PHYSICAL REVIEW D VOLUME 3, NUMBER 12 15 JUNE 1971

Inclusive Cross Sections Are Discontinuities*

Henry P. Stapp
Lawrence Radiation Laboratory, University of California, Berkeley, California 94720
(Received 15 March 1971)

Inclusive cross sections are discontinuities across certain cuts in the connected parts of appropriate multiparticle scattering amplitudes. This relationship, which is shown here to follow formally from field theory, is analogous to the well-known connection between total cross sections and total discontinuities. It has been used in recent work on the theory of high-energy reactions.

I. INTRODUCTION

Mueller's recent work[1] has initiated important new developments in the theory of high-energy reactions. The immediate achievement of that work was a simple derivation of the main features of the high-energy cross sections for reactions of the form $a+b\to c+$anything. Three important ideas have emerged from this achievement. The first is an understanding of how to obtain by direct nondynamical calculations results that had formerly been obtained from dynamical arguments. The second is the recognition that scattering functions have a cluster property in momentum space, with Regge poles playing a role similar to that played in the space-time cluster properties by ordinary particle poles. The third is the realization that unitarity entails a large set of relations between theory and experiment that had not formerly been exploited. This last point is the subject of the present work.

It is well known that the total cross section for a reaction $a+b\to$ anything is equal, apart from known factors, to the "total discontinuity" of the connected part of the amplitude for $a+b\to a+b$. The total discontinuity is the difference between the function evaluated above all cuts and the function evaluated below all cuts. Mueller's work has focused attention on the similar relationship that connects the inclusive cross section for the reaction $a+b\to c$ $+$anything to a discontinuity across a certain cut of the amplitude for the reaction[2] $a+b+\bar{c}\to a+b+\bar{c}$.

The relationships between individual discontinuities and inclusive cross sections are special cases of the "basic" discontinuity equation represented in Fig. 3 below. This basic discontinuity equation was originally studied in connection with an analysis of the analytic structure of many-particle scattering amplitudes.[3] In that work the basic discontinuity equation, and various equations derived from it by analytic continuation, were shown to be the fundamental constituents of the unitarity equations, in the sense that these individual discontinuities added up, in the cases studied, to the total discontinuity.

The importance of the basic discontinuity equation in works stemming from Mueller's work has generated interest in the question of whether it can be derived from field theory.[2] The aim of the present work is to show that this discontinuity equation does follow formally from field theory.

II. THE OFF-MASS-SHELL S MATRIX

In field theory the S matrix is related to a time-ordered product of field operators. One defines the τ function as

$$\tau(x_1, x_2,\ldots, x_n; y_1, \ldots, y_m) = \langle T(A_1(x_1)\cdots A_n(x_n)A^\dagger_{n+1}(y_1)\cdots A^\dagger_{n+m}(y_m))\rangle_0\,, \tag{2.1}$$

where the operator T on the right-hand side orders the operators $A_i(x_i)$ so that their times $t_i = x_i^0$ increase from right to left. Then the S matrix element

$$S_{\alpha\beta} \equiv (\Phi^\alpha_{\rm out}|\Phi^\beta_{\rm in}) = \langle A^{\alpha_1}_{1\,\rm out}\cdots A^{\alpha_n}_{n\,\rm out}|\,A^{\beta_1}_{n+1\,\rm in}\cdots A^{\beta_m}_{n+m\,\rm in}\rangle_0 \tag{2.2}$$

can be written as[4]

$$S_{\alpha\beta} = \int d^4x_1\cdots d^4x_n\, d^4y_1\cdots d^4y_m f^*_{\alpha_1}(x_1)\cdots f^*_{\alpha_n}(x_n) f_{\beta_1}(y_1)\cdots f_{\beta_m}(y_m)$$
$$\times(-i)^{n+m}K_{x_1}\ldots K_{x_n}K_{y_1}\ldots K_{y_m}\,\tau(x_1,\ldots,x_n;y_1,\ldots,y_m)\,. \tag{2.3}$$

Here

$$K_x = \Box_x - M_x^2 \tag{2.4}$$

is the Klein-Gordon operator associated with the variable x. The functions f_j are positive-frequency solutions to the Klein-Gordon equations:

$$K_x f_j(x) = 0 . \tag{2.5}$$

To keep things as simple as possible it is assumed that the f's have disjoint supports in velocity space, and are such that no disconnected processes contribute to $S_{\alpha\beta}$.

The momentum-space τ functions are defined by[5]

$$\begin{aligned}\tilde{\tau}(k) &= \tilde{\tau}(k_1, \ldots, k_{n+m}) \\ &= \int d^4x_1 \cdots d^4x_n\, d^4y_1 \cdots d^4y_m \exp[-i(x_1k_1 + \cdots + y_m k_{n+m})]\tau(x_1, \ldots, x_n; y_1, \ldots, y_m) .\end{aligned} \tag{2.6}$$

In the special case where the space components of all the k_i are real,

$$\vec{k}_i = \vec{q}_i = \text{real} , \tag{2.7}$$

one can write[6]

$$\tilde{\tau}(k) = \int \cdots \int \frac{dq_1^0}{2\pi} \cdots \frac{dq_{n+m}^0}{2\pi} \sum_P \left\langle A_{P1}^{(\dagger)}(q_{P1}) \frac{i}{E_1(P) - H + i\epsilon} A_{P2}^{(\dagger)}(q_{P2}) \frac{i}{E_2(P) - H + i\epsilon} \cdots A_{P(n+m)}^{(\dagger)}(q_{P(n+m)}) \right\rangle_0 . \tag{2.8}$$

Here P represents a permutation of the set of integers $(1, \ldots, n+m)$, and

$$A_{Pj}^{(*)}(q) \equiv \int d^4x A_{Pj}^{(\dagger)}(x) e^{-iqx} \tag{2.9}$$

is an operator that has nonzero matrix elements between momentum-energy eigenstates only if the momentum-energy of the state on the left-hand side is greater than that of the state on the right-hand side by the amount q:

$$\langle P | A_{Pj}^{(\dagger)}(q) | P' \rangle = 0 \quad \text{unless } P = P' + q . \tag{2.10}$$

The $^{(\dagger)}$ means that the Hermitian conjugate is taken if and only if Pj is greater than n. The quantity $E_j(P)$ is

$$E_j(P) \equiv k_{P(j+1)}^0 + k_{P(j+2)}^0 + \cdots + k_{P(n+m)}^0, \qquad j = 1, \ldots, n+m-1. \tag{2.11}$$

That is, it is the sum of the energies k_j^0 corresponding to the A's standing to its right. The quantity H is the energy operator. By virtue of (2.10), the denominator $E_j(P) - H$ is

$$E_j(P) - H = k_{P(j+1)}^0 + \cdots + k_{P(n+m)}^0 - q_{P(j+1)}^0 - \cdots - q_{P(n+m)}^0 . \tag{2.12}$$

The formula (2.8) is easily derived within the Hamiltonian framework. The factor $i/(E - H + i\epsilon)$ is the momentum-space form of the propagator

$$U(t, t') = \theta(t - t') e^{-iH(t-t')} \tag{2.13}$$

that takes the system from the time t' when the operator A standing on its right acts, to the time t when the operator A standing on its left acts. The operator $i/(E - H + i\epsilon)$ can be evaluated in the usual way by introducing a complete set of energy eigenstates:

$$\frac{i}{E - H + i\epsilon} = \sum_\alpha |\alpha\rangle \frac{i}{E - E_\alpha + i\epsilon} \langle\alpha| . \tag{2.14}$$

The momentum-space form of (2.3) and (2.8) is (see Appendix A)

$$S_{\alpha\beta} = \lim_{\epsilon \to 0^+} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_{n+m}}{(2\pi)^4} \tilde{f}_{\alpha_1}^*(k_1) \cdots \tilde{f}_{\beta_m}(k_{n+m}) (-i)^{n+m} \{[(k_1^2 - M_1^2) \cdots (k_{n+m}^2 - M_{n+m}^2)] \tilde{\tau}(k_1, \ldots, k_{n+m})\}_\epsilon , \tag{2.15}$$

where

$$\tilde{f}_j(k) = \int d^4x f_j(x) e^{ikx} , \quad k^0 > 0 \tag{2.16a}$$

and

$$\tilde{f}_j^*(k) = \int d^4x f_j^*(x) e^{ikx}, \quad k^0 < 0$$
$$= [\tilde{f}_j(-k)]^* . \tag{2.16b}$$

The braces with subscript ϵ means that the k_j^0 are to be evaluated at

$$k_{j\epsilon}^0 = k_{j0}^0(1 + i\epsilon) , \tag{2.17}$$

where k_{j0}^0 is real. The mass-shell constraint on the $f_j(k)$ can be used to write them, if desired, as

$$\tilde{f}_j(k_j) = \tilde{\phi}_j(k_j) 2\pi \delta(k_j^2 - M_j^2)\, \theta(k_j^0) . \tag{2.18}$$

Defining the off-mass-shell S matrix

$$S(k) \equiv S(k_1, \ldots, k_{n+m})$$
$$\equiv (-i)^{n+m}[(k_1^2 - M_1^2) \cdots (k_{n+m}^2 - M_{n+m}^2)]\tilde{\tau}(k) , \tag{2.19}$$

we may write (2.15) as

$$S_{\alpha\beta} = \lim_{\epsilon\to 0^+} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_{n+m}}{(2\pi)^4} \tilde{f}^*_{\alpha_1}(k_1) \cdots \tilde{f}_{\beta_m}(k_{n+m}) S(k_\epsilon) . \tag{2.20}$$

III. STEINMANN RELATIONS

Consider $S(k)$ for fixed real values of the space parts of the various k_i:

$$\vec{k}_i = \vec{q}_i = \text{real}. \tag{3.1}$$

Then Eqs. (2.20) and (2.8) give, in explicit form, the dependence of $S(k) = S_r(k)(2\pi)^4\delta^4(\sum k_j)$ on the energy components k_j^0. Equation (2.8) shows that $S_r(k)$ is analytic at all points of

$$(k_1^0, k_2^0, \ldots, k_{n+m}^0)/(\textstyle\sum k_j = 0)$$

space that satisfy

$$\text{Im}\, E_j(P) \neq 0 \quad \text{for all } j \text{ and } P . \tag{3.2}$$

That is, the singularities are confined to points where at least one of the sums of energies $E_j(P)$ defined in (2.11) is real. This type of analytic structure is called cut-plane analyticity. The function is analytic at all points that lie on none of the cuts

$$C_j(P) \equiv \{k^0 : \text{Im}\, E_j(P) = 0\} . \tag{3.3}$$

The variables $E_j(P)$ are called the channel energies associated with $S(k)$. They are the energies of the various reactions associated with $S(k)$. The function $S_r(k)$ is analytic unless at least one of the channel energies is real, provided the three-momenta are all fixed and real.

The cuts $C_j(P)$ divide the space

$$(\text{Im}k_1^0, \text{Im}k_2^0, \ldots, \text{Im}k_{n+m}^0)/(\textstyle\sum k_j = 0)$$

into a number of cones, which all meet at the origin $\text{Im}k^0 = 0$. The function $S_r(k)$ is analytic in each of these cones. The origin represents the real boundary point. In general, one expects to find a different boundary value for each cone: The value of $S_r(k)$ at $\text{Im}k^0 = 0$ will depend on the cone through which this real boundary point is approached.

In the case of a four-point function the cones in $\text{Im}k^0$ space are bounded by the three cuts $\text{Im}(k_1^0 + k_2^0) = 0$, $\text{Im}(k_1^0 + k_3^0) = 0$, and $\text{Im}(k_1^0 + k_4^0) = 0$, and the four cuts $\text{Im}k_j^0 = 0$ ($j = 1, 2, 3, 4$).

Because the energies of the intermediate states are non-negative, the singularities that could lie on these various cuts may be absent for certain values of $\text{Re}\vec{k}$. If one considers only the "connected part" (i.e., if one ignores contributions associated with the vacuum intermediate state[6,7]) and if there is a lower bound on the masses of the stable states, then there will be a region around $\text{Re}\vec{k} = 0$ such that the singularities are all absent. For these values of $\text{Re}\vec{k}$ the functions in the various cones are all parts of one single analytic function. The region in $\text{Re}\vec{k}$ for which this is true depends, of course, on the detailed nature of the spectral conditions – i.e., on the masses of the lowest states in the various channels.

The Steinmann relations are a set of properties that greatly simplify the analytic structure of the functions $S_r(k)$. They arise from the fact that $S_r(k)$ is a sum of terms corresponding to the different orderings of the A's,

$$S_r(k) = \sum_P S_{rP}(k) , \tag{3.4}$$

together with the fact that a given energy denominator $E_j(P) - H$ will occur only in certain terms of this sum.

To make this precise let $E(P', j)$ denote the set of P's such that $E_j(P)$ is identically equal to $E_j(P')$:

$$E(P', j) = \{P : E_j(P) \equiv E_j(P')\} . \tag{3.5}$$

Here $E_j(P) \equiv E_j(P')$ means that $E_j(P)$ consists of identically the same sum of energies k_i^0 as $E_j(P')$. Each of the S_P with P in $E(P', j)$ have the common energy denominator $E_j(P') - H$.

The set $E(P',j)$ evidently consists of all P such that P equals P' modulo permutations that permute the elements of the sets $\{P1, P2, \ldots, Pj\}$ and $\{P(j+1), \ldots, P(n+m)\}$ among themselves.

Let $C_j^+(P')$ be the part of $C_j(P')$ that lies at $\operatorname{Re} E_j(P') > 0$. And let $C_j'(P')$ be the set of points that lie on $C_j^+(P')$, but on no other cut:

$$C_j'(P') \equiv C_j^+(P') \bigcap_{\substack{P \in E(P',j) \\ \text{or } i \neq j}} \hat{C}_i(P). \tag{3.6}$$

Here $\hat{C}_i(P)$ is the complement of $C_i(P)$. The only terms $S_{rP}(k)$ that can have singularities on $C_j'(P')$ are those such that P belongs to $E(P',j)$.

The set

$$E(P',j) \cap E(P'',i) \equiv \{P : E_j(P) \equiv E_j(P'), E_i(P) \equiv E_i(P'')\} \tag{3.7}$$

is the set of P's such that S_{rP} can have singularities on both $C_j'(P')$ and $C_i'(P'')$. This set is clearly empty if the channels defined by (P',j) and (P'',i) are "crossed"; i.e., if the four sets

$$\begin{aligned} s^{\leftarrow\leftarrow} &= (P'1, P'2, \ldots, P'j) \cap (P''1, P''2, \ldots, P''i),\\ s^{\leftarrow\rightarrow} &= (P'1, P'2, \ldots, P'j) \cap [P''(i+1), \ldots, P''(n+m)],\\ s^{\rightarrow\leftarrow} &= [P'(j+1), \ldots, P'(n+m)] \\ &\quad \cap (P''1, P''2, \ldots, P''i),\\ s^{\rightarrow\rightarrow} &= [P'(j+1), \ldots, P'(n+m)] \\ &\quad \cap [P''(i+1), \ldots, P''(n+m)] \end{aligned} \tag{3.8}$$

are all nonempty. In this case there can be no permutation P such that both $E_j(P) \equiv E_j(P')$ and $E_i(P) = E_i(P'')$.

The important consequence of this result is that if the channels defined by (P',j) and (P'',i) are crossed, then there is no S_P that has singularities on both $C_j'(P)$ and $C_i'(P'')$. Consequently, the discontinuity of $S(k)$ across the singularity surface $C_j'(P')$ can have no discontinuity across the singularity surface $C_i'(P'')$. This is the Steinmann relation.[6]

IV. FIRST BASIC DISCONTINUITY EQUATION

The above derivation of the Steinmann relation leads directly to an important basic discontinuity equation. Let DiscS represent the discontinuity of S across some fixed cut $C_j'(P')$. Inspection of (2.8) shows that this discontinuity is obtained by making the replacement

$$\frac{i}{E_j(P) - H + i\epsilon} \rightarrow \frac{i}{E_j(P) - H + i\epsilon} - \frac{i}{E_j(P) - H + i\epsilon} = 2\pi\delta(E_j(P) - H) \tag{4.1}$$

in all terms S_P such that P belongs to $E(P',j)$, and dropping all other terms S_P.

The retained terms S_P are those corresponding to all permutations of the A's within the two separate sets $\{A_{P'1} \cdots A_{P'j}\}$ and $\{A_{P'(j+1)} \cdots A_{P'(n+m)}\}$. Thus Eqs. (2.19), (2.8), and (4.1) give

$$\begin{aligned}\operatorname{Disc}S(k) = (-i)^{n+m}\Big[\prod_i (k_i^2 - M_i^2)\Big] \int \cdots \int \prod_{i \neq j} \frac{dq_i^0}{2\pi} \sum_\alpha \sum_P \Big\langle 0 \Big| A_{P1} \frac{i}{E_{P1} - H + i\epsilon} \cdots A_{Pj} \Big| \text{out}, \alpha \Big\rangle \\ \times \sum_{P''} \Big\langle \text{out}, \alpha \Big| A_{P''(j+1)} \frac{i}{E_{P''(j+1)} - H + i\epsilon} \cdots A_{P''(n+m)} \Big| 0 \Big\rangle. \end{aligned} \tag{4.2}$$

The quantities E_{Pi} can be expressed as

$$E_{Pi} = -(k_{P1}^0 + k_{P2}^0 + \cdots + k_{Pi}^0), \quad i = 1, \ldots, j-1 \tag{4.3}$$

which is an alternative form of (2.11).

Introducing

$$|\text{out}, \alpha\rangle = \sum_\beta |\text{in}, \beta\rangle S^{-1}{}_{\beta\alpha}, \tag{4.4}$$

one obtains, after simple manipulations with the reduction formulas,[4] the discontinuity formula

$$\operatorname{Disc}S_c(k) = S_c(k') S_I^{-1} S_c(k''). \tag{4.5}$$

The variables in (4.5) are linked in the manner shown in Fig. 1.

The left-hand side of Fig. 1 represents the discontinuity of the connected part $S_c(k)$. The two plus circles on the right-hand side represent the connected parts $S_c(k')$ and $S_c(k'')$. The minus box represents S_I^{-1}, which acts on the intermediate set of particles. The shaded strips represent sets consisting of any number of lines. The two plus signs in the circle on the left-hand side indicate that the energies of channels that do not "cross" the channel associated with the discontinuity in question are all evaluated on the positive or physical sides of their cuts (i.e., $\operatorname{Re}E \operatorname{Im}E > 0$). The Steinmann relations ensure that the equation does not depend on upon which sides of the remaining cuts it is evaluated.

The occurrence of $S_c(k)$ on the left-hand side of (4.5) arises from our original restriction to processes in which only the connected parts contribute. The occurrence of the $S_c(k')$ and $S_c(k'')$ on

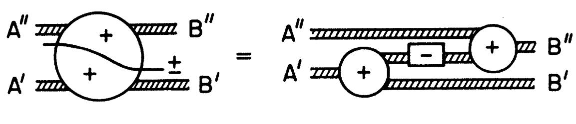

FIG. 1. Diagrammatic representation of Eq. (4.5).

the right-hand side is due to the condition that $C_j'(P')$ contains no points lying on any other cuts $C_i(P)$. [See (3.6).] This condition ensures that various disconnected terms that are potentially present in the two outer factors on the right-hand side of (4.2) are in fact absent. This is discussed further in Appendix B.

The discontinuity equation (4.5) is similar to one derived earlier within the S-matrix framework.[8] The two are identical within their common domain of definition.

V. HERMITIAN ANALYTICITY

Equations (2.2) and (2.20) give

$$\begin{aligned} S^{-1}{}_{\alpha\beta} &= (\Phi_{\text{in}}^{\alpha} \,|\, \Phi_{\text{out}}^{\beta}) \\ &= (\Phi_{\text{out}} \,|\, \Phi_{\text{in}}^{\alpha})^* \\ &= S^*_{\beta\alpha} \\ &= \lim_{\epsilon\to 0^+} \int \frac{d^4k_i}{(2\pi)^4} \cdots \frac{d^4k_{n+m}}{(2\pi)^4} \, [\tilde{f}_{\alpha_1}(k_1) \cdots \tilde{f}^*_{\beta_m}(k_{n+m})]^* \hat{S}^*(k_\epsilon) , \end{aligned} \tag{5.1}$$

where

$$\hat{S}(k) = (-i)^{n+m} [(k_1{}^2 - M_1{}^2) \cdots (k_{M+m}{}^2 - M_{n+m}{}^2)] \,\tilde{\tau}(k) \tag{5.2}$$

and

$$\tilde{\tau}(k_\epsilon) = \int \cdots \int \frac{dq_1^0}{(2\pi)} \cdots \frac{dq_{n+m}^0}{2\pi} \sum_P \left\langle \hat{A}_{P1}^{(\dagger)}(q_{P2}) \frac{i}{E_1(P) - H + i\epsilon} \hat{A}_{P2}^{(\dagger)}(q_{P2}) \cdots \hat{A}_{P(n+m)}^{(\dagger)}(q_{P(n+m)}) \right\rangle_0 . \tag{5.3}$$

Here

$$\hat{A}_j^{(\dagger)} = A_j^{\dagger} \quad \text{for } j = 1, \ldots, n \tag{5.4a}$$

and

$$\hat{A}_j^{(\dagger)} = A_j \quad \text{for } j = (n+1), \ldots, (n+m) . \tag{5.4b}$$

Using (2.16) one can write (5.1) as

$$\begin{aligned} S_{\alpha\beta}{}^{-1} &= \lim_{\epsilon\to 0^+} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_{n+m}}{(2\pi)^4} \, \tilde{f}^*_{\alpha_1}(k_1) \cdots \tilde{f}_{\beta_m}(k_{n+m}) \hat{S}^*(-k_\epsilon) \\ &= \lim_{\epsilon\to 0^+} \int \frac{d^4k_i}{(2\pi)^4} \cdots \frac{d^4k_{n+m}}{(2\pi)^4} \, \tilde{f}^*_{\alpha_1}(k_1) \cdots \tilde{f}_{\beta_m}(k_{n+m}) S(k_{-\epsilon})(-1) , \end{aligned} \tag{5.5}$$

where the second line follows from (5.2), (5.3), (5.4), and a relation similar to (2.16b):

$$[A_j(q)]^{\dagger} = A_j^{\dagger}(-q) , \tag{5.6}$$

which follows from the definition (2.9).

Comparison of (5.5) to (2.20) shows that a reversal of the signs of all the $i\epsilon$'s changes $S(k_\epsilon)$ to minus the corresponding function for S^{-1}. This result, which applies specifically to regions where only the connected part of S contributes, is called Hermitian analyticity.

Diagrammatically the connected part of S is represented by a circle with a plus sign. We shall represent this same function with a minus $i\epsilon$ prescription by a circle with a minus sign. Then Hermitian analyticity is represented by Fig. 2.

VI. SECOND BASIC DISCONTINUITY EQUATION

The second basic discontinuity equation is represented by Fig. 3. The left-hand side represents the discontinuity

$$\text{Disc}\, S(-;\,;+) \equiv S(-;\pm;+) \equiv S(-;+;+) - S(-;-;+)$$

across the cut $C_j'(P')$ corresponding to the indicated channel $(A'' + B'') \to (A' + B')$. The first argument of $S(\sigma_1;\sigma_2;\sigma_3)$ is the sign of the $i\epsilon$'s associated with the channel energies of the $(A' + B')$ part of the diagram. The third argument σ_3 is the sign of the $i\epsilon$'s of the $(A'' + B'')$ part. The second argument is the sign of the $i\epsilon$ associated with the energy of the channel $A'' + B'' \to A' + B'$.

The derivation of this formula is essentially

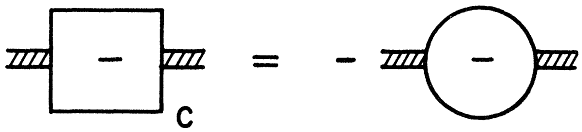

FIG. 2. Hermitian analyticity.

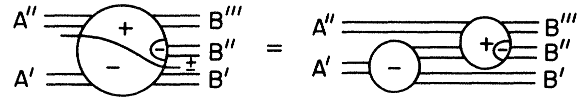

FIG. 4. Another discontinuity equation.

the same as the derivation of Eq. (4.5). The only differences are that now the $i\epsilon$'s of the first vacuum expectation value in (4.2) are preceded by minus signs, and one uses for this factor the Hermitian conjugate of the reduction formulas.

Notice that if $A' = B''$ and $B' = A''$ then the right-hand side of this discontinuity formula is, by virtue of Hermitian analyticity and unitarity, proportional to the inclusive cross section for the reaction $B'' \to A''$ + anything.

VII. CONCLUDING REMARKS

(a) Analytic continuation of the two basic discontinuity equations leads to other discontinuity equations. For example, from the basic discontinuity equation given bv Fig. 3 one obtains by analytic continuation the discontinuity equation shown in Fig. 4.

The minus sign in the little section around the set of lines B'' indicates that all the channel energies that correspond to sets of lines in the set B'' have been continued to the negative sides of their cuts. The right-hand side of Fig. 4 must, however, be interpreted with the aid of a special rule, called the "back-up rule."

The point is that the continuation around the cuts in the variables $\sim B''$ takes the function represented by the upper bubble on the right-hand side of Fig. 4 to the negative sides of certain other cuts as well. To see this consider the equation represented by Fig. 5.

The continuation of the equation represented by Fig. 3 to the negative sides of the cuts associated with B'' will take the function represented by the plus bubble on the right-hand side of Fig. 3 to the negative side both of the cuts associated with B'', and also of the cuts associated with the discontinuity indicated in Fig. 5. One way to see this is to note that intermediate energies in Fig. 4 are real (since all momenta are real) and hence the negative imaginary energy brought in at B'' forces one below the cut indicated on the right-hand side of Fig. 5.

In Ref. 3 it was shown, in some simple cases,

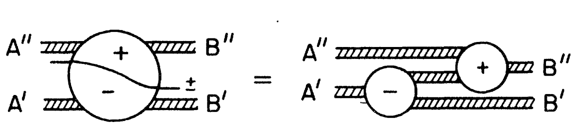

FIG. 3. The second basic discontinuity equation.

how the unitarity equation is built up out of contributions corresponding to individual discontinuities calculated in this way from the two basic discontinuity equations. It is probably possible to turn the calculation around and derive all the discontinuities from unitarity, since the discontinuities around all the singularities lying in the physical region have already been derived in this way.[8] So far it has not seemed worthwhile to reproduce in this way all the equations that come out of field theory, since the two methods appear always to give the same results.

(b) The box diagram singularities on the boundary of the double-spectral region of the four-point function correspond to points where the double discontinuity associated with crossed channels does not vanish. In the case of the mass-shell four-point function these points lie outside the physical region, and hence do not produce any violation of the Steinmann relation. On the other hand, if one replaces some of the single external lines by sets of lines, then it is possible to move the box-diagram singularity into the physical region of the larger process. This might at first appear to produce a conflict with the Steinmann relations.

However, the Steinmann relation asserts only that the double discontinuity associated with crossed channels vanishes if all the other variables are fixed, and away from their cuts. By introducing extra external lines one introduces also extra variables. These variables control, through the back-up rule, the $i\epsilon$'s associated with the cuts that conspire to give the t cut of the s-channel discontinuity formula. When this fact is taken into account, one finds that the box-diagram singularity in the physical region does not produce any conflict with the Steinmann relations. The Steinmann relations hold at *all* real points. This fact is, of course, vital to the main conclusion of this paper, which is that the discontinuity equation represented by Fig. 3 holds at *all* energies, not just near the threshold.

(c) If one invokes a locality condition (e.g., that commutators vanish outside light cones), then the analyticity in the upper and lower half E planes discussed above can be extended to analyticity in corresponding upper and lower light cones in the space of the imaginary parts of the corresponding four-vectors[6,9]:

$$E_j(P) \to k_j(P) = k_{P(j+1)} + \cdots + k_{P(n+m)}, \qquad (7.1a)$$

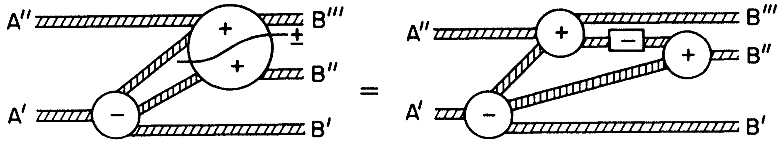

FIG. 5. Result of introducing a discontinuity formula in Fig. 3.

$$(\pm \mathrm{Im}\, E_j(P) > 0,\ \mathrm{Im}\, \vec{k}_j = 0) \rightarrow (\pm \mathrm{Im}\, k_j^0(P) > 0,\ [\mathrm{Im}\, k_j(P)]^2 > 0). \tag{7.1b}$$

The consequences of this analyticity property, together with the spectral conditions and the Steinmann relations, have been studied in detail by Bros, Epstein, and Glaser.[9] They derive by algebraic methods an equation[10] from which a variety of discontinuities across individual channel-energy cuts can be derived.

The cones of analyticity described in (7.1) do not intersect the mass shell. Indeed, one of the outstanding problems in field theory is to show that the domains of analyticity that follow from the axioms do intersect the mass shell for the n-point functions with $n > 4$. This deep problem of field theory lies outside the scope of the present paper.

In S-matrix theory the situation is simplified by the fact that the singularities are confined, by assumption, to the Landau surfaces that are singular by virtue of the unitarity equations. It seems likely that the cut-plane analyticity in $E_i(P)$ space derived from field theory will go over in S-matrix theory to local cut-plane analyticity in $S_i(P)$ space. That is, the scattering function $S_{rc}(k)$ will be analytic (in k) in some neighborhood of each real point k, except on the normal threshold cuts $\mathrm{Im} S_i(P) = 0$, where $S_i(P) \equiv [k_i(P)]^2$. One of the outstanding problems in S-matrix theory, at the fundamental level, is to prove this, together with the relations between the various boundary values that follow formally from field theory.

APPENDIX A: DISCUSSION OF Eq. (2.15)

Equation (2.15) is formally derived as follows: Let (2.8) be written as

$$\tilde{\tau}(k) = \sum_P \tilde{\tau}_P(k), \tag{A1}$$

where

$$\tau_P(x) = \int_{C_P} \left(\prod_j \frac{d^4 k_j}{(2\pi)^4} \right) e^{ik\cdot x}\, \tilde{\tau}_P(k) \tag{A2}$$

and

$$\tau(x) = \sum_P \tau_P(x). \tag{A3}$$

The contour C_P runs just above the real axis in each of the variables $E_j(P)$ $j = 1, \ldots, n+m-1$.

Next write

$$\left(\prod_j K_j\right) \tau(x)$$

$$= \sum_P \left(\prod_j K_j \right) \tau_P(x) \tag{A4}$$

$$= \sum_P \left(\prod_j K_j \right) \int_{C_P} \left(\prod_j \frac{d^4 k_j}{(2\pi)^4} \right) e^{ik\cdot x}\, \tilde{\tau}_P(k) \tag{A5}$$

$$= \sum_P \int_{C_P} \left(\prod_j \frac{d^4 k_j}{(2\pi)^4} \right) e^{ik\cdot x} \left[\prod_j (k_j^2 - M_j^2) \right] \tilde{\tau}_P(k) \tag{A6}$$

$$= \sum_P \lim_{\epsilon \to 0} \int_{C_P} \left(\prod_j \frac{d^4 k_j}{(2\pi)^4} \right) e^{ik\cdot x} \left\{ \tilde{\tau}_P(k) \prod_j (k_j^2 - M_j^2) \right\}_\epsilon \tag{A7}$$

$$= \sum_P \lim_{\epsilon \to 0} \int \left(\prod_j \frac{d^4 k_j}{(2\pi)^4} \right) e^{ik\cdot x} \left\{ \tilde{\tau}_P(k) \prod_j (k_j^2 - M_j^2) \right\}_\epsilon \tag{A8}$$

$$= \lim_{\epsilon \to 0} \int \left(\prod_j \frac{d^4 k_j}{(2\pi)^4} \right) e^{ik\cdot x} \left\{ \tilde{\tau}(k) \prod_j (k_j^2 - M_j^2) \right\}_\epsilon . \tag{A9}$$

The result (A8) is substituted in (2.3) and the limit $\epsilon \to 0$ is then moved to outside the integral.

The validity of these formal manipulations depends on the high-energy behavior of the off-mass-shell matrix elements. We do not wish to delve into such matters here. But the work of Hepp[11] indicates that the final result is all right, provided the time-ordered product is well defined in the first place.

APPENDIX B: DISCONNECTED CONTRIBUTIONS

Notice that the sum of the terms corresponding to the two orders of interaction shown in Fig. 6 leads to a factorized form of the propagator. That is,

$$\frac{1}{E_1 - H_1} \frac{1}{E_1 + E_2 - H_1 - H_2} + \frac{1}{E_2 - H_2} \frac{1}{E_1 + E_2 - H_1 - H_2} = \frac{1}{E_1 - H_1} \frac{1}{E_2 - H_2}. \tag{B1}$$

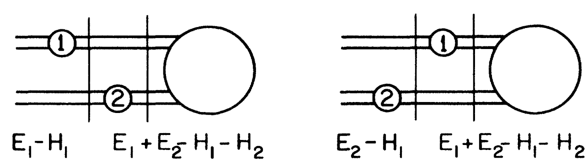

FIG. 6. Two orders of two disconnected contributions.

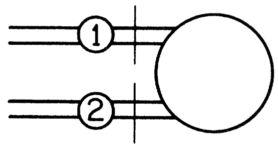

FIG. 7. An alternative representation.

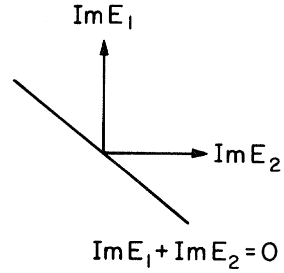

FIG. 8. The surface $\mathrm{Im}E_1+\mathrm{Im}E_2=0$.

The two parts can therefore be treated independently, as indicated in Fig. 7.

The point is that the discontinuity does not really occur in the variable E_1+E_2. It occurs in E_1 and E_2 (see Fig. 8). Thus, if one stays away from the two surfaces $\mathrm{Im}\,E_1=0$ and $\mathrm{Im}\,E_2=0$, there is no singularity. For this reason the factors $S_c(k')$ and $S_c(k'')$ in (4.5) are connected parts. The discontinuity being calculated is, by definition, to be taken at points lying away from the various other cuts.

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

[1]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[2]The interpretation of Mueller's work in terms of the basic discontinuity equations has been proposed by Professor Chung-I Tan. This idea was communicated to the author by Professor Geoffrey Chew, who informed the author of the current interest in the question of whether that equation can be derived from field theory. The basic discontinuity equation in the case $a+b+c\to a+b+c$ has been used in the following recent paper: C. E. DeTar, C. E. Jones, F. E. Low, J. H. Weis, J. E. Young, and Chung-I Tan, Phys. Rev. Letters 26, 675 (1971).

[3]H. P. Stapp, "Lectures on Analytic S-Matrix Theory" [Matscience Report No. 26, Madras, 1964, Chaps. XII and XIII (unpublished)]; "Lectures on the Analytic Structure of Many-Particle Scattering Amplitudes" [ICTP Report No. IC/65/17, Trieste, 1965 (unpublished)]. A somewhat similar, but less ambitious, work is that of D. I. Olive, Nuovo Cimento 37, 1422 (1965).

[4]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 205 (1955).

[5]S. Schweber, *An Introduction to Relativistic Quantum Theory* (Harper and Row, New York, 1962), p. 763.

[6]H. Araki, J. Math. Phys. 2, 163 (1960), Eq. (2.21).

[7]W. Zimmermann, Nuovo Cimento 13, 503 (1959), and Ref. 6, Appendix B.

[8]J. Coster and H. P. Stapp, J. Math Phys. 11, 1441 (1970); 11, 2745 (1970). The equation analogous to (4.5) derived in this reference has, in place of the connected parts $S_c(k)$, $S_c(k')$, and $S_c(k'')$, the full S matrices $S(k)$, $S(k')$, and $S(k'')$. This difference is due to the fact that (4.5) gives the discontinuity around the individual cut $C_j(P')$ alone, whereas the discontinuity given in the reference contains discontinuities around certain subenergy cuts as well. The notation of the present paper differs from that of this reference by the occurrence of the minus sign in front of the bubble in Fig. 2.

[9]J. Bros, in *High-Energy Physics and Elementary Particles* (International Atomic Energy Agency, Vienna, 1965); J. Bros, H. Epstein, and V. Glaser, Nuovo Cimento 31, 1265 (1964); Commun. Math. Phys. 1, 240 (1965).

[10]J. Bros, Ref. 9, Eq. (16).

[11]K. Hepp, Commun. Math. Phys. 1, 108 (1965).