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Complex Negative-Signature Trajectories and the Pomeranchuk Theorem*

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Within the framework of complex-angular-momentum methods, it is shown that amplitudes which violate the Pomeranchuk theorem require negative-signature trajectories which are of the form $\alpha(t) = 1 \pm \text{const}\sqrt{t} + O(t)$ near $t = 0$. There must be corresponding positive-signature trajectories. The character of the singular surfaces with negative signature is discussed briefly.

I. INTRODUCTION

In previous papers we have discussed Regge pole and branch-point trajectories $\alpha(t)$ which are complex for real $t<0$ due to left-hand cuts in the t plane of these functions.[1] Such trajectories are of interest for the description of diffraction scattering.[2,3] In particular, pole-cut systems with complex trajectories can be used for the construction of rather general and physically meaningful amplitudes which imply different, constant total cross sections for particle and antiparticle scattering.[4,5]

It is the purpose of this paper to show that amplitudes which violate the Pomeranchuk theorem[6] generally require negative-signature trajectories with square-root branch points at $t=0$. They are of the form $\alpha(t)=1\pm\text{const}\sqrt{t}+O(t)$. It then follows that there must be corresponding positive-signature trajectories, a fact which can also be proven directly.[7] The character of these singular surfaces of the continued partial-wave amplitude will be discussed briefly.

II. l-PLANE ARGUMENT

We denote the scattering amplitudes for particle and antiparticle scattering by $F(s,t)$ and $\bar{F}(s,t)$, respectively, and we introduce the combinations

$$F_\pm(s,t)=F(s,t)\pm\bar{F}(s,t)\,. \tag{1}$$

Assuming constant asymptotic total cross sections given by σ and $\bar{\sigma}$, we have for $s\to\infty$

$$\text{Im}F_\pm(s,0)\sim s\frac{\sigma\pm\bar{\sigma}}{16\pi}, \tag{2}$$

$$\text{Re}F_-(s,0)\sim-\frac{2}{\pi}s\ln s\frac{\sigma-\bar{\sigma}}{16\pi}, \tag{3}$$

with $\text{Re}F_+(s,0)$ being of the order $s(\ln s)^{-1}$.[2] These properties follow from the familar dispersion relations.[8] From Eq. (3) and the general postulates of dispersion theory or of local field theory, we obtain the bounds

$$(\sigma-\bar{\sigma})^2/4\pi^3a\leqslant\sigma_{\text{el}}\leqslant\sigma \tag{4}$$

for the elastic cross section

$$\sigma_{\text{el}}(s)\sim\frac{16\pi}{s^2}\int_{-s}^{0}dt\,|F(s,t)|^2\,. \tag{5}$$

Here a is a constant defining the maximal relevant angular momentum $L=\frac{1}{2}\sqrt{as}\ln s$ in the s-channel partial-wave expansion of $F(s,t)$ for large values of s.[4,9,6]

Let us consider the continued partial-wave amplitudes $F_\pm(t,\lambda)$. These functions have the usual analytic properties in the complex manifold (t,λ).[2] In particular, they satisfy the continued elastic unitarity condition for $4m^2\leqslant t<t_i$ (t_i = first inelastic threshold). This condition forbids certain kinds of singularities. We ask: What are the characteristic fèatures of allowed, isolated singularities of $F_\pm(t,\lambda)$ near $(t,\lambda)=(0,1)$ which are required by the special conditions (2)–(4)? We assume that $F_\pm(t,\lambda)$ has a finite number of such singularities at $\lambda=\alpha_K(t)$, $k=1,2,\ldots$. Irrespective of the character of these singularities, each one can contribute a term to the asymptotic expansion of $F_-(s,t)$ for $s\to\infty$ which, except for logarithmic factors, is of the form $s^{\alpha_k(t)}$. Equation (2) requires then that $\alpha_k(0)\leqslant1$, with $\alpha_k(0)=1$ for at least one trajectory

of each signature. We assume now that the trajectories $\alpha_k(t)$ are regular functions near $t=0$, so that the relevant ones are of the form

$$\alpha_k(t) = 1 + \alpha_k' t + O(t^2) . \qquad (6)$$

They give diffraction peaks which shrink at most like $(\ln s)^{-1}$. But if we calculate $\sigma_{el}(s)$, we find that *negative-signature* trajectories of the form (6) give rise to a logarithmically increasing contribution to σ_{el}. The logarithmic shrinkage is not sufficient for the bound (4).

We are led to consider negative-signature trajectories which are not regular at $t=0$, but have a branch point there. *A priori,* singular surfaces $\alpha(t)$ of the partial-wave amplitudes have only branch points related to certain physical thresholds in the t channel. However, there may be additional branch points due to the cross over of two or more trajectories of the *same character*.[1,2] Under these circumstances it is possible to prevent these branch points from being inherited by the continued partial-wave amplitude itself, where they should not occur. In general, a singular surface near $t=0$ can be of the form[1]

$$\alpha(t) = \alpha(0) + \sum_{i=1}^{\infty} \beta_i t^{i/n} , \qquad (7)$$

where n is an integer and the β_i are constants. It corresponds to n trajectories $\alpha_1(t), \ldots, \alpha_n(t)$ which cross at $t=0$ and form the branches of the branches of the multivalued function (7).

Suppose that the negative-signature amplitude $F_-(t,\lambda)$ has singular surfaces of the type (7) with $\alpha(0)=1$. These then give rise to terms in the asymptotic expansion of $F_-(s,t)$ for $s\to\infty$ which, as far as the power law is concerned, are of the form

$$s^{1+ct^{\kappa/n}} . \qquad (8)$$

But since we have assumed constant total cross sections, the amplitude $F_-(s,t)$ has the bound[10]

$$|F_-(s,t)| \leq O(s^{1+\sqrt{at}}) \qquad (9a)$$

for $s\to\infty$, $0\leq t < 4m^2$, and[9]

$$|F_-(s,t)| \leq O(s(\ln s)^2) \qquad (9b)$$

for $s\to\infty$, $t\leq 0$.

We have seen before that trajectories with $\kappa/n = 1$ do not give enough shrinkage for $\sigma_{el}\leq\sigma$; those with $\kappa/n > 1$ give even less shrinkage. Hence we restrict ourselves here to $\kappa/n<1$. It follows from the bound (9a) that $\kappa/n \geq \frac{1}{2}$. In addition, we find that for $n\geq 3$, $\kappa<n$ there are three or more branches of $\alpha(t)$ corresponding to the roots of $(t^\kappa)^{1/n}$. The amount of the phase difference between two such roots is at most $\frac{2}{3}\pi$. Hence there is always a root so that for $t<0$ and small values of $|t|$ we have

$$\mathrm{Re}\,\alpha(t) = 1 + \beta |t|^{\kappa/n}$$

with $\beta>0$. Such a trajectory would violate the unitarity requirement

$$|F(s,t)| \leq \text{const}\, s(\ln s)^2 \quad \text{for } t<0 .$$

Our considerations indicate that singular surfaces of the form

$$\alpha_{1,2}(t) = 1 \pm \text{const}\sqrt{t} + O(t) \qquad (10)$$

are a characteristic feature of negative-signature trajectories which violate the Pomeranchuk theorem.

III. s-CHANNEL ARGUMENT

It may be of interest to indicate an alternative but related argument for the presence of singular surfaces of the type (10) in the negative-signature amplitude. For large values of s, we can write $F(s,t)$ in the form of a Bessel transform[4,5,11]:

$$F(s,t) \sim s\int_0^1 d\xi\, \psi(\xi, \ln s) J_0(\xi\sqrt{-at}\ln s) , \qquad (11)$$

where a is again determined by the maximal angular momentum $L=\frac{1}{2}\sqrt{as}\ln s$. There are corresponding expressions for $\bar{F}$ and for $F_\pm$. If we can show that the function

$$\psi_-(\xi, \ln s) = \psi(\xi, \ln s) - \bar{\psi}(\xi, \ln s)$$

has support for nonzero values of ξ in the interval $0\leq\xi\leq 1$ and in the limit $s\to\infty$, then it follows that the asymptotic expansion of $F_-(s,t)$ contains terms of the form

$$s^{1\pm i\bar{\xi}\sqrt{-at}} \qquad (12)$$

as far as the power law is concerned. In the complex angular momentum plane, these terms correspond again to negative-signature trajectories of the type (10). As a contrasting example, we mention that a function like

$$\psi(\xi, \ln s) \propto \xi \ln s\, e^{-\xi^2 \ln s}$$

would give rise to an asymptotic term with the power behavior $\sim s^{1+at}$ corresponding to a trajectory which is regular at $t=0$.

From Eqs. (5) and (11) we obtain

$$\sigma_{el}(s) \sim \frac{32\pi}{a}\int_0^1 d\xi\, \xi^{-1}\rho(\xi, \ln s) , \qquad (13)$$

where

$$\rho(\xi, \ln s) \equiv (\ln s)^{-2} |\psi(\xi, \ln s)|^2 . \qquad (14)$$

Using the inequality (4) we conclude then the following:

(1) The lower bound implies that the function $\rho(\xi, \ln s)$ must be positive for some values of ξ in the interval $0 \leq \xi \leq 1$.

(2) The upper bound requires that the support of ρ is not restricted to the point $\xi = 0$ for $\ln s \to \infty$. Because the integral (5) must converge, we find that ρ has to vanish for $\xi \to 0$, and hence it must be positive for some nonzero values of $\xi \leq 1$.

It remains to determine the signature of the positive contributions to ρ. From Eqs. (3) and (11) we find

$$\int_0^1 d\xi \,\mathrm{Re}\psi_-(\xi, \ln s) = -\frac{2}{\pi}\frac{\sigma - \bar{\sigma}}{16\pi}\ln s, \qquad (15)$$

and hence

$$(\ln s)^{-2} |\mathrm{Re}\psi_-(\xi, \ln s)|^2$$

must make a finite contribution to $\rho(\xi, \ln s)$ for $\ln s \to \infty$. But then it follows from the previous arguments that $(\ln s)^{-1}\mathrm{Re}\psi_-$ has support for $\ln s \to \infty$ and nonzero values of $\xi \leq 1$. This is what we wanted to show.

The argument described above can also be given in terms of the integrated cross section

$$\sigma^{(-)}(s) = \frac{16\pi}{s^2}\int_{-s}^0 dt\, |F_-(s,t)|^2,$$

which is proportional to charge-exchange or regeneration cross sections, and which satisfies an inequality corresponding to Eq. (4).

With the dispersive part of $F_-(s,t)$ having terms of the form (12) in the asymptotic expansion, it follows from the dispersion relations that the same must be true for the absorptive part. Hence also $\mathrm{Im}\psi_-(\xi, \ln s)$ has support for positive $\xi \leq 1$. We can use the presence of asymptotic terms like

$$s^{1+\mathrm{const}\sqrt{t}} \qquad (16)$$

in the high-energy limit of $\mathrm{Im}F_-(s,t)$ in order to show that also the *positive-signature* amplitude must have trajectories of the type (10): It can be seen from the partial-wave expansions that $\mathrm{Im}F(s,t)$ and $\mathrm{Im}\bar{F}(s,t)$ are positive for $0 \leq t < 4m^2$. Because $\mathrm{Im}F_-(s,t)$ has asymptotic terms like (16) for $s \to \infty$, these positivity constraints require that corresponding terms are present in $\mathrm{Im}F_+(s,t)$.

IV. DISCUSSION

We see that amplitudes which violate the Pomeranchuk theorem and which instead satisfy Eqs. (2) and (3) must have complex trajectories of the form $\alpha(t) = 1 \pm \mathrm{const}\sqrt{t} + O(t)$ in the negative- and the positive-signature partial-wave amplitudes. Although our arguments are not completely rigorous from the mathematical point of view, they are sufficient as far as the usual complex-angular-momentum methods are concerned.

There remains the question of the character of the trajectories (10) as singular surfaces of $F_\pm(t,\lambda)$. We restrict ourselves here to a few remarks. It is well known that simple-pole surfaces in F_- lead to an increasing elastic cross section ($\sim\ln\ln s$),[12] and hence violate the bound (4). Logarithmic branch points are possible near $t = 0$. But in order to be effective, they generally must appear in forms like

$$\ln[(\lambda - \alpha_1(t))(\lambda - \alpha_2(t))],$$

so that $F_- \to \infty$ for $\lambda \to \alpha_{1,2}$. Terms of this type are incompatible with the continued elastic unitarity condition for $4m^2 \leq t < t_i$, because branch-point trajectories usually do not have thresholds at $t = 4m^2$ as do pole trajectories.[2,5] As we have pointed out in previous papers,[2,5] this conclusion can be circumvented by introducing very special shielding cuts.

We had found earlier that rather natural pole-cut systems with square-root branch points can give rise to acceptable amplitudes[1,5]; these branch points are given in terms of the poles $\alpha_{1,2} = 1 \pm \mathrm{const}\sqrt{t} + O(t)$ by

$$\alpha_{c1,2}(t) = n\alpha_{1,2}(t/n^2) - n + 1 = \alpha_{1,2}(t) + O(t). \qquad (17)$$

In the neighborhood of $(t, \lambda) = (0, 1)$, they degenerate to forms like

$$F_-(t,\lambda) \propto \int_0^1 d\xi \frac{\rho_-(\xi)}{[(\lambda-1)^2 - a\xi^2 t]^{1/2}}. \qquad (18)$$

We must impose the subsidiary condition

$$\int_0^1 d\xi \frac{\rho_-(\xi)}{\xi} = 0, \qquad (19)$$

in order to prevent a singularity at $t = 0$. The terms (18) may be considered as a superposition of Regge poles and associated square-root branch points which only coincide near $t = 0$ as indicated in Eq. (17). At $t = 4m^2$, the pole trajectories must develop a threshold, while the branch points do this only at $t = t_i$. The pole–cut relationship can be such that the unwanted branches of the pole trajectories are removed from the physical sheet for $t > 0$.[1,5] In Eq. (18) we assume that the weight function $\rho_-(\xi)$ is sufficiently well behaved so as not to introduce unwanted singularities.

In the s channel, the amplitude (18) gives rise to expressions like Eq. (11) with

$$\mathrm{Im}\psi_-(\xi, \ln s) = \rho_-(\xi),$$

$$\mathrm{Re}\psi_-(\xi, \ln s) = \frac{2}{\pi}\ln s\left(\int_0^\xi dx \frac{\rho_-(x)}{x}\right). \qquad (20)$$

Of special interest is the question of oscillations in the differential cross section, which can be present with amplitudes of the type discussed here.[13] We hope to come back to this problem elsewhere.

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Ghost-Eliminating Modifications in Multipion Amplitudes: Factorization on Leading Trajectories

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The modification of B_n to eliminate the ghost at $\alpha_\rho = 0$ is investigated. The $J_1J_2J_3$ leading three-particle vertex in B_n is calculated. Using this form, it is shown that *no* finite number of term modification of B_n without trajectory depression satisfies consistent factorization in all multipion amplitudes. Allowing trajectory depression, although daughter levels still presumably do not factorize, a solution is found in which (a) all leading trajectories factorize, (b) are nondegenerate, and (c) the $\rho\rho\rho$ vertex need not be zero. We believe this to be the suitable generalization of the Lovelace 4-π amplitude.

I. INTRODUCTION

Since the introduction of the Veneziano 4-point[1] and n-point[2] amplitudes, a great deal of work has been done deriving the properties of planar dual amplitudes. From the work of Hopkinson and Chan[3] and others, and with the factorization results of Fubini and Veneziano,[4] and Bardakci and Mandelstam,[5] the function B_n appears to be a suitable approximation for the n-σ amplitude with all identical internal trajectories (σ: $J^{PIG} = 0^{+0+}$). There are a number of modifications which must be made to B_n to obtain an appropriate form for, say, the n-π amplitude. Among these are (a) the ghost at $\alpha_\rho = 0$, which would be a tachyon due to the positive intercept of the real ρ trajectory, must be eliminated, (b) positivity constraints arising from the requirement that all three-particle couplings be real must be imposed, and (c) the so-called abnormally coupling leading trajectories (ω-A_2 trajectory in odd number of pion channels) must be included.

The ω-A_2 trajectory inclusion has been discussed in the literature by Dorren *et al.*[6] and by Gabarró and González Mestres[7] for the 6-π amplitude, and by Canning and Jacobs[8] for the 8-π and n-π amplitudes. The ρ ghost has been discussed by Lovelace[9]