New Dual N-Point Functions

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As an example of the group-theoretical approach to the construction of new dual amplitudes, we discuss a model pion N-point function in which the π trajectory lies $\frac{1}{2}$ unit below the leading (ρ) trajectory. The model is completely factorizable, contains a natural G parity, and obeys the Adler condition approximately. The degeneracy of the spectrum and the ghost problem are essentially the same as in the conventional Veneziano N-point function. In a separate section, we discuss a dual interaction of an $SU(3)$ nonet of scalar mesons with mass splitting.

I. INTRODUCTION

Recently, by abstracting the symmetry of the Veneziano N-point function, a general group-theoretical prescription for factorizable dual amplitudes has been given,[1] opening the way for the construction of a wide variety of new dual amplitudes, some of which will hopefully be closer to nature than the conventional multi-Veneziano expression. As an example of this technique, we propose in Sec. II a model pion N-point function incorporating the ρ trajectory, G parity, and the Adler condition approximately. We note here that pion N-point functions based on a generalization[2] of the Lovelace amplitude[3] or on a relativistic quark model[4] are not dual in the group-theoretic sense since the cyclic symmetry is put in by hand. Since it contains all the diseases of the conventional amplitude, the major interest in the present model is the example it provides of one dual vertex carrying two trajectories. In addition to the πA_1 trajectory containing the external particles, there is a leading trajectory one half unit higher containing the ρ and f. States on the parent πA_1 trajectory have negative G parity and are decoupled from even numbers of pions. In Sec. III, we discuss a model for the N-point function of a nonet of scalar mesons with $SU(3)$ mass splittings. The model contains a leading trajectory a variable distance above the scalar-meson trajectory.

For completeness, we summarize here the basic rules for a factorizable, dual amplitude given in Ref. 1. In terms of suitable creation and destruction operators, one constructs a representation of the $SU(1,1)$ generators L_0, $L_\pm$, and a vertex operator $V(z)$, z on the unit circle, transforming under the $SU(1,1)$ algebra as some spin-J_S representation. That is,

$$[L_0, L_\pm] = \pm L_\pm\,, \tag{1.1}$$

$$[L_+, L_-] = -L_0\,, \tag{1.2}$$

$$[L_0, V(z)] = -z\frac{d}{dz}V(z)\,, \tag{1.3}$$

$$[L_\pm, V(z)] = -\frac{z^{\pm1}}{\sqrt{2}}\left(z\frac{d}{dz} \mp J_S\right)V(z)\,. \tag{1.4}$$

$V(z)$ represents the vertex for the absorption of a particle and may depend on all of the quantum numbers of that particle and, in particular, on its four-momentum k_μ. The significance of J_S as the $SU(1,1)$ spin of the particle is clear since, if we define the Casimir operator

$$L^2 = L_0{}^2 - L_+L_- - L_-L_+\,, \tag{1.5}$$

then using (1.3) and (1.4)

$$L^2V(z)|0\rangle = J_S(J_S+1)V(z)|0\rangle\,. \tag{1.6}$$

If one has N such vertices for the absorption of N (in general, different) particles, each of which transforms with the same $SU(1,1)$ spin J_S, a factorizable, dual N-point function is

$$A_N = \frac{1}{C}\oint \left\langle 0\left|\prod_{i=1}^{N}\left\{\frac{dz_i}{z_i}|z_i - z_{i+1}|^{-1-J_S} \times \theta(\arg z_i - \arg z_{i+1})V_i(k_i z_i)\right\}\right|0\right\rangle, \tag{1.7}$$

where

$$C = \oint \frac{dz dz' dz''\,\theta(\arg z - \arg z')\,\theta(\arg z' - \arg z'')}{zz'z''|z-z'||z'-z''||z''-z|}\,. \tag{1.8}$$

The contours are all taken around the unit circle and the z's are defined cyclically, $z_{N+1} \equiv z_1$.

The conventional multi-Veneziano amplitude employs an infinite set of boson creation and destruction operators[5] satisfying

$$[a_\mu^n, a_\nu^{m\dagger}] = \delta_{mn}g_{\mu\nu}\,, \tag{1.9}$$

with metric $g_{\mu\nu} = (1,1,1,-1)$. Under the $SU(1,1)$ algebra generated by

$$L_0=\sum_{m=0}^{\infty}(m+\tfrac{1}{2}\epsilon)a^{m\dagger}a^m\,,$$
$$L_+=\sum_{m=0}^{\infty}\left[\frac{(m+\epsilon)(m+1)}{\sqrt{2}}\right]^{1/2}a^{m+1\dagger}a^m\,, \qquad (1.10)$$
$$L_-=L_+^\dagger\,,$$

the operator

$$Q_\mu(z)=\sum_{m=0}^{\infty}\left[\frac{(m-1+\epsilon)!}{m!}\right]^{1/2}(a_\mu^m z^{m+\epsilon/2}+a_\mu^{m\dagger}z^{-m-\epsilon/2}) \qquad (1.11)$$

transforms effectively as an $SU(1,1)$ scalar in the limit $\epsilon\to 0$. In that limit, the operator

$$V(k,z)=:e^{ik\cdot Q(z)}: \qquad (1.12)$$

transforms with[6,7] $J_s=-\frac{1}{2}k^2=-\alpha_0$.

Taking Eq. (1.12) to represent the vertex for the absorption of a scalar meson of momentum k_μ and substituting into Eq. (1.5) with $J_s=-\alpha_0$, one obtains the usual N-point function in Koba-Nielsen form. It is interesting to note that the two points of special simplicity in the model, $\alpha_0=0$ and $\alpha_0=1$, are the null points of the Casimir operator, Eq. (1.6).

Because of the projective invariance, one can, in general, reduce Eq. (1.5) to the explicitly factorized form by the Fubini-Veneziano technique.[6] Define the ground-state bra and ket:

$$|k_N\rangle=\lim_{z_N\to\infty}z_N{}^{-J_s}V(k_N,z_N)|0\rangle\,, \qquad (1.13)$$

$$\langle k_1|=\lim_{z_1\to 0}z_1{}^{J_s}\langle 0|V(k_1,z_1)\,. \qquad (1.14)$$

Then

$$A_N=\langle k_1|V(k_2,1)\prod_{i=1}^{N-3}[\Delta_i(L_0)V(k_{i+2},1)]|k_N\rangle\,, \qquad (1.15)$$

where the propagator is given by

$$\Delta_i(L_0)=\int_0^1 dx_i x_i{}^{-1+L_0+J_s}(1-x)^{-1-J_s}\,. \qquad (1.16)$$

In the multi-Veneziano amplitude, the external particle lies on the leading trajectory. We now proceed to discuss a generalization in which the external particle lies $\frac{1}{2}$ unit below the leading trajectory. Such a configuration is close to the physical situation in which the π trajectory is $\frac{1}{2}$ unit below the ρ trajectory. In Sec. II, we will allow the incorporation of the Paton-Chan[8] isospin factors to be understood although we will not discuss them explicitly.

II. PION N-POINT FUNCTION

We would now like to consider a model for the pion N-point function which includes the πA_1, ρ, and f trajectories. To this end, we introduce two infinite sets of spinless Fermi operators satisfying the anticommutation relations

$$\{b^m,b^{n\dagger}\}=\{d^m,d^{n\dagger}\}=\delta_{mn}\,,$$
$$\{b^m,b^n\}=\{b^m,d^n\}=\{d^m,d^n\}=\{b^m,d^{n\dagger}\}=0\,, \qquad (2.1)$$

and commuting with the a's of Eq. (1.9). We construct the following $SU(1,1)$ generators:

$$L_0=\sum_{m=0}^{\infty}(m+\tfrac{1}{2}\epsilon)a^{m\dagger}a^m+\sum_{m=0}^{\infty}(m+\tfrac{1}{4})(b^{m\dagger}b^m+d^{m\dagger}d^m)\,, \qquad (2.2)$$

$$L_+=\sum_{m=0}^{\infty}\left[\frac{(m+\epsilon)(m+1)}{2}\right]^{1/2}a^{m+1\dagger}a^m+\sum_{m=0}^{\infty}\left[\frac{(m+\frac{1}{2})(m+1)}{2}\right]^{1/2}(b^{m+1\dagger}b^m+d^{m+1\dagger}d^m)\,, \qquad (2.3)$$

$$L_-=L_+^\dagger\,, \qquad (2.4)$$

and the operator

$$H(z)=\sum_{m=0}^{\infty}\left[\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\right]^{1/2}(b^m z^{m+1/4}+d^{m\dagger}z^{-m-1/4})\,. \qquad (2.5)$$

Under the $SU(1,1)$ algebra defined by Eqs. (2.2)–(2.4), $H(z)$ transforms as $J_s=-\frac{1}{4}$. We could as well have taken the b's and d's to satisfy commutation relations instead of the anticommutation relations of Eq. (2.1), but by use of Fermi operators, we will be able to keep the degeneracy from increasing appreciably over that of the conventional model. Similarly, we could have taken spinor b's and d's, but we avoid doing so, mostly for simplicity but partly to avoid a negligible increase in the number of ghosts in the theory.

We now define the vertex for pion absorption as

$$V(k_j,z_j)=:e^{ik_jQ(z_j)}H^\dagger(z_j)H(z_j):\,, \qquad (2.6)$$

where $Q(z)$ is still given by Eq. (1.11) in terms of the a operators. Under the $SU(1,1)$ algebra, $V(k,z)$ transforms as a $J_s=-\frac{1}{2}k^2-\frac{1}{2}=-\alpha_0-\frac{1}{2}$ representation. As before, we obtain a dual factorizable amplitude by substituting Eq. (2.6) into Eq. (1.7) with $J_s=-\alpha_0-\frac{1}{2}$. The four-point function can be immediately written down using the facts that (ignoring constant factors)

$$\langle 0|\prod_{i=1}^{4}:e^{ik_iQ(z_i)}:|0\rangle=\prod_{i=1}^{3}\prod_{j=i+1}^{4}|z_i-z_j|^{k_ik_j} \qquad (2.7)$$

and

$$\{H^+(z_i),H^-(z_j)\}=|z_i-z_j|^{-1/2}\,, \qquad (2.8)$$

where the + and − refer to positive- and negative-frequency parts of H. Then putting

$$x=\frac{|z_1-z_2|\,|z_4-z_3|}{|z_4-z_2|\,|z_1-z_3|}\,,$$
$$1-x=\frac{|z_4-z_1|\,|z_2-z_3|}{|z_4-z_2|\,|z_1-z_3|}\,, \qquad (2.9)$$

we find

$$A_4=\int_0^1 dx\,x^{-1-\alpha(s)}(1-x)^{-1-\alpha(t)}\left[2-2\sqrt{x}-2(1-x)^{1/2}+\left(\frac{1-x}{x}\right)^{1/2}+\left(\frac{x}{1-x}\right)^{1/2}+[x(1-x)]^{1/2}\right]$$

$$=2\left[\frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t)}-\frac{\Gamma(-\alpha_s+\frac{1}{2})\Gamma(-\alpha_t)}{\Gamma(-\alpha_s-\alpha_t+\frac{1}{2})}-\frac{\Gamma(-\alpha_s)\Gamma(-\alpha_t+\frac{1}{2})}{\Gamma(-\alpha_s-\alpha_t+\frac{1}{2})}\right]$$

$$+\frac{\Gamma(-\alpha_s-\frac{1}{2})\Gamma(-\alpha_t+\frac{1}{2})}{\Gamma(-\alpha_s-\alpha_t)}+\frac{\Gamma(-\alpha_s+\frac{1}{2})\Gamma(-\alpha_t-\frac{1}{2})}{\Gamma(-\alpha_s-\alpha_t)}+\frac{\Gamma(-\alpha_s+\frac{1}{2})\Gamma(-\alpha_t+\frac{1}{2})}{\Gamma(-\alpha_s-\alpha_t+1)}. \tag{2.10}$$

The first point to be noted is that there is no pole at $\alpha(s)=0$, corresponding to the absence of a pion pole in elastic $\pi\pi$ scattering. Furthermore at $\alpha(s)=n$, for $n\geq 1$, the residue contains only spins $0, 1, \ldots, n-1$. Hence, the parent A_1 trajectory decouples from the elastic scattering as it should. There are, however, contributing poles on the daughter trajectories (e.g., a 0^+ particle at the A_1 mass, etc.).

The asymptotic behavior of A_4 is

$$\lim_{s\to\infty} A_4=[-\alpha(s)]^{\alpha(t)+1/2}, \tag{2.11}$$

corresponding to a leading trajectory $\frac{1}{2}$ unit above the π trajectory. Thus, if we put $\alpha_\rho(t)=\alpha(t)+\frac{1}{2}$, the asymptotic behavior for $\pi\pi$ scattering is $s^{\alpha_\rho(t)}$ as it should be. It is to be noted that, unlike the recent model of Bardakci and Halpern,[9] the Pomeranchuk trajectory has no place in the Born term of the present model. However, as in the conventional amplitude, diffraction might arise from higher-order nonplanar loop graphs.

The 4-point function of Eq. (2.10) also contains poles at the ρ-meson mass and its recurrences. That is, at $\alpha(s)=n-\frac{1}{2}$[or $\alpha_\rho(s)=n$], there are poles with spins $0, 1, \ldots, n$. The Paton-Chan factors identify these poles as the positive-G-parity ρ and f trajectories. The chiral-symmetry prediction $m_{A_1}^2=2m_\rho^2$ is also built into the model. A presently unavoidable difficulty is the existence of a spin-zero pole at $\alpha_\rho(s)=0$ which becomes a tachyon in the physical case of $\alpha_\rho(0)>0$.

To examine the pole structure of the general N-point function, it is convenient to use the propagator Eq. (1.16) with $J_s=-\alpha_0-\frac{1}{2}$,

$$\Delta(L_0)=\frac{\Gamma(L_0-\alpha_0-\frac{1}{2})\Gamma(\alpha_0+\frac{1}{2})}{\Gamma(L_0)}. \tag{2.12}$$

If we write Eq. (2.2) as

$$L_0=\tfrac{1}{2}p_0^2+R, \tag{2.13}$$

with

$$R=\sum_{m=1}^{\infty} m a^{m\dagger}a^m+\sum_{m=0}^{\infty}(m+\tfrac{1}{4})(b^{m\dagger}b^m+d^{m\dagger}d^m), \tag{2.14}$$

the propagator can be written

$$\Delta=\frac{\Gamma(-\alpha(-p_0^2)+R-\frac{1}{2})\Gamma(\alpha_0+\frac{1}{2})}{\Gamma(\frac{1}{2}p_0^2+R)}. \tag{2.15}$$

The physical eigenvalues of R are integral and half integral. To see this, it is sufficient to prove that on the physical states

$$\tfrac{1}{4}\sum_{m=0}^{\infty}(b^{m\dagger}b^m+d^{m\dagger}d^m)=\tfrac{1}{2}\sum_{m=0}^{\infty}b^{m\dagger}b^m. \tag{2.16}$$

To prove Eq. (2.16), it is sufficient to note that the operator

$$\delta=\sum_{m=0}^{\infty}(b^{m\dagger}b^m-d^{m\dagger}d^m) \tag{2.17}$$

commutes with the vertex Eq. (2.6) and with the propagator Eq. (2.12) and hence annihilates the physical states, i.e.,

$$\delta|\Psi^n_{\text{phys}}\rangle=0, \tag{2.18}$$

where

$$|\Psi^n_{\text{phys}}\rangle=\lim_{z\to\infty} z^{\alpha_0+1/2}V(k_1,1)\Delta(L_0)\times V(k_2,1)\Delta(L_0)\cdots V(k_{n-1},1)V(k_n,z)|0\rangle. \tag{2.19}$$

Thus the poles of Eq. (2.15) occur at $\alpha(-p_0^2)=n$ for $R=\frac{1}{2}, \frac{3}{2}, \ldots, n+\frac{1}{2}$ and at $\alpha(-p_0^2)=n-\frac{1}{2}$ for $R=0, 1, \ldots, n$. The first set of poles corresponds to the πA_1 trajectory and the second to the ρ, f trajectory a half unit above. It is a simple matter to construct the intermediate states in terms of the occupation-number states of the a's, b's, and d's. The b's and d's introduce no new ghosts into the theory, since they are scalar operators; since their number operators have eigenvalues zero and one only, they do not contribute significantly to the degeneracy already in the model due to the a's and $a^\dagger$'s. Since the new $L_0-\sqrt{2}L_-$ annihilates the physical states, Eq. (2.19), the usual Ward identities are still in force.

The conservation of G parity is also easily demonstrable in the model. We define the G-parity operator

$$G=\exp[i\pi\sum_{m=0}^{\infty}(b^{m\dagger}d^m+d^{m\dagger}b^m)], \tag{2.20}$$

with the property that

$$GVG^\dagger = -V \qquad (2.21)$$

and

$$[G, L_0] = 0\,, \qquad (2.22)$$

so that operating on Eq. (2.19),

$$G|\psi^n_{\text{phys}}\rangle = (-1)^n |\psi^n_{\text{phys}}\rangle\,. \qquad (2.23)$$

Inserting the identity operator $1 = G^\dagger G$ at any point in the N-point amplitude of Eq. (1.7) [or (1.15)] yields

$$A_N = (-1)^N A_N\,, \qquad (2.24)$$

so that amplitudes for the scattering of an odd number of pions vanish identically.

We would now like to investigate the behavior of A_4 at the Adler point $\alpha_s = \alpha_t = \alpha_u = 0$. Putting $\alpha_s = \alpha_t = \alpha$, Eq. (2.10) becomes

$$A_4 = 2\Gamma(-\alpha)\left[\frac{\Gamma(-\alpha)}{\Gamma(-2\alpha)} - 2\frac{\Gamma(-\alpha+\frac{1}{2})}{\Gamma(-2\alpha+\frac{1}{2})}\right] + \Gamma(-\alpha+\tfrac{1}{2})\left[2\frac{\Gamma(-\alpha-\frac{1}{2})}{\Gamma(-2\alpha)} + \frac{\Gamma(-\alpha+\frac{1}{2})}{\Gamma(-2\alpha+1)}\right]. \qquad (2.25)$$

Using the logarithmic derivative $\Psi(x) = d\ln\Gamma(x)/dx$ and expanding to first order in α, we have

$$\frac{\Gamma(\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-2\alpha)} \cong 1 + \alpha\Psi(\tfrac{1}{2})\,, \qquad (2.26)$$

$$\frac{\Gamma(-\alpha)}{\Gamma(-2\alpha)} = \frac{2^{2\alpha+1}\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\alpha)} \cong 2[1 + 2\alpha\ln 2 + \alpha\Psi(\tfrac{1}{2})]\,, \qquad (2.27)$$

so that

$$\lim_{\alpha\to 0} A_4(\alpha_s = \alpha_t = \alpha) = -8\ln 2 + \pi\,. \qquad (2.28)$$

Thus there is a partial cancellation between the ρ, f and πA_1 trajectories at the Adler point. The value at threshold ($\alpha_s \cong 0.05$, $\alpha_t \cong 0$) is approximately twenty times greater than Eq. (2.28). We have not been able to demonstrate a similar suppression in the N-point amplitude. In addition, the value of A_4 at the Adler point is uncomfortably sensitive to the way the limit is approached. For example, if instead of the symmetric approach above, one first takes $\alpha_s \to 0$ and then $\alpha_t \to 0$, the amplitude diverges.

Because of this ambiguity and the ghost problem discussed above, and because of the unsatisfactory features of the Paton-Chan scheme for isospin, the amplitude proposed here is not completely acceptable. Nevertheless, it has several obvious features in common with empirical observations that the conventional multi-Veneziano model lacks.

III. INCLUSION OF $SU(3)$

In this section, we discuss a possible dual interaction of a nonet of scalar mesons with broken-mass degeneracy. Following the method of the Appendix of Ref. 7, for any $\eta > 0$ we can construct the $SU(1,1)$ algebra

$$L_0 = \sum_{m=0}^{\infty} (m + \tfrac{1}{2}\eta) b^{m\dagger} b^m\,, \qquad (3.1)$$

$$L_+ = \sum_{m=0}^{\infty} \frac{1}{\sqrt{2}} [(m+\eta)(m+1)]^{1/2} b^{m+1\dagger} b^m\,, \qquad (3.2)$$

$$L_- = L_+^\dagger\,, \qquad (3.3)$$

and under this algebra the field

$$B(z) = \sum_{m=0}^{\infty} \left[\frac{\Gamma(m+\eta)}{\Gamma(m+1)}\right]^{1/2} b^m z^{m+\eta/2} \qquad (3.4)$$

transforms covariantly with $J_s = -\eta/2$.

Following Bardakci and Halpern,[9] we would like to make b^m an $SU(3)$ triplet and write

$$L_0 = \sum_{m=0}^{\infty} (m + \tfrac{1}{2}\eta) b^{m\dagger} (\alpha_1 \lambda^0 + \alpha_2 \lambda^8) b^m\,, \qquad (3.5)$$

$$L_+ = \sum_{m=0}^{\infty} \frac{1}{\sqrt{2}} [(m+\eta)(m+1)]^{1/2} b^{m+1\dagger} (\beta_1 \lambda^0 + \beta_2 \lambda^8) b^m\,, \qquad (3.6)$$

$$L_- = L_+^\dagger\,. \qquad (3.7)$$

However, we are not free to have arbitrary symmetry breaking.

Requiring the operators of Eq. (3.4) to satisfy the $SU(1,1)$ algebra gives us the nonlinear constraints

$$\alpha_1\beta_1 + \alpha_2\beta_2 = (\tfrac{3}{2})^{1/2}\beta_1\,, \qquad (3.8a)$$

$$\alpha_2\beta_1 + \alpha_1\beta_2 - (\tfrac{1}{2})^{1/2}\alpha_2\beta_2 = (\tfrac{3}{2})^{1/2}\beta_2\,, \qquad (3.8b)$$

$$\beta_1^2 + \beta_2^2 = (\tfrac{3}{2})^{1/2}\alpha_1\,, \qquad (3.8c)$$

$$2\beta_1\beta_2 - \frac{\beta_2^2}{\sqrt{2}} = (\tfrac{3}{2})^{1/2}\alpha_2\,. \qquad (3.8d)$$

Besides the trival solution $\alpha_1 = \beta_1 = (\frac{3}{2})^{1/2}$, $\alpha_2 = \beta_2 = 0$, there are two solutions with nonvanishing symmetry breaking:

$$\alpha_1 = (\tfrac{2}{3})^{1/2} = \beta_1\,, \quad \alpha_2 = (\tfrac{1}{3})^{1/2} = \beta_2\,, \qquad (3.9a)$$

$$\alpha_1 = (\tfrac{1}{6})^{1/2} = \beta_1\,, \quad \alpha_2 = -(\tfrac{1}{3})^{1/2} = \beta_2\,, \qquad (3.9b)$$

corresponding to the two matrices

$$A_1 = (\tfrac{2}{3})^{1/2}\lambda^0 + (\tfrac{1}{3})^{1/2}\lambda^8\,, \qquad (3.10a)$$

$$A_2 = (\tfrac{1}{6})^{1/2}\lambda^0 - (\tfrac{1}{3})^{1/2}\lambda^8\,, \qquad (3.10b)$$

where A_1 and A_2 can be seen to be the projection operators onto the space of nonstrange and strange quarks, respectively. We would now like to build a dual amplitude based on the algebra

$$L_0=\sum_{m=0}^{\infty}(m+\tfrac{1}{2}\epsilon)a^{m\dagger}a^m+\sum_{m=0}^{\infty}(m+\tfrac{1}{2}\eta_1)(b^{m\dagger}A_1b^m+d^{m\dagger}A_1d^m)+\sum_{m=0}^{\infty}(m+\tfrac{1}{2}\eta_2)(b^{m\dagger}A_2b^m+d^{m\dagger}A_2d^m)\,, \tag{3.11a}$$

$$L_+=\frac{1}{\sqrt{2}}\sum_{m=0}^{\infty}[(m+\epsilon)(m+1)]^{1/2}a^{m+1\dagger}a^m+\frac{1}{\sqrt{2}}\sum_{m=0}^{\infty}[(m+\eta_1)(m+1)]^{1/2}(b^{m+1\dagger}A_1b^m+d^{m+1\dagger}A_1d^m)$$
$$+\frac{1}{\sqrt{2}}\sum_{m=0}^{\infty}[(m+\eta_2)(m+1)]^{1/2}(b^{m+1\dagger}A_2b^m+d^{m+1\dagger}A_2d^m)\,, \tag{3.11b}$$

$$L_-=L_+^{\dagger}\,. \tag{3.11c}$$

The parameter ϵ is to be taken to zero at the end of all operations, but η_1 and η_2 are constants to be determined later by the meson masses. The model of Ref. 9 is obtained in the limit $\eta_1\to 1$, $A_1\to 1$, $A_2\to 0$. However, terms in A_2 are important for a consistent dual theory with symmetry breaking. We now consider the spin-zero $SU(3)$ triplet

$$B_r(z)=\sum_{s=1}^{3}\sum_{m=0}^{\infty}\frac{1}{\Gamma^{1/2}(m+1)}[\Gamma^{1/2}(m+\eta_1)A_{1rs}b_s^m z^{m+\eta_1/2}+\Gamma^{1/2}(m+\eta_2)A_{2rs}b_s^m z^{m+\eta_2/2}]\,, \tag{3.12}$$

with z on the unit circle. Similarly,

$$D_r(z)=\sum_{s=1}^{3}\sum_{m=0}^{\infty}\frac{1}{\Gamma^{1/2}(m+1)}[\Gamma^{1/2}(m+\eta_1)A_{1rs}d_s^m z^{m+\eta_1/2}+\Gamma^{1/2}(m+\eta_2)A_{2rs}d_s^m z^{m+\eta_2/2}]\,. \tag{3.13}$$

We now define the quark operator

$$H_r(z)=B_r(z)+D_r^{\dagger}(z)\,. \tag{3.14}$$

Commuting H with the $SU(1,1)$ generators of Eqs. (3.11) and comparing with Eqs. (1.3) and (1.4), we see that H_r transforms with $J_s=-\frac{1}{2}\eta_1$ if $r=1$, or 2, and with $J_s=-\frac{1}{2}\eta_2$ if $r=3$.

The vertices for meson absorption are now written

$$\Phi^{\alpha}(k,z)=:e^{ik\cdot Q(z)}H^{\dagger}(z)\lambda^{\alpha}H(z):\,. \tag{3.15}$$

Commuting with the generators, we find

$$[L_0,\Phi^{\alpha}(k,z)]=-z\frac{d}{dz}\Phi^{\alpha}(k,z)\,, \tag{3.16}$$

$$[L_{\pm},\Phi^{\alpha}(k,z)]=-\frac{z^{\pm1}}{\sqrt{2}}\left(z\frac{d}{dz}\Phi^{\alpha}\pm\tfrac{1}{2}k^2\Phi^{\alpha}\pm:e^{ik\cdot Q}H^{\dagger}\{\tfrac{1}{2}(\eta_1A_1+\eta_2A_2),\lambda^{\alpha}\}H:\right). \tag{3.17}$$

$SU(1,1)$ covariance requires

$$\{\tfrac{1}{2}(\eta_1A_1+\eta_2A_2),\lambda^{\alpha}\}=h^{\alpha}\lambda^{\alpha}\quad(\text{no sum on }\alpha)\,. \tag{3.18}$$

Using Eqs. (3.10a), (3.10b), and the familiar anticommutation relations of the λ's, we find the following eigenstates of Eq. (3.18):

$$\Phi^{\pi^{\pm},\pi^0}:\ \frac{\lambda^1\pm i\lambda^2}{\sqrt{2}}\,,\ \lambda^3\,, \tag{3.19a}$$

$$\Phi^{K^{\pm},K^0,\bar{K}^0}:\ \frac{\lambda^4\pm i\lambda^5}{\sqrt{2}}\,,\ \frac{\lambda^6\pm i\lambda^7}{\sqrt{2}}\,, \tag{3.19b}$$

$$\Phi^{\eta}:\ (\tfrac{2}{3})^{1/2}\lambda^0+(\tfrac{1}{3})^{1/2}\lambda^8\,, \tag{3.19c}$$

$$\Phi^{\eta'}:\ -(\tfrac{1}{3})^{1/2}\lambda^0+(\tfrac{2}{3})^{1/2}\lambda^8\,, \tag{3.19d}$$

with eigenvalues

$$h_{\alpha}=(\tfrac{2}{3}\eta_1+\tfrac{1}{3}\eta_2)+(\tfrac{1}{3})^{1/2}(\eta_1-\eta_2)d_{8\alpha\alpha}\,. \tag{3.20}$$

Substituting Eq. (3.18) into (3.17), we see that the vertex Φ^{α} transforms under $SU(1,1)$ with

$$J_s(\alpha)=-\tfrac{1}{2}k_{\alpha}^{\,2}-h_{\alpha}\,. \tag{3.21}$$

According to criterion IV of Ref. 1, in order for particles with different quantum numbers to interact dually, they must have the same J_s. Thus if the mass splittings of the Φ^α are octet-dominated, i.e.,

$$k_\alpha^2 = -m_0^2 - \sqrt{3}\,\delta m^2 d_{8\alpha\alpha}, \tag{3.22}$$

we must have

$$\eta_2 - \eta_1 = -\tfrac{3}{2}\delta m^2 = m_K^2 - m_\pi^2 . \tag{3.23}$$

Then J_s becomes independent of α, and depends only on the central mass m_0 and the parameters η_1 and η_2:

$$J_s = \tfrac{1}{2}m_0^2 - (\tfrac{2}{3}\eta_1 + \tfrac{1}{3}\eta_2) = \tfrac{1}{2}m_\pi^2 - \eta_1 . \tag{3.24}$$

The dual amplitude for the scattering of n scalar mesons with $SU(3)$ quantum numbers $\alpha_1\alpha_2\cdots\alpha_n$ is, according to (1.7),

$$A_{\alpha_1\alpha_2\cdots\alpha_n} = \frac{1}{C}\oint\left(\prod_{i=1}^{n}\frac{dz_i}{z_i}\,\theta(\arg z_i - \arg z_{i+1})\,|z_i - z_{i+1}|^{-1-J_s}\right)\langle 0|\Phi^{\alpha_1}\Phi^{\alpha_2}\cdots\Phi^{\alpha_n}|0\rangle . \tag{3.25}$$

In the case of the four-point function, (3.25) is easy to evaluate using the anticommutation relation

$$\{B_r(z_i), B_s^\dagger(z_j)\} = e^{i\pi\eta_1/2}\Gamma(\eta_1)A_{1rs}|z_i - z_j|^{-\eta_1} + e^{i\pi\eta_2/2}\Gamma(\eta_2)A_{2rs}|z_i - z_j|^{-\eta_2}, \tag{3.26}$$

although some may prefer to use the factorized form of Eqs. (1.15) and (1.16). With either method one finds, for example, that in the case of $\pi^0\pi^0$ elastic scattering the st term in the amplitude takes the form

$$\begin{aligned} A_{3333} = {} & \frac{\Gamma(-\alpha_\pi(s))\Gamma(-\alpha_\pi(t))}{\Gamma(-\alpha_\pi(s)-\alpha_\pi(t))} - \frac{\Gamma(-\alpha_\pi(s))\Gamma(-\alpha_\pi(t)+\eta_1)}{\Gamma(-\alpha_\pi(s)-\alpha_\pi(t)+\eta_1)} - \frac{\Gamma(-\alpha_\pi(t))\,\Gamma(-\alpha_\pi(s)+\eta_1)}{\Gamma(-\alpha_\pi(s)-\alpha_\pi(t)+\eta_1)} \\ & + \frac{\Gamma(-\alpha_\pi(s)+\eta_1)\,\Gamma(-\alpha_\pi(t)+\eta_1)}{\Gamma(-\alpha_\pi(s)-\alpha_\pi(t)+2\eta_1)} + \frac{\Gamma(-\alpha_\pi(s)-\eta_1)\,\Gamma(-\alpha_\pi(t)+\eta_1)}{\Gamma(-\alpha_\pi(s)-\alpha_\pi(t))} + \frac{\Gamma(-\alpha_\pi(s)+\eta_1)\,\Gamma(-\alpha_\pi(t)-\eta_1)}{\Gamma(-\alpha_\pi(s)-\alpha_\pi(t))} . \end{aligned} \tag{3.27}$$

If we choose $\eta_1 = \frac{1}{2}$, we recover the $\pi\pi$ amplitude of Eq. (2.10) (apart from some factors of 2 due to the trace of A_1). There is a leading trajectory one half unit above the π trajectory. The isospin content of the present model is, however, entirely different from the model of the preceding section which relied on the Paton-Chan formalism. As can be seen by examining Eq. (3.25) for $\pi\pi$ scattering, in other charge states the ρ and A_2 trajectories decouple from the model leaving only isospin-zero trajectories (ω, f). By taking $\eta_1 = 1$, we can move the leading trajectory to one unit above the π trajectory, thus obtaining an isospin-zero Pomeranchuk trajectory as in Ref. 9.

In the case of scalar $K\pi$ scattering, Eq. (3.25) yields

$$A_{K^+\pi^0K^-\pi^0} = \frac{\Gamma(-\alpha_K(s))\Gamma(-\alpha_\pi(t))}{\Gamma(-\alpha_K(s)-\alpha_\pi(t))} - \frac{\Gamma(-\alpha_K(s)+\eta_1)\Gamma(-\alpha_\pi(t))}{\Gamma(-\alpha_K(s)-\alpha_\pi(t)+\eta_1)} + 2\,\frac{\Gamma(-\alpha_K(s)+\eta_1)\Gamma(-\alpha_\pi(t)-\eta_1)}{\Gamma(-\alpha_K(s)-\alpha_\pi(t))}, \tag{3.28}$$

where

$$\alpha_K(s) = -\tfrac{1}{2}m_K^2 + \tfrac{1}{2}s = \alpha_\pi(s) + \tfrac{1}{2}(\eta_1 - \eta_2) . \tag{3.29}$$

If $\eta_1 = \frac{1}{2}$, the asymptotic behavior is $s^{\alpha_\rho(t)}$ as desired, and poles appear in the s channel at the mass of the K^* and its recurrences with

$$m_{K^*}^2 - m_\rho^2 = m_K^2 - m_\pi^2 . \tag{3.30}$$

However, the parent trajectory in the K^* family decouples. Thus one might prefer to take $\eta_1 = 1$ and adopt the Pomeranchuk interpretation of the leading pole.

By examining scalar K^+K^+ scattering, one finds the existence of exotics on low-lying daughter trajectories, as expected.

The general four-point function can be written as follows:

$$A_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{1}{C}\oint\prod_{i=1}^{4}\left(\frac{dz_i}{z_i}|z_i - z_{i+1}|^{-1+\eta_1 - m_\pi^2/2}\,\theta(\arg z_i - \arg z_{i+1})\right)\left(\prod_{i=1}^{3}\prod_{j=i+1}^{4}|z_i - z_j|^{k_i\cdot k_j}\right)T_4, \tag{3.31}$$

where

$$T_4 = \langle 0 | \prod_{i=1}^{4} : H^\dagger(z_i)\lambda^{\alpha_i} H(z_i) : | 0 \rangle \,. \tag{3.32}$$

We adopt the notation

$$\langle ijk \cdots n \rangle \equiv \mathrm{Tr}(\lambda^{\alpha_i} C_{ij} \lambda^{\alpha_j} C_{jk} \lambda^{\alpha_k} \cdots \lambda^{\alpha_n} C_{ni}) \,, \tag{3.33}$$

with

$$(C_{ij})_{rs} = \{B_r(z_i),\, B_s^\dagger(z_j)\} \,. \tag{3.34}$$

Then

$$\begin{aligned} T_4 = \langle 1234 \rangle + \langle 4321 \rangle - \langle 1243 \rangle \\ - \langle 2314 \rangle - \langle 3421 \rangle - \langle 4132 \rangle \\ + \langle 12 \rangle \langle 34 \rangle + \langle 23 \rangle \langle 41 \rangle + \langle 13 \rangle \langle 24 \rangle \,. \end{aligned} \tag{3.35}$$

The cyclic symmetry of the four-point function is evident from Eqs. (3.31) and (3.35). We assume that the cyclic character of the higher N-point functions can be similarly demonstrated although we have not developed a general proof of this.

Finally, we note that we have taken the field $H(z)$ to be spin-zero for simplicity. It is natural of course to make H a Lorentz spinor, in which case one could form the pseudoscalar octet

$$\Phi^\alpha = : e^{ik\cdot Q} \bar{H} \gamma_5 \lambda^\alpha H : \,. \tag{3.36}$$

The $SU(1,1)$ transformation properties Φ^α are not altered by this generalization, but the scattering amplitudes above are modified by the appropriate traces of γ matrices [e.g., in the expression (3.33) each λ matrix is multiplied by γ_5]. The odd N-point functions are then identically zero. Additional ghosts appear in the model because of the third and fourth Dirac components of the b's and d's. In view of the fact that the $SU(3)$-breaking mechanism discussed in this section forces $\pi^0\eta$ degeneracy and the canonical quark-model mixing angle [Eqs. (3.19) and (3.22)], it does not seem worthwhile to pursue this possibility without more drastic modifications of the model. It is interesting, however, that phenomenological attempts to construct a dual $\pi^0\eta$ scattering amplitude have also been forced to assume a $\pi^0\eta$ degeneracy. This fact makes it additionally interesting to try to construct, from the group-theoretical point of view, a dual model with a different symmetry-breaking mechanism in which π^0 and η do not have equal masses.

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This work arose out of a model for nucleons based on the vertex

$$V_\Psi(k,z) = : e^{ik\cdot Q} u_r B_r(z) + \bar{u}_r D_r^\dagger(z) : \,,$$

where B_r and D_r were fermion spinors with $J_s = -\frac{1}{2}$. After some preliminary investigations were made of this field interacting with itself and with the scalar mesons of the conventional Veneziano model, we received the paper of Bardakci and Halpern[9] where essentially the same operators and the same representation of $SU(1,1)$ were used to define a quark field. For that reason, and because the model can be easily reconstructed by the reader, we do not discuss it here. The author is indebted to Professor S. Mandelstam for providing a copy of Ref. 9. He would also like to acknowledge several interesting discussions with Dr. P. Ramond.

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