Violation of the Pomeranchuk Theorem and Zeros of the Scattering Amplitudes*

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Assuming constant but unequal asymptotic total cross sections for particle-particle and particle-antiparticle collisions, it is proved within axiomatic field theory that the scattering amplitude must have infinitely many zeros in a certain narrow angular region of the t plane containing the physical region $t \leq 0$. It is shown further that this region cannot be made narrower without additional assumptions. These results are also valid for a more general class of scattering amplitudes, including those saturating the Froissart bound.

I. INTRODUCTION

The data on high-energy total cross sections obtained at Serpukhov[1] have raised the possibility of experimental violation of the Pomeranchuk theorem,[2] and have led several authors to investigate its implications and to propose models of scattering amplitudes incorporating such a violation.[3,4] Some of these models[3] have the feature that they predict an oscillation of the differential cross sections at near-forward angles. If such an oscillation were an intrinsic feature of all these amplitudes, it would of course be of considerable experimental interest. More recently, however, other models[4] have been proposed in which such an oscillation does not appear in the cross section, although it still appears in the derivative of the cross section with respect to t (momentum transfer squared). In any case, in all these models, the scattering amplitudes have infinitely many zeros, all collapsing onto the origin $t=0$ at the rate $(\ln s)^{-2}$ as $s\rightarrow\infty$, either along the negative real axis,[3] or through the complex region.[4] Thus it will be interesting to see to what extent the properties of zeros of the Pomeranchuk-theorem-violating amplitudes can be determined in a model-independent fashion. The main purpose of this paper is to give an answer to this question within the framework of axiomatic field theory.

The first useful information on the zeros of the scattering amplitude was obtained several years ago by Bessis,[5] who showed that, within axiomatic field theory, any scattering amplitude $F(s,t)$ (not restricted to the Pomeranchuk-theorem-violating one), which does not become purely real in the high-energy limit, cannot have zeros within a circle of radius $C_0(\ln s)^{-2}$, where C_0 is a positive constant determined by the asymptotic behavior of $F(s,t)$ for $s\rightarrow+\infty$, $|t|<t_0$. By adapting Bessis's result to the Pomeranchuk-theorem-violating amplitude, Eden and Kaiser[6] have recently shown that such a constant C_0 can be found in this case, too. They have shown further that one can find another constant C_1, which is a finite multiple of C_0, such that $F(s,t)$ has at least one zero within the ring

$$\frac{C_0}{(\ln s)^2}<|t|<\frac{C_1}{(\ln s)^2}. \tag{1}$$

Their proof is based on the observation that, if the domain (1) has no zero of $F(s,t)$ for arbitrarily large C_1, one inevitably runs into contradiction with unitarity. They have shown further that the number of zeros in (1) will increase as C_1 increases. Unfortunately, this result does not tell us in what part of the t plane these zeros are located. By sharpening the technique of Ref. 6, however, we have been able to show that some of these zeros must lie on the left half t plane.[7,8]

Through these investigations it has become increasingly clear that the requirement that the scattering amplitude violates the Pomeranchuk theorem is so restrictive that it determines the analytic property of $F(s,t)$ to a considerable extent. For example, as was shown by Arafune and Sugawara,[9] in the $0<t<t_0$ region, $\mathrm{Im}F(s,t)$ has a lower bound which is qualitatively very similar to the well-known upper bound.

The most striking manifestation of this strong restriction, however, is that the function defined by

$$f(\tau)=\lim_{s\to+\infty}\frac{F(s,-t_0\tau(\ln s)^{-2})}{F(s,0)}, \quad (2)$$

where $\mathrm{Re}F(s,0)\sim C's\ln s$, $\mathrm{Im}F(s,0)\sim C''s$, is not only analytic in τ but is actually an entire function of order $\frac{1}{2}$. (The precise meaning of $\lim_{s\to+\infty}$ will be specified later.) This result is derived from analyticity and unitarity of axiomatic field theory and does not require any extra assumption. As far as this point is concerned, Casella's guess[3] has therefore been justified. It is an immediate consequence of the nonintegral order that $f(\tau)$ must have infinitely many zeros.[10] Thus we have recovered (and generalized) in a very simple way the result of Eden and Kaiser.[6]

Actually these results are not restricted to the amplitudes that violate the Pomeranchuk theorem. The function $f(\tau)$ defined by (2) is entire for any scattering amplitude that satisfies the condition[11]

$$s(\ln s)^2\frac{\mathrm{Im}F(s,0)}{|F(s,0)|^2}\leqslant \text{const} \quad \text{for } s>s_0. \quad (3)$$

In particular, scattering amplitudes saturating the Froissart bound [i.e, $\sigma_{\mathrm{tot}}(s)\sim C(\ln s)^2$] belong to this class.

In Sec. II we show that $f(\tau)$ of this class is an entire function of order $\frac{1}{2}$ and of finite type $\neq 0$. The fact that $f(\tau)$ is square-integrable (a consequence of unitarity) enables us to express it in terms of certain integral representations. One of these representations can be readily turned into an eikonal-like representation. This is studied in Sec. III. We examine the distribution of (infinitely many) zeros of $f(\tau)$ in Sec. IV. In particular we show that $f(\tau)$ must have infinitely many zeros in the neighborhood of the positive τ axis (i.e., negative t axis) defined by

$$|\theta|\leqslant\frac{1}{(\ln|\tau|)(\ln\ln|\tau|)\cdots(\ln\ln\cdots\ln|\tau|)}, \quad (4)$$

where $\theta=\arg\tau$, and the logarithm is taken n times in the last (nth) factor, n being any positive integer. Furthermore we show that it is not possible to improve this domain without making some additional assumptions. This means that scattering amplitudes which are subject to no condition other than (3) are not required in general to have zeros on the negative t axis, or produce visible oscillations in the differential cross section. Of course this does not prevent us from finding oscillating cross sections for some amplitudes of the class defined by (3) which are subject to additional constraints. For instance, Roy[12] has noticed that, if the ratio $(\Delta\sigma_{\mathrm{tot}})^2/\sigma_{\mathrm{el}}$ is larger than some critical value,[13] oscillations must be present in the cross section. He has found in particular that, if the scattering amplitude saturates the Froissart bound strongly, i.e., it is as large as is allowed theoretically,[14] all zeros of the limit function $f(\tau)$ must lie on the real τ axis. We shall give an alternative proof of this result of Roy's at the end of Sec. IV.

In Appendix A we sketch the proof of the "Paley-Wiener" theorem for the Hankel transform stated in Sec. II. Appendix B gives a derivation of Eq. (28) needed in Sec. III. Appendix C is devoted to a sketch of an explicit construction of the entire functions discussed in Sec. IV.

II. ANALYTIC PROPERTY OF $f(\tau)$

In order to avoid unnecessary complications we shall restrict ourselves to the elastic scattering of spin-zero particles of equal mass. Let $F(s,t)$ be the invariant scattering amplitude normalized as

$$\frac{d\sigma}{dt}\simeq\frac{1}{16\pi s^2}|F(s,t)|^2 \quad (5)$$

for sufficiently large s. It is shown within axiomatic field theory that[15]

(i) $F(s,t)$ is holomorphic in the disk $|t|<t_0$ for any s in the cut s plane, where t_0 is a constant less than or equal to the t-channel threshold;

(ii) $F(s,t)$ is bounded by Cs^N for $|t|<t_0$ and $s\to+\infty$;

(iii) $F(s,t)$ satisfies unitarity in the s channel.

From these properties it follows that $F(s,t)$ satisfies the bound[6]

$$|F(s,t)|\leqslant[4\pi\sigma_{\mathrm{el}}(s)/t_0]^{1/2}\, s\ln s\exp[(|t|/t_0)^{1/2}\ln s] \quad (6)$$

for $|t|<t_0$.

As is well known, the Pomeranchuk-theorem-violating amplitudes behave asymptotically as[2]

$$F(s,0)\simeq i\sigma s+Cs\ln s,\quad 0<|C|<(4\pi\sigma/t_0)^{1/2}, \quad (7)$$

where σ is the total cross section. Actually we may weaken this asymptotic behavior and require it only for some sequence of real points $\{s_i|\ s_i\to+\infty\}$ dense at infinity.[16]

We note that $F(s,0)$ with the asymptotic behavior (7) satisfies also the inequality

$$s(\ln s)^2\frac{\mathrm{Im}F(s,0)}{|F(s,0)|^2}\leqslant C_0 \quad \text{for } s>s_0, \quad (8)$$

where C_0 is a positive constant. As is shown in the following, this is the crucial relation in determining the analytic property of Pomeranchuk-theorem-violating amplitudes in the high-energy limit. In fact, insofar as (8) is satisfied, the result of this paper applies to any scattering amplitude, whether it is of the form (7) or not. Besides the Pomeranchuk-theorem-violating amplitudes, the class of amplitudes defined by (8) contains scattering amplitudes that saturate the Froissart bound. In this case the inequality (8) is satisfied in another way:

$$F(s,0)\simeq(i\alpha+\beta)s(\ln s)^2,$$

$$0<\alpha\le\frac{4\pi}{t_0},\quad |\beta|\le\left(\frac{4\pi\alpha}{t_0}\right)^{1/2},\quad C_0=\frac{\alpha}{\alpha^2+\beta^2}. \tag{9}$$

Let us now introduce

$$f(s,\tau)=\frac{F(s,-t_0\tau(\ln s)^{-2})}{F(s,0)}. \tag{10}$$

From Eqs. (6), (8), and (10) together with $\sigma_{\rm el}(s) \le \sigma_{\rm tot}(s)\simeq s^{-1}{\rm Im}F(s,0)$, we deduce the bound on $f(s,\tau)$,

$$|f(s,\tau)|\le\left(\frac{4\pi C_0}{t_0}\right)^{1/2}e^{\sqrt{|\tau|}},\quad |\tau|\le(\ln s)^2-\epsilon. \tag{11}$$

Thus, the set $\Phi_S\equiv\{f(s,\tau)|s>S\}$ is a family of analytic functions of τ uniformly bounded in the disk $|\tau|\le(\ln S)^2-\epsilon$. As a consequence, Φ_S is a normal family[17]: it contains at least one sequence $\{f(s_n,\tau)|n=1,2,\ldots\}$ converging uniformly in $|\tau| \le(\ln S)^2-\epsilon$ to a function $f_S(\tau)$ analytic in the same disk. Moreover $f_S(\tau)$ satisfies the bound (11). But this is true for any S. Hence by choosing, for example, $S=\text{const}\times N$ $(N=1,2,\ldots)$, and using a classical diagonal procedure, we can extract another sequence $\{f(s_n',\tau)|n=1,2,\ldots\}$ converging uniformly to an analytic function in any compact set of the τ plane. An alternative procedure is to use again the sequence s_n and note that Vitali's theorem applies to the functions $f(s_n,\tau)$ which tend to a limit in $|\tau| <(\ln S)^2$ and are bounded in $|\tau|<(\ln S)^2$. In both ways one obtains a function $f(\tau)=\lim_{s\to\infty}f(s,\tau)$ which is *entire* and has the properties

$$f(0)=1, \tag{12}$$

$$|f(\tau)|<\left(\frac{4\pi C_0}{t_0}\right)^{1/2}e^{\sqrt{|\tau|}}\quad\text{for all }\tau. \tag{13}$$

The last property implies that $f(\tau)$ is of order $\rho \le\frac12$ (actually $\rho=\frac12$, as is shown below).

Although (12) ensures that the limit we have defined does not vanish identically, there is no need for this limit to be unique. It must be realized that uniqueness can be obtained only by imposing extra assumptions which prevent the amplitude from oscillating indefinitely in s for every t in a *fixed* neighborhood of $t=0$. We shall not discuss this point further here since it does not affect the following considerations.

The most important restriction on the property of $f(\tau)$ results from the unitarity condition written in the form

$$\frac{1}{16\pi s^2}\int_{-s+4m^2}^{0}dt|F(s,t)|^2\le\sigma_{\rm tot}(s). \tag{14}$$

In fact, inserting Eqs. (8) and (10) in (14), we obtain

$$\int_0^T d\tau|f(s_n,\tau)|^2<\frac{16\pi C_0}{t_0}+\epsilon,$$

and taking the limit $s_n\to\infty$, we get

$$\int_0^T d\tau|f(\tau)|^2<\frac{16\pi C_0}{t_0}$$

for arbitrary $T>0$. Hence

$$\int_0^\infty d\tau|f(\tau)|^2\le\frac{16\pi C_0}{t_0}. \tag{15}$$

Thus, for any scattering amplitude belonging to the class defined by (8), we have $f(\tau)\in L_2(0,\infty)$. In particular, $f(\tau)$ cannot be identically equal to 1. We shall also need another consequence of unitarity,

$$|f(\tau)|<\text{const}\quad\text{for }\tau>0 \tag{16}$$

(this constant is not necessarily unity).

We have already noted that the order ρ of $f(\tau)$ cannot exceed $\frac12$ because of the bound (13). We shall now show that unitarity requires that $f(\tau)$ is at least of order $\frac12$. To see this, suppose that the order of $f(\tau)$ is less than $\frac12$. Then, from (16) and the Phragmén-Lindelöf theorem,[18] $f(\tau)$ is bounded everywhere and hence is a constant. This contradicts Eq. (15).

A more precise result is obtained from the theorem[19] for a nonconstant entire function $f(\tau)$ of growth $(\frac12, 0)$ that

$$\lim_{r\to\infty}\sup m(r)=\infty, \tag{17}$$

where $m(r)$ is the minimum modulus of $f(\tau)$ for $|\tau|=r$. This again contradicts unitarity. In fact $f(\tau)$ must be an entire function of order exactly equal to $\frac12$ and of *finite* type $\ne 0$. This last statement on the type can also be obtained from the inequalities established by Arafune and Sugawara.[9] As was noted already, $f(\tau)$ must have infinitely many zeros, being of nonintegral order.[10]

The fact that $f(\tau)$ belongs to $L_2(0,\infty)$ enables us to make use of a modified form[20] of the Paley-Wiener theorem (see Appendix A): An entire function is of order $\rho=\frac12$ and belongs to $L_2(0,\infty)$ if and only if its Hankel transform of zeroth order[21] has its support contained in $[0,a]$ and belongs to $L_2(0,a)$, a (the type) being finite. Hence $f(\tau)$ has an integral representation

$$f(\tau)=\tfrac12\int_0^1 du\,h(u)J_0(\sqrt{\tau u}), \tag{18}$$

where

$$h(u)\in L_2(0,1) \tag{19}$$

[note that $a\le1$ according to (13)], and

$$\tfrac12\int_0^1 du\,h(u)=1 \tag{20}$$

from Eq. (12). Let us note that Eq. (18) can be inverted as the usual Fourier transform,

$$h(u)=\lim_{T\to\infty}\tfrac{1}{2}\int_0^T d\tau\, f(\tau)J_0(\sqrt{\tau u}) \tag{21}$$

(strong convergence), and that, the Parseval relation still being true [see Eq. (A.3)], we have from Eq. (15)

$$\int_0^1 du\,|h(u)|^2 \leq \frac{16\pi C_0}{t_0}. \tag{22}$$

It is then easy to improve the bound (16) by applying the Schwarz inequality to Eq. (18),

$$|f(\tau)| \lesssim (8C_0/t_0)^{1/2}\tau^{-1/4} \quad \text{for } \tau \to +\infty.$$

An equally useful approach is to introduce a new function

$$\psi(z) \equiv f(\tau), \quad z=\sqrt{\tau}\,. \tag{23}$$

Obviously $\psi(z)$ is an entire function of order 1 (i.e., a function of exponential type) in z and has the following properties:

(a) $\psi(z)$ is even in z,

(b) $\int_{-\infty}^{\infty}|\psi(x)|^2 x\,dx$ is convergent,

(c) $|\psi(z)| < \left(\frac{4\pi C_0}{t_0}\right)^{1/2} e^{|\mathrm{Im}\,z|}$.

The property (b) follows from (15), while (c) is a general property of even entire functions of order 1 bounded on the real axis.[22] [Another property, $\psi(0)=1$, is not essential in the following.]

It follows from (b) and analyticity of $\psi(z)$ at $z=0$ that

$$\int_{-\infty}^{\infty}|\psi(x)|^2 dx < \infty. \tag{24}$$

Thus $\psi(z)$ satisfies all conditions of the standard Paley-Wiener theorem,[23] and hence has the integral representation of the form

$$\psi(z)=\int_{-a}^{a} dp\,\tilde{\psi}(p)e^{ipz}, \tag{25}$$

where a is the type of $\psi(z)$ (= 1 in our case) and $\tilde{\psi}(p)\in L_2(-a,a)$, $\tilde{\psi}(p)=\tilde{\psi}(-p)$.

It should be noted that the representations (18) and (25) are not equivalent. Though (18) implies (25), the converse is not true.

Finally, let us quote a theorem[24] which is crucial in our later discussion: For any given real positive nondecreasing even function $C(x)$ such that

$$\int_1^{\infty}\frac{C(x)}{x^2}\,dx < \infty, \tag{26}$$

one can find an entire function $\psi(z)$, not identically equal to zero, of the form (25), such that

$$|\psi(x)| < e^{-C(x)}. \tag{27}$$

III. EIKONAL-LIKE REPRESENTATION OF $F(s, t)$ IN THE HIGH-ENERGY LIMIT

Starting from Eq. (18), we can derive rigorously an eikonal-like representation of $F(s,t)$ in the limit $s\to+\infty$. For this purpose we have only to establish a relation between the weight function $h(u)$ and the partial-wave amplitude $f_l(s)$. As is shown in Appendix B, this relation is given by[25]

$$f_l(s)\big|_{l=(us/t_0)^{1/2}(\ln s)/2} \underset{s\to\infty}{\simeq} \frac{t_0}{8\pi}\frac{F(s,0)}{s(\ln s)^2}\left[\tfrac{1}{3}h(u-0)+\tfrac{2}{3}h(u+0)\right], \tag{28}$$

under the additional assumption that $h(u)$ is a function of bounded variation in $[u-\epsilon, 1]$. Solving (28) for $h(u)$ and substituting the result in (18), we obtain the eikonal-like representation of $F(s,t)$. We shall discuss in particular the two cases of special interest: (A) the case where the Pomeranchuk theorem is violated, and (B) the case where the Froissart bound is saturated.

(A) In this case, using $|\mathrm{Im}F(s,t)| < \mathrm{Im}F(s,0)$ for $-s+4m^2<t<0$, we obtain

$$\begin{aligned}|\mathrm{Im}f(\tau)| &= \lim_{s\to\infty}\left|\frac{\mathrm{Im}F(s,-t_0\tau(\ln s)^{-2})}{Cs\ln s}\right| \\ &= \lim_{s\to\infty}\frac{\sigma}{C\ln s}\left|\frac{\mathrm{Im}F(s,-t_0\tau(\ln s)^{-2})}{\mathrm{Im}F(s,0)}\right| \\ &= 0\end{aligned} \tag{29}$$

for all $\tau<0$, where C and σ are defined by (7). Thus we have $f(\tau^*)=f^*(\tau)$ and $h(u)$ is a real function by (21). According to (7), (10), (18), and (28), we can therefore express $F(s,t)$ in the eikonal form

$$F(s,t) \underset{s\to\infty}{\simeq} 8\pi s\int_0^{(t_0)^{-1/2}\ln s} b\,db\;\delta(b,s)J_0(b\sqrt{-t}), \tag{30}$$

where $b=2l/\sqrt{s}=(u/t_0)^{1/2}\ln s$ is the impact parameter and

$$\delta(b,s)\simeq f_l(s)\simeq(C\,t_0/8\pi\ln s)h(u).$$

(B) In this case $f(\tau)$ is no longer real on the real axis [even if $\beta=0$, $f(\tau)$ is not necessarily real there]. We have

$$F(s,t) \underset{s\to\infty}{\simeq} 8\pi s\int_0^{(t_0)^{-1/2}\ln s} b\,db\,\frac{e^{2i\delta(b,s)}-1}{2i}J_0(b\sqrt{-t}), \tag{31}$$

with

$$\frac{e^{2i\delta(b,s)}-1}{2i} \simeq \frac{(i\alpha+\beta)t_0}{8\pi}h\left(\frac{t_0 b^2}{(\ln s)^2}\right). \tag{32}$$

The unitarity property $\mathrm{Im}\,\delta(b, s) \geq 0$ implies

$$\alpha \mathrm{Re} h(u) + \beta \mathrm{Im}\, h(u) \geq \frac{t_0}{8\pi}(\alpha^2+\beta^2)|h(u)|^2 \qquad (33)$$

for $0 \leq u \leq 1$.

IV. DISTRIBUTION OF ZEROS OF $\psi(z)$

Turning now to the problem of the distribution of zeros z_i of $\psi(z)$, let us first show that $\psi(z)$ must have infinitely many zeros in the small domain around the real axis defined by[26]

$$|\theta| \leq \frac{1}{(\ln|z|)(\ln\ln|z|)\cdots(\ln\ln\cdots\ln|z|)}. \qquad (34)$$

Here $\theta = \arg z$, and the logarithm is taken n times in the last (nth) factor, where n is any fixed positive integer. To prove this, we note that, if $\psi(z)$ is of exponential type such that

$$\limsup \frac{\ln|\psi(z)|}{|z|} > 0 \quad \text{for } z \to \pm i\infty \qquad (35)$$

and

$$\int_{-\infty}^{\infty} \frac{\ln^+\psi(x)}{1+x^2}\,dx < \infty, \qquad (36)$$

where

$$\ln^+ y = \begin{cases} \ln y & \text{for } y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

[(36) follows from (16)], then the number of zeros, $n(r)$, in $|z| \leq r$ must be of order r [i.e., $n(r) \sim Ar$, $A \neq 0$] as $r \to \infty$.[27] On the other hand, $\psi(z)$ has the property that[27, 28]

$$\sum_i \frac{|\sin\theta_i|}{|z_i|} < \infty, \qquad (37)$$

where the summation is over all zeros of $\psi(z)$. Suppose now that the number of zeros in the domain defined by (34) is finite. This would mean that

$$\sum_i{}' \frac{1}{|z_i|(\ln|z_i|)\cdots(\ln\cdots\ln|z_i|)} < \infty, \qquad (38)$$

where there are n logarithms in the last term as before, and the summation is over all zeros outside the domain (34). But this is not compatible with $n(r) \sim Ar$. Thus the domain (34) must contain infinitely many zeros of $\psi(z)$. By a standard application of Hurwitz's theorem, one then finds zeros in the amplitude $F(s_i, -t_0\tau(\ln s_i)^{-2})$ for s_i large enough, with locations as close as one wishes from the limiting zeros within any compact set in τ.

Let us now show that the domain (34) is essentially the best available and cannot be improved without additional assumptions on $\psi(z)$. Obviously, it is sufficient to construct a function $\psi(z)$ which has only a finite number of zeros within a domain slightly smaller than (34), for example,[26]

$$|\theta| < \frac{1}{(\ln|z|)(\ln\ln|z|)\cdots(\ln\ln\cdots\ln|z|)^{1+\epsilon}}, \quad \epsilon > 0. \qquad (39)$$

To achieve this we first construct an auxiliary even entire function $\chi(z)$ such that

$$|\chi(z)| \leq |\chi(0)| \qquad (40)$$

in the domain (39). Once such a function is found, the desired function $\psi(z)$ may be defined by

$$\psi(z) = [\chi(z) - \chi(0)]/z^2. \qquad (41)$$

Thus the problem is reduced to that of finding an appropriate entire function $\chi(z)$. Since it is sufficient to construct one such example, we shall further assume that $\chi(z)$ is square integrable on the real axis. Then we can apply the theorem of Ref. 24 mentioned at the end of Sec. II. Namely, if we choose a nondecreasing function

$$C_1(x) = \frac{|x|}{|[\ln(|x|+ic)][\ln\ln(|x|+ic)]\cdots[\ln\ln\cdots\ln(|x|+ic)]^{1+\epsilon}|}, \qquad (42)$$

it satisfies the condition (26), and hence there exists an entire function $\chi(z)$ such that

$$|\chi(x)| < e^{-C_1(x)} \qquad (43)$$

for real x. For complex z we have[22]

$$|\chi(z)| < e^{|\mathrm{Im}\, z|}. \qquad (44)$$

A sketch of an explicit method of construction of such a function will be given in Appendix C.

Now, if we consider the function defined by

$$\chi(z)e^{iz + C_2(z)}, \qquad (45)$$

with

$$C_2(z) = \frac{z}{[\ln(z+ic) - \frac{1}{2}i\pi]\{\ln[\ln(z+ic) - \frac{1}{2}i\pi]\}\cdots\{\ln\cdots\ln[\ln(z+ic) - \frac{1}{2}i\pi]\}^{1+\epsilon}}, \qquad (46)$$

where the positive constant c is chosen big enough to avoid singularities of the logarithms, this function is holomorphic in the first quadrant $0 \leq \arg z \leq \frac{1}{2}\pi$, and is bounded by 1 on the positive real axis [from (43)] and on the positive imaginary axis [from (44)]. Applying the Phragmén-Lindelöf theorem[18] to the function (45), we therefore find that (45) is bounded by 1 everywhere in the first quadrant. This means that

$$|\chi(z)| < |\exp[-iz - C_2(z)]|, \quad 0 \leq \arg z \leq \tfrac{1}{2}\pi. \tag{47}$$

Similar bounds can be found for other quadrants, too.

The region where $|\chi(z)|$ is less than unity, namely where the real part of the exponent in (47) is negative, is given for large $|z|$ by

$$\left|\frac{\mathrm{Im}z}{\mathrm{Re}z}\right| \lesssim \frac{1}{(\ln|z|)\cdots(\ln\ln\cdots\ln|z|)^{1+\epsilon}}. \tag{48}$$

Thus in the region[26]

$$|\theta| < \frac{\alpha}{(\ln|z|)(\ln\ln|z|)\cdots(\ln\ln\cdots\ln|z|)^{1+\epsilon}}, \tag{49}$$

where $\alpha < 1$, $\chi(z)$ decreases very rapidly as $|z| \to \infty$. Therefore, in the domain (49) the function

$$\psi(z) = \frac{\chi(z) - \chi(0)}{z^2} \tag{50}$$

can have at most a *finite* number of zeros. This completes the proof that the result (34) on the location of zeros cannot be improved.

The absence of oscillations of $\psi(z)$ could also be shown. But it is most simply seen in the example proposed by Okun and Popov and independently by one of us (G.A.),[4,29]

$$\psi(z) = \frac{1}{z^2}\left(1 - \frac{\sin z}{z}\right), \tag{51}$$

where, however, the zeros are much closer to the real axis. Hence the existence of a nonzero difference between asymptotic particle-particle and antiparticle-particle cross sections does not necessarily imply oscillations of the differential cross sections. However, as was mentioned in the Introduction, Roy has noted that too strong a quantitative violation of the Pomeranchuk theorem induces oscillations in the cross section.[30] He has found in particular that, in the case where the Froissart bound is strongly saturated [namely, not only $\sigma_{\mathrm{tot}}(s) \sim B(\ln s)^2$, but B is the largest possible constant[14]], the function $f(\tau)$ is completely determined and given by

$$f(\tau) = \frac{2J_1(\sqrt{\tau})}{\sqrt{\tau}}. \tag{52}$$

This means, in particular, that all zeros of $f(\tau)$ are on the real axis and there are no complex zeros.

We shall close this section by giving an alternative proof of (52). Strong saturation of the Froissart bound means that $\alpha = 4\pi/t_0$ in Eq. (9). Let us put

$$g(u) \equiv \frac{t_0}{8\pi}(i\alpha + \beta)h(u) = \left(\tfrac{1}{2}i + \frac{\beta t_0}{8\pi}\right)h(u). \tag{53}$$

Then we obtain

$$1 \geq \mathrm{Im}g(u) \geq |g(u)|^2 \tag{54}$$

from (33), and

$$\int_0^1 du\, g(u) = i + \frac{\beta t_0}{4\pi} \tag{55}$$

and

$$\int_0^1 du\, \mathrm{Im}g(u) = 1 \tag{56}$$

from (20). But (56) and $\mathrm{Im}g(u) \leq 1$ imply that $\mathrm{Im}g(u) = 1$ almost everywhere. It follows from (54) that $\mathrm{Re}g(u) = 0$. This means that $\beta = 0$ and hence $h(u) = 2$. This result enables us to evaluate $f(\tau)$:

$$\begin{aligned} f(\tau) &= \tfrac{1}{2}\int_0^1 du\, h(u) J_0(\sqrt{\tau u}) \\ &= \int_0^1 du\, J_0(\sqrt{\tau u}) \\ &= \frac{2J_1(\sqrt{\tau})}{\sqrt{\tau}}. \end{aligned} \tag{57}$$

This proves (52).

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APPENDIX A: PROOF OF THE PALEY-WIENER THEOREM FOR HANKEL TRANSFORMS

The theorem we stated just before Eq. (18) contains two independent parts: a Plancherel-type theorem (including the Parseval relation) and a support property.

(i) *"Plancherel" theorem.* If $f(\tau)\in L_2(0,\infty)$, then there exists a function $h(u)\in L_2(0,\infty)$ such that (all limits here are strong-convergence limits)

$$h(u)=\lim_{T\to\infty}\tfrac{1}{2}\int_0^T d\tau\, f(\tau)\, J_0(\sqrt{\tau u}\,), \tag{A.1}$$

$$f(\tau)=\lim_{U\to\infty}\tfrac{1}{2}\int_0^U du\, h(u)\, J_0(\sqrt{\tau u}\,), \tag{A.2}$$

and

$$\|h(u)\|=\|f(\tau)\|. \tag{A.3}$$

More generally

$$(h_1,h_2)=(f_1,f_2). \tag{A.4}$$

(ii) *Support property.* If $f(\tau)$ is moreover an entire function of order $\frac{1}{2}$ and type $\leqslant a$, then $h(u)=0$ for $u>a^2$, and vice versa.

The proof of (i) is quite similar to the proof of the analogous theorem for Fourier transforms, and we shall omit it. That it must be true is intuitively clear from the properties of Hankel's repeated integral[21] and from the fact that (A.1) takes the form

$$\sqrt{v}\,h(v^2)\sim(2/\pi)^{1/2}\int_0^\infty dx\sqrt{x}\,f(x^2)\cos(vx-\tfrac{1}{4}\pi),$$

if we replace $J_0(\sqrt{\tau u}\,)$ by its asymptotic form. Then $f(\tau)\in L_2(0,\infty)$ and $h(u)\in L_2(0,\infty)$ are equivalent to $\sqrt{x}\,f(x^2)\in L_2(0,\infty)$ and $\sqrt{v}\,h(v^2)\in L_2(0,\infty)$, respectively, and (i) is "equivalent" to the Plancherel theorem for Fourier transforms.

The direct proof of (ii) is more delicate, but we can start from the standard Paley-Wiener theorem for the function $\psi(z)=f(z^2)$. Let us suppose that $f(\tau)$ is an entire function of order $\frac{1}{2}$ and type $\leqslant a$, such that $f(\tau)\in L_2(0,\infty)$ on the real axis. Then we have, from Eq. (25),

$$f(\tau)=2\int_0^a dp\,\tilde{\psi}(p)\cos(p\sqrt{\tau}\,). \tag{A.5}$$

Let us consider

$$H(u_1)\equiv\int_0^{u_1} du\, h(u)=\int_0^\infty du\, h(u)\theta(u_1-u). \tag{A.6}$$

Since $h(u)\in L_2(0,\infty)$ and $\theta(u_1-u)\in L_2(0,\infty)$, we can apply the Parseval relation (A.4) and obtain

$$H(u_1)=\int_0^\infty du\, h(u)\theta(u_1-u)=\int_0^\infty d\tau\, f(\tau)\hat{\theta}(\tau),$$

where

$$\hat{\theta}(\tau)=\lim_{U\to\infty}\tfrac{1}{2}\int_0^U du\,\theta(u_1-u)\,J_0(\sqrt{\tau u}\,)$$
$$=\tfrac{1}{2}\int_0^{u_1} du\, J_0(\sqrt{\tau u}\,)=\left(\frac{u_1}{\tau}\right)^{1/2} J_1(\sqrt{\tau u_1}).$$

Thus we have

$$H(u_1)=\int_0^\infty d\tau\, f(\tau)\left(\frac{u_1}{\tau}\right)^{1/2} J_1(\sqrt{\tau u_1}). \tag{A.7}$$

Let us define $G(u_2)$ by performing a further integration,

$$G(u_2)\equiv\int_0^{u_2} du_1 H(u_1)$$
$$=\int_0^{u_2} du_1\int_0^\infty d\tau\, f(\tau)\left(\frac{u_1}{\tau}\right)^{1/2} J_1(\sqrt{\tau u_1}). \tag{A.8}$$

The repeated integral in (A.8) is now absolutely convergent [this results from $f(\tau)\in L_2(0,\infty)$ and $J_1(\sqrt{\tau u_1})\underset{\tau\to\infty}{\simeq}(2/\pi)^{1/2}(\tau u_1)^{-1/4}\cos(\sqrt{\tau u_1}-\frac{3}{4}\pi)$], and the Tonelli-Hobson theorem allows us to interchange the order of integration:

$$G(u_2)=\int_0^\infty d\tau\frac{f(\tau)}{\sqrt{\tau}}\int_0^{u_2} du_1\sqrt{u_1}\,J_1(\sqrt{\tau u_1})$$
$$=2\int_0^\infty d\tau\, f(\tau)\frac{u_2}{\tau}J_2(\sqrt{\tau u_2}). \tag{A.9}$$

At this stage we can insert the representation (A.5) and use the Tonelli-Hobson theorem once again:

$$G(u_2)=4\int_0^\infty d\tau\,\frac{u_2}{\tau}J_2(\sqrt{\tau u_2})\int_0^a dp\,\tilde{\psi}(p)\cos(p\sqrt{\tau}\,)$$
$$=4u_2\int_0^a dp\,\tilde{\psi}(p)\int_0^\infty\frac{d\tau}{\tau}J_2(\sqrt{\tau u_2})\cos(p\sqrt{\tau}\,).$$

Now, for $u_2>p^2$,[31]

$$\int_0^\infty\frac{d\tau}{\tau}J_2(\sqrt{\tau u_2})\cos(p\sqrt{\tau}\,)=1-2\frac{p^2}{u_2},$$

so that

$$G(u_2)=u_2\left(4\int_0^a dp\,\tilde{\psi}(p)\right)-\left(8\int_0^a dp\,p^2\tilde{\psi}(p)\right)\quad\text{for } u_2>a^2. \tag{A.10}$$

But $H(u_1)$ is by definition a continuous function. Thus,

$$H(u_2)=G'(u_2)=4\int_0^a dp\,\tilde{\psi}(p)=\text{const}\quad\text{for any } u_2>a^2, \tag{A.11}$$

which implies

$$H(u'')-H(u')=\int_{u'}^{u''} du\, h(u)=0\quad\text{for any } u',u''>a^2.$$

As a consequence,[32]

$$h(u)=0\ \text{almost everywhere for } u>a^2. \tag{A.12}$$

This proves the first part of (ii). The second part (converse) is trivial.

Finally, we give the *formal* connection between $h(u)$ and $\tilde{\psi}(p)$:

$$\tilde{\psi}(p) = \frac{1}{2\pi}\int_{p^2}^{a^2} du\, \frac{h(u)}{(u-p^2)^{1/2}},$$

$$h(u) = -4\frac{d}{du}\int_{\sqrt{u}}^{a} dp\, \frac{p\tilde{\psi}(p)}{(p^2-u)^{1/2}},$$

where $0 \leq p \leq a,\ 0 \leq u \leq a^2$.

APPENDIX B: PROOF OF Eq. (28)

The partial-wave amplitude $f_l(s)$ may be expressed as

$$f_l(s) \simeq \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, F(s,t)P_l(\cos\theta)$$
$$\simeq \frac{1}{32\pi}\,\frac{2t_0}{s(\ln s)^2}\,F(s,0)$$
$$\times\int_0^{(s/t_0)(\ln s)^2} d\tau\, f(\tau)P_l\left(1-\frac{2t_0\tau}{s(\ln s)^2}\right) \quad \text{(B.1)}$$

in the large-s limit where $f(\tau)$ is defined by (2). If we put $l = \frac{1}{2}(us/t_0)^{1/2}\ln s$ $(0 < u \leq 1)$, this relation becomes

$$f_l(s)\Big|_{s\to\infty} \simeq \frac{t_0}{16\pi}\,\frac{F(s,0)}{s(\ln s)^2}\,X_l(u)\Big|_{l\to\infty}, \quad \text{(B.2)}$$

where

$$X_l(u) = \int_0^{4l^2/u} d\tau\, f(\tau)P_l\left(1-\frac{\tau u}{2l^2}\right). \quad \text{(B.3)}$$

Using Eq. (18), this may be rewritten as[33]

$$X_l(u) = \tfrac{1}{2}\int_0^1 dv\, h(v)\int_0^{4l^2/u} d\tau\, J_0(\sqrt{\tau v})P_l\left(1-\frac{\tau u}{2l^2}\right)$$
$$= 2l\int_0^1 dv\, \frac{h(v)}{\sqrt{uv}}\, J_{2l+1}(2l(v/u)^{1/2}). \quad \text{(B.4)}$$

If we introduce

$$x = (v/u)^{1/2}, \quad \nu = 2l, \quad g(x) = 2h(ux^2), \quad \text{(B.5)}$$

then

$$\lim_{l\to\infty} X_l(u) = \lim_{\nu\to\infty} X(\nu, u), \quad \text{(B.6)}$$

where

$$X(\nu,u) = \int_0^{1/\sqrt{u}} dx\, g(x)\nu J_\nu(\nu x). \quad \text{(B.7)}$$

In order to evaluate $X(\nu, u)$, it is convenient to split the domain of integration into four parts $(0, 1-\epsilon)$, $(1-\epsilon, 1)$, $(1, 1+\epsilon)$, and $(1+\epsilon, 1/\sqrt{u})$. We shall evaluate the corresponding integrals X_1, X_2, X_3, and X_4 with the help of the following formulas[34]:

$$\nu J_\nu(\nu x) \underset{\nu\to\infty}{\simeq} \left(\frac{\nu}{2\pi\tanh\alpha}\right)^{1/2} e^{\nu(\tanh\alpha-\alpha)}$$
$$\times[1+O(1/\nu\tanh\alpha)], \quad x = 1/\cosh\alpha\ (<1), \quad \text{(B.8)}$$

$$\nu J_\nu(\nu x) \underset{\nu\to\infty}{\simeq} \left(\frac{2\nu}{\pi\tan\beta}\right)^{1/2} \cos[\nu(\tan\beta-\beta)-\tfrac{1}{4}\pi]$$
$$\times[1+O(1/\nu\tan\beta)], \quad x = 1/\cos\beta\ (>1), \quad \text{(B.9)}$$

$$\int_0^1 dx\,\nu J_\nu(\nu x) \underset{\nu\to\infty}{\simeq} \tfrac{1}{3} + O(\nu^{-2/3}), \quad \text{(B.10)}$$

$$\int_0^\infty dx\,\nu J_\nu(\nu x) = 1. \quad \text{(B.11)}$$

Noting that $\tanh\alpha - \alpha < 0$ in (B.8) for $0 < x < 1-\epsilon$, it is easily seen that

$$\lim_{\nu\to\infty} X_1 = 0 \quad \text{for any } \epsilon > 0. \quad \text{(B.12)}$$

To evaluate X_4, we may rewrite it as

$$X_4 \sim \sqrt{\nu}\int_{y_1}^{y_2} dy\,\tilde{g}(y)\cos(\nu y - \tfrac{1}{4}\pi) \quad \text{(B.13)}$$

taking account of (B.9), where $y = \tan\beta - \beta$ and

$$\tilde{g}(y) = \left(\frac{2}{\pi\tan\beta}\right)^{1/2}\frac{1}{\sin\beta}\,g\left(\frac{1}{\cos\beta}\right). \quad \text{(B.14)}$$

Since $\tilde{g}(y)$ is of bounded variation in y by assumption, the integral over y in (B.13) is $O(1/\nu)$,[35] and hence

$$X_4 = O(1/\sqrt{\nu}) \quad \text{for any } \epsilon > 0. \quad \text{(B.15)}$$

Using the second mean-value theorem,[36] we may express the integral X_2 in the form

$$X_2 = g(1-\epsilon+0)\int_{1-\epsilon}^{c_\nu} dx\,\nu J_\nu(\nu x) + g(1-0)\int_{c_\nu}^{1} dx\,\nu J_\nu(\nu x), \quad \text{(B.16)}$$

where $1-\epsilon \leq c_\nu \leq 1$. Rewriting this as

$$X_2 = g(1-0)\int_{1-\epsilon}^{1} dx\,\nu J_\nu(\nu x)$$
$$+[g(1-\epsilon+0)-g(1-0)]\int_{1-\epsilon}^{c_\nu} dx\,\nu J_\nu(\nu x), \quad \text{(B.17)}$$

and noting that

$$\lim_{\nu\to\infty}\int_{1-\epsilon}^{1} dx\,\nu J_\nu(\nu x) = \tfrac{1}{3} \quad \text{for any } \epsilon > 0 \quad \text{(B.18)}$$

and

$$\int_{1-\epsilon}^{c_\nu} dx\,\nu J_\nu(\nu x) \text{ is bounded uniformly in } \epsilon,\ \nu, \quad \text{(B.19)}$$

which follow from (B.8) and (B.10), and that

$$\lim_{\epsilon\to 0}[g(1-\epsilon+0)-g(1-0)] = 0, \quad \text{(B.20)}$$

we obtain

$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty} X_2 = \tfrac{1}{3}g(1-0). \tag{B.21}$$

In a similar fashion we find

$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty} X_3 = \tfrac{2}{3}g(1+0). \tag{B.22}$$

Putting (B.12), (B.15), (B.21), and (B.22) together, we finally obtain

$$\lim_{\nu\to\infty} X(\nu, u) = \lim_{\epsilon\to 0}\lim_{\nu\to\infty} X(\nu, u) = \tfrac{1}{3}g(1-0) + \tfrac{2}{3}g(1+0). \tag{B.23}$$

APPENDIX C: EXPLICIT CONSTRUCTION OF $\chi(z)$

Since the theorem of Ref. 24 might not be familiar to most mathematical physicists, we shall sketch here one way of constructing the entire function $\chi(z)$ explicitly. Our starting point is the Paley-Wiener theorem,[23] which guarantees that any entire function of exponential type square-integrable on the real axis can be expressed by the formula (25). This theorem reduces our problem to that of finding a suitable weight function $\tilde{\chi}(p)$. Since $\tilde{\chi}(p)$, if it is discontinuous at the boundaries of its support, is likely to produce oscillations in $\chi(z)$ similar to the diffraction pattern by a disk with a sharp edge, we should look for a $\tilde{\chi}(p)$ which vanishes very smoothly at the boundaries of its support.

As an example of such a $\tilde{\chi}(p)$, let us take

$$\tilde{\chi}(p) = \tfrac{1}{2}\exp\left[-\left(\frac{1}{1-p^2}\right)^n\right], \quad -1 \leqslant p \leqslant 1, \quad n \gg 1 \tag{C.1}$$

and show that the entire function $\chi(z)$, defined by

$$\chi(z) = \tfrac{1}{2}\int_{-1}^{1} dp \exp\left[-\left(\frac{1}{1-p^2}\right)^n\right] e^{ipz}, \tag{C.2}$$

is bounded by $\exp(-|z|^{1-\epsilon})$, $\epsilon = 1/(n+1)$, for real z. In order to prove this, we shall take advantage of the fact that

$$\exp\left[-\left(\frac{1}{1-p^2}\right)^n\right] \to 0 \tag{C.3}$$

for $p\to\pm 1$ and $|\arg(1-p^2)| < \pi/2n$, and shift the contour of integration into the complex path. For instance, one may take the contour made of the segments

$$\arg(1-p) = -\pi/4n, \quad 0 \leqslant \mathrm{Re}p \leqslant 1,$$

$$\arg(1+p) = \pi/4n, \quad -1 \leqslant \mathrm{Re}p \leqslant 0. \tag{C.4}$$

It is then easy to show that the maximum of the integrand takes the form

$$\exp(-C|x|^{1-\epsilon}), \quad C > 0 \tag{C.5}$$

on this contour. This result and the property

$$|\chi(z)| < e^{|\mathrm{Im}\, z|} \tag{C.6}$$

obtained from (C.2) lead us to the bound

$$|\chi(z)| < \left|\exp\left[\frac{-iC(-iz)^{1-\epsilon}}{\cos\frac{1}{2}\pi\epsilon} - iz\right]\right|, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{2}\pi, \tag{C.7}$$

with the help of the Phragmén-Lindelöf theorem[18] as in the derivation of (47).

In order to obtain finer examples, one must take the weight function to be

$$\exp\left[-\left(\frac{1}{1-p^2}\right)^{1/(1-p^2)}\right], \text{ etc.,}$$

and distort the integration contour very carefully.

We note, incidentally, that these examples give existence proofs for the test functions used in Jaffe's field theory.[37]

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[8]Suppose $F(s,t)$, for large fixed and positive s, has n zeros $t_1, \ldots, t_n$ in the disk of radius R. Let us define the function $\varphi(t)$ by

$$\varphi(t) = \prod_{i=1}^{n}\left|\frac{R(t-t_i)}{R^2 - tt_i^*}\right|\left|\frac{R}{t_i}\right|^{(R-t)/(R+t)}.$$

Then we obtain the inequality

$$\sigma_{\mathrm{tot}} > \frac{1}{16\pi s^2}|F(s,0)|^2\int_{-R}^{0} dt\, \varphi^2(t) \exp\left(\frac{4tC\sqrt{R}\ \ln s}{R+t}\right),$$

from the unitarity condition, by refining the method of Ref. 6. Now, if all zeros of $F(s,t)$ were on the right half t plane, $\varphi(t)$ would be greater than 1 and this inequality would break down for sufficiently large s. This proves that $F(s,t)$ must have zeros on the left half t plane.

[9]J. Arafune and H. Sugawara, Phys. Rev. Letters 25, 1516 (1970).

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[11]A class of scattering amplitudes similar to (3) has been investigated by H. Cornille, Lett. Nuovo Cimento 4, 267 (1970).

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[17]See, for example, A. E. Taylor, *Introduction to Functional Analysis* (Wiley, New York, 1961), p. 150.

[18]Reference 10, p. 4.

[19]Reference 10, p. 39.

[20]A similar approach has also been considered by L. B. Okun and V. S. Popov (private communication).

[21]G. N. Watson, *Theory of Bessel Functions* (Cambridge Univ. Press, London, 1945), Chap. 14.4.

[22]Reference 10, p. 82.

[23]Reference 10, p. 103.

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[25]The s dependence of the partial-wave amplitudes given by (28) is in agreement with that obtained by a more primitive consideration by one of us (T.K.). See formula (23) of the third article quoted in Ref. 2.

[26]Strictly speaking, this inequality characterizes the domain in question only for $|z| > \exp\exp\cdots\exp e$, where "exp" is taken n times.

[27]M. L. Cartwright, *Integral Functions* (Cambridge Univ. Press, London, 1962), p. 87.

[28]Reference 10, p. 86.

[29]For this example, we obtain $h(0)=\frac{1}{2}\int_0^\infty d\tau f(\tau)=\infty$. If we want to avoid this, we may take as $\psi(2z)$ the square of (51). The same argument applies to (50) and the example constructed in Appendix C.

[30]This point is under further investigation by S. M. Roy and one of us (A.M.).

[31]*Table of Integral Transforms,* edited by A. Erdelyi (McGraw-Hill, New York, 1954), Vol. 2, p. 36.

[32]E. C. Titchmarsh, *The Theory of Functions* (Oxford Univ. Press, London, 1939), p. 360.

[33]Reference 31, p. 13.

[34]Reference 21, pp. 243, 244, 259, and 386.

[35]Reference 32, p. 426.

[36]In applying the second mean-value theorem, it is understood that the function of bounded variation g is first split into real monotonic components,

$$g=(g_1-g_2)+i(g_3-g_4),$$

and that each component g_1, g_2, g_3, and g_4 is treated separately.

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