.

We now illustrate how factorization is obtained in these modified amplitudes by considering the multiperipheral configurations of the 6-π amplitude for the α factors $\alpha_{12}\alpha_{45}$. We look at the triple pole $\alpha_{12}=J_1$, $\alpha_{123}=J_2$, $\alpha_{56}=J_3$. This pole occurs in the orderings 123456, 123465, 213456, 213465, 123564, 123654, 213564, and 213654. For cyclic (and anticyclic) symmetry we have the amplitude with the ordering 123456 given as $\alpha_{12}\alpha_{45}B_6(-\alpha_{12}, 1-\alpha_{34}, -\alpha_{45}, 1-\alpha_{56}, 1-\alpha_{61}; -\alpha_{123}, 2-\alpha_{234}, -\alpha_{345}) + \alpha_{23}\alpha_{56}B_6' + \alpha_{34}\alpha_{61}B_6''$, where B_6' and B_6'' have appropriately modified arguments to restore correct Regge behavior in all channels. Corresponding expressions hold for the other seven orderings. We consider the ordering 123456. A factor α_{23}, say, changes the arguments $\alpha_{12}\to\alpha_{12}-1$, $\alpha_{123}\to\alpha_{123}$, $\alpha_{56}\to\alpha_{56}$, but does not change the number of contractions at any of the four vertices in the tree graph. Thus α_{23} contributes a factor $(J_1-\beta_{12})$ to the residue of the triple pole [remember that a factor of Eq. (2) for each vertex has been removed *a priori*]. When we evaluate the complete residue from all eight orderings with the correct Paton-Chan[15] factors, we find (a) the particles on the 12 and 56 trajectories alternate in isospin, i.e., $I=0$ if J is even, $I=1$ if J is odd (Bose statistics), (b) the residue when the 123 trajectory has $I=1$, J even or $I=0$, J odd, is

$$4[(2J_1-\beta_{12})(2J_3-\beta_{23}) + \beta_{12}\beta_{23}(J_2-\beta_{12})(J_2-\beta_{23})/J_2(J_2-1)], \quad (3a)$$

and (c) the residue when the 123 trajectory has $I=1$, J odd or $I=0$, J even, is

$$-4\beta_{12}\beta_{23} \quad (3b)$$

[four factors of Eq. (2) have been removed from Eqs. (3a) and (3b)].

The minimum degeneracy implied by these forms is a nondegenerate ρ-f trajectory, a nondegenerate A_1 trajectory ($I=1$, J odd or $I=0$, J even, beginning at $J=1$), and a doubly degenerate π trajectory ($I=1$, J even or $I=0$, J odd, beginning at $J=0$) except that the $J=0$ and 1 particles on the π trajectory are nondegenerate. There is the usual ambiguity of spurious degeneracies here, but the important restriction of reality of the couplings bounds the magnitudes of the couplings. Thus, in the 4-π Lovelace amplitude[9] the minimal factorization has a vertex $\Gamma(\pi\pi\rho^J)=[J^{1/2}$ times Eq. (2)]. It is also possible to have a doubly degenerate $\rho(1)^J$, $\rho(2)^J$ with vertices $\Gamma(\pi\pi\rho(1)^J)=(J-\ln J)^{1/2}$, $\Gamma(\pi\pi\rho(2)^J)=(\ln J)^{1/2}$, but not $\Gamma(\pi\pi\rho(1)^J)=(J^2+J)^{1/2}$, $\Gamma(\pi\pi\rho(2)^J)=(-J^2+J)^{1/2}$. This bound on the ambiguity will be used in the next section.

III. NONEXISTENCE OF FACTORIZATION

In this section we show that factorization cannot be made consistent among all n-π amplitudes with only a finite number of terms in each n-π amplitude and without depressing some internal trajectories to below leading contributions. We assume the existence of at least one π and one ρ particle and one nonzero $\pi\pi\rho$ vertex.

In the preceding section we saw how the α factors $\alpha_{12}\alpha_{45}$ (+cyclic) produced particle sequences starting at $J=0$, 1, and 2. As an α, say α_{12}, is cycled around the tree graph, it produces spin-1 correlations on various internal lines – sometimes on more than one line at once. Then it is possible that more than one α, say J_0 of them, raise the starting spin on the same internal line, causing the existence of a sequence of particles starting at J_0. In addition, the vertex of this sequence with other particles (Γ) involves a polynomial in J (the spin of the particle) and β_{31} and β_{12} (the correlations with the other two particles) of order J_0. Because of the reality of residues, one can never obtain this order of polynomial with fewer than J_0 α-factors. Thus we need prove only that there are sequences of particles starting at increasing values of J_0 to show that an infinite-order α factor is needed in, say, the 6-π amplitude.

We consider the n-3 pole in trajectories (12), (123), (1234), ..., $(n-1, n)$, shown in Fig. 1. For this tree graph to have a nonvanishing value when all of the π trajectory poles are evaluated at $\alpha_\pi=0$, there must exist one α factor (with its B_n) of the form

$$\alpha_{12}^{a_1}\alpha_{23}^{a_2}\alpha_{14}^{b_1}\alpha_{45}^{b_2}\alpha_{16}^{c_1}\alpha_{67}^{c_2}\cdots\alpha_{n-1,n}^{z_1}\alpha_{n-2,n-1}^{z_2}. \quad (4)$$

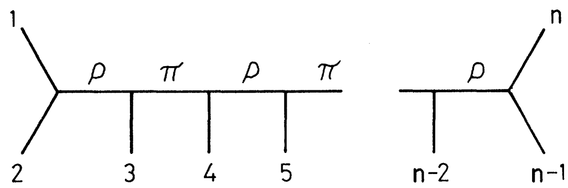

FIG. 1. Multiperipheral tree graph considered in the text. π and ρ label the type of trajectory function which occurs in the indicated channel.

This form must provide a ghost-eliminating mechanism for all $n(n-2)/4$ internal ρ trajectories; in particular it must eliminate ghosts on the following trajectory subset: (2, 5), (4, 7), (6, 9), ..., (n-4, n-1). This restricts Eq. (4) so that

$$a_1+b_1\geq 1,\quad b_1+c_1\geq 1,\quad \ldots,\quad y_1+z_1\geq 1.$$

From cyclic symmetry there exists an α factor like Eq. (4) but with all subscripts incremented by 1. On the tree graph of Fig. 1 this new α factor provides at least a J_0 = integer part of $\frac{1}{4}(n-2)$ starting value in trajectory (12). Therefore there always exists an n-π amplitude which starts a new sequence at a point higher than that at which any finite polynomial in α's can start in the 6-π amplitude, and consistent factorization among all n-π amplitudes cannot hold.

IV. FACTORIZATION WITH DEPRESSION OF TRAJECTORIES ALLOWED

At this point we have two choices available for attempting factorization. We can either allow an infinite number of terms in each n-π amplitude or permit depressing certain internal trajectories to below leading behavior (in discrete terms of the amplitude, although all internal trajectories must have leading particles in the complete amplitude). The first choice may involve radical changes in the basic properties of the amplitude and questions of convergence of infinite series of functions, so we shall investigate only the latter. This problem has been considered by Olive and Zakrzewski,[14] and by Dorren *et al.*[13] Both of these papers require factorization of leading trajectories contributions with particles of nonleading behavior in nondual channels. By a process of reasoning similar to that in Sec. III, it is obvious that these daughter particles do not satisfy a consistent factorization. Therefore, in our solution we require factorization of residues only for all particles of leading contributions.

In the Olive and Zakrzewski[14] solution, there is a contribution to the amplitude from each tree graph. For each internal trajectory line, the integrand of B_n is modified by a factor $(1-U_i)$, where U_i is the conjugate (integration) variable to α_i. This factor has the effect of lowering all trajectories dual to it. They find a solution which factorizes to a finite degeneracy (π trajectory nondegenerate, ρ trajectory doubled) when they have no $\rho\rho\rho$ vertices. Since this implies at least one π internal line ($N>4$), some trajectories are depressed. If we take their obverse solution which has terms arising from tree graphs with only $\rho\rho\rho$ internal couplings, they claim a factorizing solution (both π and ρ trajectories have degeneracies increasing with J). However, with only these terms, there are *no* internal pions with spin less than 2. This is clearly an unphysical solution.

In the Dorren *et al.* model,[13] the dotted lines or dotted lines with crosses both depress some internal trajectories. Their solution always involves at least one of these lines. They find that the n-π amplitude factorizes with the ρ trajectory doubled if the $\rho\rho\rho$ couplings are zero or trebled otherwise. We shall show that the doubled ρ trajectory is actually nondegenerate when factorization is considered only for leading particles, and that allowing a nonzero $\rho\rho\rho$ vertex does not further increase the degeneracy if additional terms are included in the amplitude.

Their D-type solution for the 6-π amplitude is $\alpha_{12}\alpha_{56}$ with trajectories dual to (123) depressed. This turns out to be nothing but the portion to the left of the plus sign in Eq. (3a) and all of Eq. (3b), calculated in Sec. II. These triple-pole residues obviously can be factorized to

$$\Gamma_{\pi(1)a^{\alpha}\,\pi(1)b^{\beta}\,\rho(2)c^{\gamma}}=g_0(2c-\beta-\gamma)c^{-1/2}, \tag{5a}$$

$$\Gamma_{\pi(1)a^{\alpha}\,\pi(2)b^{\beta}\,\rho(2)c^{\gamma}}=g_0 i(\beta+\gamma)c^{-1/2}, \tag{5b}$$

and

$$\Gamma_{\pi(2)a^{\alpha}\,\pi(2)b^{\beta}\,\rho(2)c^{\gamma}}=g_0(2c-\beta-\gamma)c^{-1/2} \tag{5c}$$

[times Eq. (2)], where a particle $p(1)^J$ has spin J and isospin $I=1$ if J is even, $I=0$ if J is odd; $p(2)^J$ has opposite isospin assignment.[16] This form has, therefore, nondegenerate π and ρ trajectories. In the generalization to the n-π amplitudes given by Dorren *et al.* for the D-type solution, we find no increase in degeneracy on the π and ρ trajectories and no change in the vertices (5a)-(5c). With only these terms, the $\rho\rho\rho$ vertices are predicted to be zero. To correct this they suggest adding a B'-type solution which, for 6-π, is $\alpha_{12}\alpha_{34}\alpha_{56}B_6'$ in which all trajectories in B_6' are depressed except (12), (34), and (56). This predicts a vertex

$$\Gamma_{\rho(2)a^{\alpha}\,\rho(2)b^{\beta}\,\rho(2)c^{\gamma}}=g_1 abc \tag{6}$$

with ρ nondegenerate,[16] and $\pi\pi\rho$ vertices as in Eqs. (5). Note that the two types of α factors, D and B', correspond to contributions in the 6-π

amplitude from the two classes of tree graphs: (1) two ρ trajectories, one π trajectory, no $\rho\rho\rho$ vertices and (2) three ρ trajectories, no π trajectories, one $\rho\rho\rho$ vertex.

In the 8-π amplitude there are three classes of tree graphs: (1) three ρ and two π trajectories, no $\rho\rho\rho$ vertices, (2) four ρ and one π trajectories, one $\rho\rho\rho$ vertex, and (3) five ρ and no π trajectories, two $\rho\rho\rho$ vertices. The D-type solution covers *all/only* class-(1) tree graphs with vertices given by Eqs. (5). The B'-type solution covers *all/only* class-(3) tree graphs with vertices given by Eqs. (5) and (6). The additional degeneracy needed by Dorren *et al.* occurs because no class-(2) tree graph is covered. Thus, the ρ-trajectory line in 4-π channels couples to an external pion and internal π trajectory at both "ends" [class (1)] or to two internal ρ trajectories at both "ends" [class (3)], but not to two π's at one "end" and two ρ's at the other "end" [class (2)]. It is possible to single out class-(2) tree graphs by depressing trajectories dual to the dashed lines of Fig. 2 (notation of Dorren *et al.*). The $\rho\rho\rho$ vertices are assured by the α factors, $\alpha_{12}\alpha_{34}\alpha_{14}$, while the proper $\pi\pi\rho$ vertices are obtained from the factor α_{56}. One immediately recognizes this as a "combination" of a 4-π portion of the 6-π B' solution with a 4-π portion of the 6-π D solution. It is easily seen that this general combination can always be made for the different classes of tree graphs for higher n-π amplitudes. Thus, by including these new terms in the amplitude, we have *no* degeneracy on leading trajectories.

Although we do not present the construction here, it is obvious that other solutions exist which satisfy factorization. In fact, any degeneracy scheme for leading particles with vertices Γ polynomial in J_1, J_2, J_3, β_{12}, β_{23}, and β_{31} can be incorporated (Γ must still satisfy certain symmetry requirements). The solution corresponding to the Lovelace[9] minimal 4-π solution would have the smallest number of terms and the least modification of B_n. The D-type solution satisfies these criteria but has the unpleasant property that all $\rho\rho\rho$ vertices are zero. The simplest solution involving nonzero $\rho\rho\rho$ coupling is the one we have presented involving B', D, and combinations of $B'D$ α-factors. There are no degeneracies and the vertices are given by Eqs. (5) and (6).

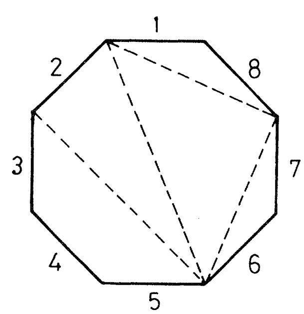

FIG. 2. Pictorial representation of trajectory depression in the 8-π amplitude which depresses all class (1) and (3) tree graphs to nonleading behavior. Any trajectory dual to a dashed line is depressed. The notation is that of Ref. 13.

V. SUMMARY

We have presented an analysis of the factorization properties of leading particles within the modified B_n amplitude. The advantage of our method is that we are able to consider three-particle vertices with all leading particles only, rather than vertices in which only one particle is known to be leading and the other two particles may have leading or daughter status. We show that any finite-term modification of B_n without trajectory depression cannot satisfy factorization simultaneously in all n-π amplitudes. It follows by a similar proof that the jth daughter level does not factorize without at least $(j+1)$-order trajectory depression. We believe, therefore, that imposing factorization conditions on vertices of leading particles (which are made to factorize) with daughter-level particles (which do not factorize) is overrestrictive.

Within our model we are able to find the natural extension of the Lovelace 4-π amplitude to higher pion amplitudes when trajectory depression is allowed. It has the properties that no degeneracies occur at any spin on leading trajectories; no three-particle vertices need to be zero – although ratios among the independent three-particle vertices are fixed; for these two properties the form involves the smallest number of modified B_n terms. We have explicit expressions for all leading vertices; in particular, the $A_1\rho\pi$ decay is predicted to be pure s wave. Using our technique and the results of Ref. 8, the extension to include leading, abnormally coupled trajectories (the ω-A_2 trajectory in an odd number of pion channels) is straightforward. This minimal dual model with all three π, ρ, and ω trajectories included has three arbitrary constants which could be fixed from the $\rho\to\pi\pi$, $f\to 4\pi$, and $A_2\to\rho\pi$ decay analyses. Nevertheless, because the daughter levels do not factorize, there are presumably important corrections to the amplitude which come from unitarity. It is possible to include separate external K particles or the complete $SU(3)$ pseudoscalar multiplet into the amplitude.

A significant negative result of this paper is that the leading trajectories need not be given degeneracies because of duality constraints in n-π amplitudes. If the famous splitting of the A_2 meson is due to duality at all, it must come from the strong restrictions of the minimal dual form of vertices in conjunction with other things, such as inclusion of the remainder of the $SU(3)$ multiplets of leading mesons as external and internal particles; it does not come from n-π factorization alone, as previously claimed.

Note. During the final stages of preparation of this paper, the author received a paper by Balachandran, Chang, and Frampton,[17] and a paper by Frampton,[18] proving a much weaker nonexistence theorem. They show that a one- or two-term modification of B_6 without trajectory depression is not consistent with a singly (that is, non-) degenerate π and ρ.

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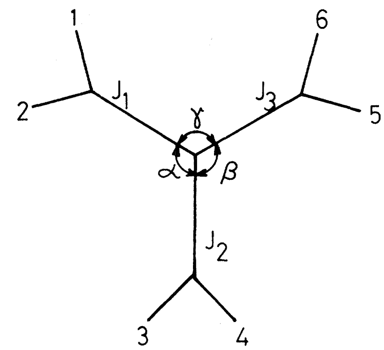

FIG. 3. A tree graph in the 6-π amplitude. The leading particle on line (12), (34), or (56) has spin J_1, J_2, or J_3 respectively; α indices of the wave function for J_1 are contracted with indices of J_2, β indices of J_2 with those of J_3, and γ indices of J_1 with those of J_3.

APPENDIX

We describe here the derivation of Eq. (2). The simplest occurrence of the general $J_1J_2J_3$ vertex is in the tree graph of Fig. 3 in B_6. We use Eqs. (4) and (5) of Hopkinson and Plahte[19] to reduce B_6 to

$$B_6=\sum_{a_1a_2a_3=0}^{\infty}(-1)^{a_1+a_2+a_3}\binom{X_{61}-X_{234}-X_{345}+X_{34}}{a_1}\binom{X_{345}-X_{34}-X_{45}}{a_2}\binom{X_{24}-X_{34}-X_{23}}{a_3}$$
$$\times B_4(X_{12}+a_1+a_3,X_{23})B_4(X_{34},X_{13}+a_1+a_2+a_3)B_4(X_{45},X_{56}+a_1+a_2), \tag{A1}$$

where $\binom{X}{a}$ is the binomial coefficient. Poles occur explicitly in B due to the relation

$$\text{residue of } B_4(X,y)|_{X=-J}=(-1)^J\binom{y-1}{J}.$$

The residue of the triple pole of Fig. 3 is polynomial in the X's and contains contributions from particles of spin $0,\ldots,J_1$ on trajectory (12), $0,\ldots,J_2$ on (34) and $0,\ldots,J_3$ on (56). On the tree graph of Fig. 3, we can write

$$\begin{aligned}
X_{12}&=-J_1,\\
X_{23}&=\tfrac{1}{2}\{(P_1-P_2)\cdot V_1^h\}\{V_1^h\cdot V_2^{h'}\}\{V_2^{h'}\cdot(P_3-P_4)\}+\tfrac{1}{4}\{(P_1-P_2)\cdot V_1^h\}\{V_1^h\cdot(P_3+P_4-P_5-P_6)\}+\text{lower terms},\\
X_{34}&=-J_2,\\
X_{45}&=\tfrac{1}{2}\{(P_3-P_4)\cdot V_2^h\}\{V_2^h\cdot V_3^{h'}\}\{V_3^{h'}\cdot(P_5-P_6)\}+\tfrac{1}{4}\{(P_1+P_2-P_3-P_4)\cdot V_3^h\}\{V_3^h\cdot(P_5-P_6)\}+\text{lower terms},\\
X_{56}&=-J_3,\\
X_{61}&=\tfrac{1}{2}\{(P_1-P_2)\cdot V_1^h\}\{V_1^h\cdot V_3^{h'}\}\{V_3^{h'}\cdot(P_5-P_6)\}+\text{lower terms},\\
X_{123}&=\tfrac{1}{2}\{(P_3-P_4)\cdot V_2^h\}\{V_2^h\cdot(P_5+P_6-P_1-P_2)\}-y-1,\\
X_{234}&=\tfrac{1}{2}\{(P_1-P_2)\cdot V_1^h\}\{V_1^h\cdot(P_3+P_4-P_5-P_6)\}+\text{lower terms},\\
X_{345}&=\tfrac{1}{2}\{(P_5-P_6)\cdot V_3^h\}\{V_3^h\cdot(P_1+P_2-P_3-P_4)\}+\text{lower terms}.
\end{aligned} \tag{A2}$$

The V_i^h represent three-vectors under $O(3)$ transformations in the rest frame of the $i=(12)$, (34), or (56) channel; the indices h and h' are helicity or equivalent indices. Sums over h and h' are intended in Eq. (A2) and can be done with the identity

$$\sum_h V_i^{h\mu}V_i^{h\nu}=g^{\mu\nu}-P_i^\mu P_i^\nu/m_i^2.$$

Equation (A2) explicitly contains the highest spin correlations needed and the correlations of lower-spin content are referred to as "+ lower terms," except for X_{123}, where the lower terms are defined to be $-y-1$.

The terms in the polynomial residue contributing to spin content of J_1 or J_3 involve X_{61}, X_{23}, and X_{234} or X_{45}, X_{61}, and X_{345}. The highest power present is J_1 or J_3, respectively, and we extract this spin portion by evaluating these highest terms in Eq. (A1) with substitution for these X's given by Eq. (A2). The spin correlations in the (34) channel involve X_{23}, X_{45}, and X_{123}. Spin correlations as high as $J_1+J_2+J_3-2a_1$ can occur in specific terms here although in the final answer, of course, all terms of spin greater than J_2 cancel. Nevertheless at this point we must keep all terms in the (34)-channel spin. When we expand our form we have

(Residue with spin J_1 and J_3)

$$= \sum_{a_1=0}^{a_{\max}} \sum_{a_2=0}^{a_{\max}} \sum_{a_3=0}^{a_{\max}} \sum_{b_1=0}^{a2} \sum_{b_2=0}^{a_3} \sum_{b_3=0}^{J_2} \sum_{b_4=0}^{b_3} \sum_{c_1=0}^{J_1-a_1-b_2} \sum_{c_2=0}^{J_3-a_1-b_1} \sum_{c_3=0}^{a_1} (-1)^{J_1+J_2+J_3+a_1+a_2+a_3+b_1+b_2} (1/2)^{2J_1+2J_3-2a_1-b_1-b_2+b_4-c_1-c_2-c_3}$$

$$\times \frac{S_{J_2}^{b_3}(a_1+a_2+a_3+y)^{b_3-b_4} b_3!(J_1-a_1-b_2)!(J_3-a_1-b_1)!}{J_2!(b_3-b_4)!b_1!b_2!b_4!(J_1-a_1-a_3)!(a_3-b_2)!(J_3-a_1-a_2)!} \frac{\{(P_1-P_2)\cdot V_1^h\}^{J_1}\{(P_3-P_4)\cdot V_2^{h'}\}^{J_2}\{(P_5-P_6)\cdot V_3^{h''}\}^{J_3}}{(a_2-b_1)!c_1!(J_1-a_1-b_2-c_1)!c_2!(J_3-a_1-b_1-c_2)!c_3!}$$

$$\times \frac{\{V_1^h\cdot(P_3+P_4-P_5-P_6)\}^{J_1-c_1-c_3}\{V_2^{h'}\cdot(P_5+P_6-P_1-P_2)\}^{b_4}}{(a_1-c_3)!}$$

$$\times \{V_3^{h''}\cdot(P_1+P_2-P_3-P_4)\}^{J_3-a_1-c_2}\{V_1^h\cdot V_2^{h'}\}^{c_1}\{V_2^{h'}\cdot V_3^{h''}\}^{c_2}\{V_1^h\cdot V_3^{h''}\}^{c_3}. \tag{A3}$$

The upper limits on the a_1, a_2, a_3 summations, $a_{\max}$, are given by $a_1+a_2 \leq J_3$, $a_1+a_3 \leq J_2$; S_J^b are Sterling numbers of the first kind[20]; summations are also intended over h, h', and h''. Among other things, this residue involves a sum over the various $J_1J_2J_3$ couplings defined in Sec. II. The portion of this residue having α, β, and γ contractions as shown in Fig. 3 is obtained by requiring $c_1=\alpha$, $c_2=\beta$, $c_3=\gamma$, $a_1=\gamma$, $b_4=J_2-\alpha-\beta$. We now disregard the kinematic couplings and discuss only the coefficient in the residue, $V_{\alpha\beta\gamma}^{J_1J_2J_3}$. On the remaining quintuple sum for V, consider specifically the portion

$$\sum_{a_2=0}^{J_3-\gamma} \sum_{b_1=0}^{a_2}.$$

If we interchange the a_2, b_1 summation order and substitute $A_2=a_2+b_1$, the A_2 sum can be done symbolically using powers of the forward difference operator.[21] A corresponding interchange and substitution on the a_3, b_2 summations allows the new A_3 sum to be done with further powers of the difference operator. The resulting triple sum is over b_2, b_1, and b_3 – in that order. Changing the b_2 summation to a sum over $B=b_2+b_1$ and then bringing the b_1 sum "inside" of the B sum, we find that the b_1 summation can be done explicitly using binomial coefficient identities.[22]

The resultant double summation still cannot be done explicitly; however, from it we obtain the recurrence relation

$$V_{\alpha+1\,\beta-1\,\gamma}^{J_1J_2J_3} = V_{\alpha\beta\gamma}^{J_1J_2J_3} \frac{\beta(J_1-\alpha-\gamma)}{(\alpha+1)(J_3-\beta-\gamma+1)}.$$

From iteration of this we obtain

$$V_{\alpha\beta\gamma}^{J_1J_2J_3} = \frac{(\alpha+\beta)!(J_3-\gamma)!(J_1-\alpha-\beta-\gamma)!}{\alpha!\beta!(J_3-\gamma-\beta)!(J_1-\alpha-\gamma)!} V_{\alpha+\beta\,0\,\gamma}^{J_1J_2J_3}. \tag{A4}$$

At this point we use the cyclic symmetry in V which is obscured in our equations because of our originally nonsymmetric reduction of B_6. Then, reusing Eq. (A4), we obtain

$$V_{\alpha\beta\gamma}^{J_1J_2J_3} = \frac{(\alpha+\beta+\gamma)!(J_1-\alpha-\beta-\gamma)!J_2!(J_3-\alpha-\beta-\gamma)!}{\alpha!\beta!\gamma!(J_1-\alpha-\gamma)!(J_2-\alpha-\beta)!(J_3-\beta-\gamma)!} V_{00\,\alpha+\beta+\gamma}^{J_1J_2J_3}.$$

For $V_{00\sigma}^{J_1J_2J_3}$, the double sum degenerates to a single term with $b_3=J_2$, $B=J_1+J_3-\sigma$, and we obtain

$$V_{\alpha\beta\gamma}^{J_1J_2J_3} = \frac{(-1/2)^{J_1+J_2+J_3-\alpha-\beta-\gamma}}{\alpha!\beta!\gamma!(J_1-\alpha-\gamma)!(J_2-\alpha-\beta)!(J_3-\beta-\gamma)!}.$$

Factorizing this form into contributions from each of the four vertices of Fig. 3, we obtain Eq. (2).

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[22]See Ref. 20, p. 822.