Spontaneous Breakdown of Conformal and Chiral Invariance*

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We discuss the implications of a theory in which scale and chiral invariance are spontaneously broken, and the dilaton appears as a mixture of the two isoscalar members of the scalar nonet. The usual assumptions for the conformal properties of the axial-vector current constrain the low-energy behavior of the spin-2 form factor, $F_1(t)$, of the pionic matrix element of the stress-energy tensor. In the limit of scale invariance, we find $F_1'(0)=F_{\sigma\pi}'(0)/F_{\sigma\pi}(0)$, where $F_{\sigma\pi}(t)$ is the axial-vector form factor obtained from the coupling of the dilaton to a pion via the axial-vector current, and the prime denotes differentiation. This relation connects the assumptions of f dominance of $F_1(t)$ and A_1 dominance of $F_{\sigma\pi}(t)$. Using the method of collinear dispersion relations, we estimate the effects of violation of scale invariance. A result previously obtained in the limit of scale invariance becomes $F_\sigma F_{\sigma\pi}(m_\sigma^2)f_\pi \simeq \frac{1}{2}$, where f_π is the decay constant of the pion, and F_σ couples the dilaton to the vacuum via the stress-energy tensor. Similar corrections to the scale-invariant prediction for $F_1'(0)$ are calculated. The magnitudes of the corrections are controlled by the $A_1\sigma\pi$ coupling constant. According to the usual estimates of this constant, the predicted width of the dilaton is compatible with the Adler-Weisberger sum rule for $\pi\pi$ scattering and phenomenological estimates of the σNN coupling constant. While the relation for $F_1'(0)$ obtained in the limit of scale invariance is compatible with the assumption of f dominance, the effects of symmetry breaking are large. In the real world, we find that f dominance is a poor approximation, a conclusion which is supported by recent estimates of the fNN coupling constants. We discuss the relation of our work to the magnitude of parameters measuring symmetry violation in the energy density. Our interpretation of a recent result of Cheng and Dashen is that scale invariance is spontaneously broken, and chiral $SU(2)\times SU(2)$ is a much better symmetry than $SU(3)$.

I. INTRODUCTION

The theory of broken scale invariance has received considerable attention[1] since the discovery[2] of "scaling laws" in deep-inelastic electroproduction and e^+e^- annihilation. By influencing the short-distance behavior of operator-product expansions,[3] broken scale invariance provides constraints on cross sections for a limited number of such high-energy processes.[4] Another aspect of scaling theory, first considered by Kastrup[5] and Mack,[6] is the possibility that scale invariance is spontaneously broken, so that there is also a connection to low-energy phenomena.

In this paper we continue an investigation[7] of the spontaneous breakdown of scale and conformal invariance, making use of constraints imposed by the theory of broken chiral symmetry. The central objects of our analysis are the stress-energy tensor, $\Theta_{\mu\nu}(x)$, and the vector and axial-vector currents, $\mathfrak{F}^a_\mu(x)$ and $\mathfrak{F}^a_{5\mu}(x)$ ($a=1,\ldots,8$) from which current algebra is constructed. The charges

$$F^a(x_0)=\int d^3x\,\mathfrak{F}^a_0(x), \tag{1.1a}$$

$$F^a_5(x_0)=\int d^3x\,\mathfrak{F}^a_{50}(x) \tag{1.1b}$$

generate chiral $SU(3)\times SU(3)$ transformations, while

$$D(x_0)=\int d^3x\,x^\mu\Theta_{0\mu}(x), \tag{1.2a}$$

$$K_\mu(x_0)=\int d^3x\,[2x_\mu x^\lambda\Theta_{0\lambda}(x)-x^2\Theta_{0\mu}(x)] \tag{1.2b}$$

are the generators of scale and special conformal transformations.[8,9] Unlike the exact $SU(3)\times SU(3)$ algebra formed by F^a and F^a_5, symmetry violation affects the conformal algebra obeyed by D, K_μ, and the Poincaré generators P_μ and $M_{\mu\nu}$; the simplest example is[6]

$$i[P_0,D]=\dot{D}-P_0. \tag{1.3}$$

If $\phi(x)$ denotes a local operator with known behavior under Lorentz transformations, the operators

$$d_\phi(x)=i[D(x_0)-x\cdot P,\,\phi(x)], \tag{1.4}$$

$$k_\phi{}^\mu(x)=i[K^\mu(x_0)-2x^\mu D(x_0)+2x_\alpha M^{\alpha\mu}+2x^\mu x\cdot P-x^2P^\mu,\,\phi(x)] \tag{1.5}$$

are local, but may have obscure properties under boosts because of the violation of conformal symmetry. If it exists, the dimension l_ϕ of $\phi(x)$ is given by

$$d_\phi(x)=-l_\phi\phi(x)+\cdots, \tag{1.6}$$

where the dots indicate the possible existence of gradient terms which vanish when integrated over 3-space. We are concerned with a few of the physical implications of popular assumptions about $d_\phi(x)$ and $k^\mu_\phi(x)$ for $\phi = \mathfrak{F}^a_\mu,\ \mathfrak{F}^a_{5\mu},\ \Theta_{\mu\nu},\ \partial^\alpha \mathfrak{F}^a_\alpha$, and $\partial^\alpha \mathfrak{F}^a_{5\alpha}$.

In the limit of conformal invariance, $\Theta_\mu{}^\mu \to 0$, matrix elements of $\Theta_{\mu\nu}$ have poles at zero momentum transfer due to the presence of massless scalar particles, called "dilatons"; we assume that just one dilaton is present. Most particles retain their mass in this limit. However, scale invariance is supposed to be accompanied by chiral invariance, so the members of the pseudoscalar octet become massless. All other particles, including the ninth pseudoscalar meson η', remain massive. Physical $SU(3)$ symmetry, according to which particle multiplets are classified, is not spontaneously violated: As $\Theta_\mu{}^\mu$ vanishes, the dilaton state $|\sigma\rangle$ becomes an $SU(3)$ singlet, and the $SU(3)$ symmetry of the vacuum is retained.

In order that scale-symmetric relations for amplitudes involving soft dilatons remain approximately true as scale invariance is broken, the low-mass dilaton state is assumed to dominate an unsubtracted dispersion relation for $\langle\Theta_\mu{}^\mu\rangle$ at small values of the square of the momentum transfer.[6,9-12] This condition is the hypothesis of partial conservation of the dilation current (PCDC).[6] Normalizing states invariantly, baryons and mesons couple to the dilaton according to the PCDC relations

$$F_\sigma g_{\sigma BB} = M_B + O(m_\sigma{}^2/2M_B), \tag{1.7}$$

$$F_\sigma G_{\sigma MM} = 2m_M{}^2 + O(m_\sigma{}^2), \tag{1.8}$$

where the definition of F_σ is

$$\langle\sigma(k)\,|\,\Theta_{\mu\nu}(0)\,|\,0\rangle = -\tfrac{1}{3}F_\sigma(k_\mu k_\nu - g_{\mu\nu}k^2). \tag{1.9}$$

Equations (1.7) and (1.8) are useful only if the mass involved, M_B or m_M, is large enough to swamp the scale-invariance-breaking effects. For example, it has been shown[7,13] that such effects cannot be neglected for pseudoscalar mesons.

There may be scalar mesons such as $\epsilon'(1060)$, called η_{0^+} or S^* by the Particle Data Group,[14] which retain their masses as the dilaton mass vanishes, and which can mix with the dilaton as conformal invariance is broken. Thus, in the real world, the dilaton quality may be distributed among two or more mesons with $(J^P,\ I^G) = (0^+, 0^+)$. The least massive $(0^+, 0^+)$ state is supposed to become the massless dilaton state in the limit of conformal invariance. In the real world, we continue to use the notation $|\sigma\rangle$ for it, and assume that m_σ is significantly lower than 1 GeV. We expect the magnitude of m_σ^2 to indicate the inaccuracy of conformal-symmetric results only if mixing is properly taken into account.

The main body of this paper consists of Secs. II–V and our conclusions are summarized in Sec. VI. Section II contains a discussion of the limits of conformal and chiral symmetry. The main result is that the slope of the spin-2 form factor of $\langle\pi|\Theta_{\mu\nu}|\pi\rangle$, $F_1{}'(0)$, is determined by the conformal properties of the axial-vector current. Section III is concerned with the effects of mixing on soft-dilaton theorems. After noting some examples in which the contribution of the ϵ' pole to $\langle\Theta_\mu{}^\mu\rangle$ is significant, its effect on calculations of the $\sigma\pi\pi$ coupling constant is considered. In Sec. IV we use the method of collinear dispersion relations to estimate the effects of the violation of conformal invariance. A new estimate of the $\sigma\pi\pi$ coupling constant is obtained, and found to be consistent with the Adler-Weisberger sum rule for pions. A similar calculation yields $F_1{}'(0)$. Meson-baryon scattering and the result of Cheng and Dashen[15] are discussed in Sec. V.

II. LIMITS OF CHIRAL AND CONFORMAL SYMMETRY

The nature of the breakdown of a symmetry can be specified by giving the symmetry properties of the appropriate current divergences. In Gell-Mann's theory of broken chiral symmetry,[10] the energy density is written

$$\Theta_{00} = \overline{\Theta}_{00} - u_0 - cu_8, \tag{2.1}$$

where $\overline{\Theta}_{00}$ is chiral-invariant, u_0 and u_8 belong to a set of scalars u_b and pseudoscalars v_b $(b = 0,\ldots, 8)$ which form a $(\underline{3}, \underline{3}^*)\oplus(\underline{3}^*, \underline{3})$ representation of $SU(3) \times SU(3)$, and $SU(2)\times SU(2)$ violation is measured by the deviation of $c \simeq -1.25$ from $-\sqrt{2}$.[16,17] The assumed Lorentz properties of u_b and v_b lead to the relations

$$i[\Theta_{00}, F^a] = \partial^\mu \mathfrak{F}^a_\mu, \tag{2.2a}$$

$$i[\Theta_{00}, F^a_5] = \partial^\mu \mathfrak{F}^a_{5\mu}, \tag{2.2b}$$

from which the $SU(3)\times SU(3)$ behavior of $\partial^\mu \mathfrak{F}^a_\mu$ and $\partial^\mu \mathfrak{F}^a_{5\mu}$ automatically follows. The feature of this theory which influences our work is the *absence* of a term[18]

$$S_{\mu\nu} = (\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)s(x) \tag{2.3}$$

in $\Theta_{\mu\nu}$, where $s(x)$ is not invariant under chiral transformations. The existence of $S_{\mu\nu}$ would imply that Eq. (2.2b) is invalid, and the axial-vector current would not have dimension −3. Unless explicitly stated otherwise, we assume that Eq. (2.1) is correct.

The breakdown of scale invariance has been similarly treated.[3,9] The scale-invariance-breaking terms w_n in the energy density

$$\Theta_{00}=\hat{\Theta}_{00}+\sum_n w_n \tag{2.4}$$

have dimension $l_n \neq -4$, and $\hat{\Theta}_{00}$ is the scale-invariant part of Θ_{00}. The assumption that $\sum_n (l_n+4)w_n$ is a scalar operator gives the virial theorem[9]

$$\Theta_\mu{}^\mu=\sum_n (l_n+4)w_n. \tag{2.5}$$

Following Wilson,[3] we assume that $\hat{\Theta}_{00}$ is $SU(3)\times SU(3)$-invariant, and write

$$\Theta_{00}=\hat{\Theta}_{00}+\delta+u, \tag{2.6}$$

where δ violates scale invariance but not chiral invariance, and $u=-u_0-cu_8$ breaks both scale and chiral invariance.

An important matrix element for our work is

$$\begin{aligned}\mathfrak{M}_{\mu\nu}&=\langle\pi(P+\tfrac{1}{2}k)\,|\,\Theta_{\mu\nu}(0)\,|\,\pi(P-\tfrac{1}{2}k)\rangle\\&=[2P_\mu P_\nu-\tfrac{1}{6}(k_\mu k_\nu-g_{\mu\nu}k^2)]F_1(k^2)\\&\quad+(k_\mu k_\nu-g_{\mu\nu}k^2)F_2(k^2),\end{aligned} \tag{2.7}$$

with $F_1(0)=1$. This expansion for $\mathfrak{M}_{\mu\nu}$ was chosen[7] because the dispersion theory of $F_1(t)$ and $F_2(t)$ is simplified when the mass of the pion is neglected; intermediate states with (J^P, I^G) equal to $(2^+, 0^+)$ and $(0^+, 0^+)$ contribute to $\mathrm{Im}F_1(t)$ and $\mathrm{Im}F_2(t)$, respectively. Application of the condition of scale invariance, $\Theta_\mu{}^\mu\to 0$ with $m_\pi^2\to 0$, to the trace of Eq. (2.7),

$$\mathfrak{M}_\mu{}^\mu=2m_\pi{}^2F_1(k^2)-3k^2F_2(k^2), \tag{2.8}$$

leads to the result[4]

$$F_2(k^2)=0. \tag{2.9}$$

Thus the effects of scale violation are responsible for the presence of the induced scalar form factor $F_2(t)$ in Eq. (2.7).

Equation (2.7) may also be investigated in the limit of chiral $SU(2)\times SU(2)$ invariance, with scale invariance broken. In this limit, Eq. (2.2b) requires that the energy density, not just the Hamiltonian and other Poincaré generators, be chiral-invariant; thus, $\dot{F}_5=0$ implies

$$[F_5, \Theta_{\mu\nu}]=0, \tag{2.10}$$

from which the constraint[7,19]

$$F_2(0)=-\tfrac{1}{3} \tag{2.11}$$

follows.

A brief derivation of Eq. (2.11) follows from the formula[20]

$$\partial^\lambda R^*_{\lambda\mu\nu}(x)=0, \tag{2.12}$$

where

$$R^*_{\lambda\mu\nu}(x)=[\mathfrak{F}_{5\lambda}(x), \Theta_{\mu\nu}(0)]_{\text{ret}}\,. \tag{2.13}$$

is a retarded commutator made covariant by careful treatment of the singularity at $x=0$. Then

$$q^\lambda\int d^4x\, e^{-iq\cdot x}\langle\pi(q')\,|R^*_{\lambda\mu\nu}(x)\,|\,0\rangle=0 \tag{2.14}$$

implies

$$\lim_{q\to 0}\langle\pi(q')\,|\,\Theta_{\mu\nu}(0)|\pi(q)\rangle=0, \tag{2.15}$$

which may be combined with Eq. (2.7) to yield Eq. (2.11).

The behavior of $F_2(t)$ at $t=k^2=0$ indicated by Eqs. (2.9) and (2.11) is characteristic of theories involving a Nambu-Goldstone boson. The relation

$$F_2(t)\simeq-\tfrac{1}{3}m_\sigma{}^2/(m_\sigma{}^2-t), \tag{2.16}$$

which holds for $|t|\lesssim m_\sigma{}^2$, shows that the limits $t\to 0$, $m_\sigma{}^2\to 0$ are not interchangeable. However, the author regards the difference between Eqs. (2.9) and (2.11) as significant only for dilaton theories. In the scale-invariant limit of a theory with no dilatons, all masses vanish, so chiral invariance is no longer realized in the Goldstone manner and has no connection with soft-meson amplitudes. Therefore, the difference between Eqs. (2.9) and (2.11) is significant only if the zero-mass limit of Eq. (2.11) is supposed to be smooth, in spite of the infrared problem.[21] This limit might be smooth for amplitudes which are vacuum expectation values of an operator expansion near the light cone. However, Eq. (2.11) is a low-energy result arising from the behavior of the operator product $R^*_{\lambda\mu\nu}(x)$ at large x^2, so we expect that the difference between Eqs. (2.9) and (2.11) is generated by infrared effects if no dilatons are present.[22]

Returning to dilaton theory, another important matrix element is

$$\begin{aligned}\langle\sigma(k)\,|\,\mathfrak{F}^3_{5\lambda}(0)\,|\pi^0(q)\rangle&=-i(k+q)_\lambda F_{\sigma\pi}(t)\\&\quad+i(k-q)_\lambda G_{\sigma\pi}(t),\end{aligned} \tag{2.17}$$

with divergence

$$\begin{aligned}D_{\sigma\pi}(t)&=\langle\sigma(k)\,|\,\partial^\lambda\mathfrak{F}^3_{5\lambda}(0)\,|\pi^0(q)\rangle\\&=(m_\sigma{}^2-m_\pi{}^2)F_{\sigma\pi}(t)-tG_{\sigma\pi}(t),\end{aligned} \tag{2.18}$$

where $t=(q-k)^2$ is the momentum transfer squared. In the limit of $SU(2)\times SU(2)$ invariance, $F_{\sigma\pi}$ and $G_{\sigma\pi}$ are related:

$$m_\sigma{}^2F_{\sigma\pi}(t)=tG_{\sigma\pi}(t). \tag{2.19}$$

Apart from the relation[23]

$$F_{\sigma\pi}(m_\sigma{}^2)=G_{\sigma\pi}(m_\sigma{}^2) \tag{2.20}$$

obtained from Eq. (2.19) at $t=m_\sigma{}^2$, we have

$$m_\sigma{}^2[F_{\sigma\pi}{}'(m_\sigma{}^2)-G_{\sigma\pi}{}'(m_\sigma{}^2)]=G_{\sigma\pi}(m_\sigma{}^2), \tag{2.21}$$

$$m_\sigma{}^2[F_{\sigma\pi}{}''(m_\sigma{}^2)-G_{\sigma\pi}{}''(m_\sigma{}^2)]=2G_{\sigma\pi}{}'(m_\sigma{}^2), \tag{2.22}$$

and so on, when m_π vanishes. (Here the prime denotes differentiation.) The Goldberger-Treiman relation[24]

$$(2f_\pi)^{-1}G_{\sigma\pi\pi} = m_\sigma^2 F_{\sigma\pi}(0) + O(m_\pi^2) \qquad (2.23)$$

is obtained at $t=0$. In the limit of scale invariance,[7]

$$G_{\sigma\pi}(t) = 0 \qquad (2.24)$$

follows from Eq. (2.18). In spite of Eq. (2.20), $F_{\sigma\pi}(0)$ survives in this limit. The limits $t \to m_\sigma^2$, $m_\sigma^2 \to 0$ are not interchangeable in Eq. (2.19) because of the zero-mass pion pole in $G_{\sigma\pi}(t)$:

$$G_{\sigma\pi}(t) \simeq m_\sigma^2 F_{\sigma\pi}(0)/t \qquad (2.25)$$

holds for $|t| \lesssim m_\sigma^2$.

We have previously shown[7] that, in the limit of scale invariance, the formula

$$F_\sigma F_{\sigma\pi}(0) f_\pi = \tfrac{1}{2} \qquad (2.26)$$

follows from the requirements $\Theta_\mu{}^\mu = 0$, $\partial^\lambda \mathfrak{F}_{5\lambda} = 0$, and is consistent with the assumption that $\mathfrak{F}_{5\lambda}$ has dimension -3:

$$i[D(0), \mathfrak{F}_{5\lambda}(0)] = 3\mathfrak{F}_{5\lambda}(0). \qquad (2.27)$$

This means that the term $S_{\mu\nu}$ of Eq. (2.3) must be absent in the limit of scale invariance, and explains why Eqs. (2.23) and (2.26) are consistent with the low-energy result[7, 13, 25]

$$F_\sigma G_{\sigma\pi\pi} \simeq m_\sigma^2, \qquad (2.28)$$

which we also obtained using Eq. (2.16).

There is a conformal calculation corresponding to the analysis based on Eq. (2.27). Again we work in the limit of scale invariance, postponing the consideration of scale-invariance-breaking effects to Sec. IV. We assume the validity of the standard equal-time commutation relation[26]

$$[K_\mu(0), \mathfrak{F}_{5\nu}(0)] = 0; \qquad (2.29)$$

in the language of Eq. (1.5), we are supposing that $k^\mu_\phi(x)$ vanishes for $\phi = \mathfrak{F}_{5\nu}$. Then, defining the conformal current

$$\mathcal{K}_{\mu\nu}(x) = (2x_\mu x^\lambda - \delta_\mu^\lambda x^2)\Theta_{\nu\lambda}, \qquad (2.30)$$

the amplitude

$$M_{\mu\nu\lambda}(k,q) = i\int d^4x\, e^{ik\cdot x}\langle 0|T(\mathcal{K}_{\mu\nu}(x)\mathfrak{F}_{5\lambda}{}^3(0))|\pi^0(q)\rangle \qquad (2.31)$$

obeys the identity

$$k^\nu M_{\mu\nu\lambda}(k,q) = O(k). \qquad (2.32)$$

As k^ν approaches zero, the only contributions to $k^\nu M_{\mu\nu\lambda}$ come from terms in $M_{\mu\nu\lambda}$ which are singular in k. Applying Eqs. (1.9), (2.7), (2.9), (2.17), and (2.24), the pion- and dilaton-pole diagrams shown in Fig. 1 represent the amplitude

$$P_{\alpha\nu\lambda}(k,q) = -\tfrac{1}{3}i(k+q)_\lambda F_\sigma F_{\sigma\pi}((k-q)^2)\frac{k_\alpha k_\nu - g_{\alpha\nu}k^2}{k^2} - \frac{iF_1(k^2)}{2f_\pi}(k-q)_\lambda \frac{2(q-\frac{1}{2}k)_\alpha(q-\frac{1}{2}k)_\nu - \frac{1}{6}(k_\alpha k_\nu - g_{\alpha\nu}k^2)}{2q\cdot k - k^2}. \qquad (2.33)$$

The part of $M_{\mu\nu\lambda}$ which is singular in k may be obtained by differentiating $P_{\alpha\nu\lambda}(k,q)$ twice with respect to k:

$$k^\nu M_{\mu\nu\lambda}(k,q) = -k^\nu\left(2\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k_\alpha} - \delta_\mu^\alpha\frac{\partial}{\partial k^\beta}\frac{\partial}{\partial k_\beta}\right)P_{\alpha\nu\lambda}(k,q) + O(k) \qquad (2.34)$$

follows from Eqs. (2.30) and (2.31). Calculations of this nature are always simplified by applying the identities

$$k^\mu\frac{\partial}{\partial k^\nu} = \frac{\partial}{\partial k^\nu}k^\mu - \delta_\nu^\mu, \qquad (2.35)$$

$$k^\nu\left(2\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\alpha} - g^{\mu\alpha}\frac{\partial}{\partial k^\beta}\frac{\partial}{\partial k_\beta}\right) = \left(2\frac{\partial}{\partial k_\mu}\frac{\partial}{\partial k_\alpha} - g^{\mu\alpha}\frac{\partial}{\partial k^\beta}\frac{\partial}{\partial k_\beta}\right)k^\nu - 2g^{\alpha\nu}\frac{\partial}{\partial k_\mu} + 2\left(g^{\mu\alpha}\frac{\partial}{\partial k_\nu} - g^{\mu\nu}\frac{\partial}{\partial k_\alpha}\right). \qquad (2.36)$$

[Equation (2.35) is useful when handling expressions involving the dilation current.] The last term of Eq. (2.36) is antisymmetric in the indices (α, ν), so it does not contribute to Eq. (2.34). Combining Eqs. (2.32)–(2.34) and (2.36), we obtain

$$2ig_{\mu\lambda}[F_\sigma F_{\sigma\pi}(0) - (2f_\pi)^{-1}] = 4iq_\mu q_\lambda[F_\sigma F_{\sigma\pi}'(0) - (2f_\pi)^{-1}F_1'(0)], \qquad (2.37)$$

which implies Eq. (2.26) together with a new relation

$$F_{\sigma\pi}'(0)/F_{\sigma\pi}(0) = F_1'(0). \qquad (2.38)$$

The primes denote differentiation with respect to the momentum transfer squared.

If poles due to the $A_1(1070)$ meson and the spin-2 $SU(3)$ singlet, which corresponds to a mixture of the $f(1260)$ and $f'(1515)$ states, are supposed to dominate $F_{\sigma\pi}(t)$ and $F_1(t)$, respectively, then

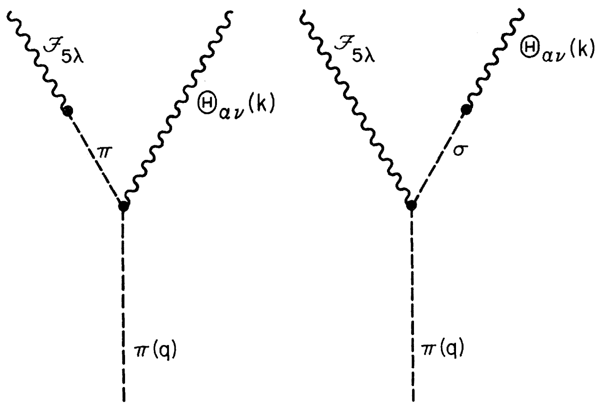

FIG. 1. Pole diagrams for Eq. (2.33).

Eq. (2.38) implies

$$m^2(2^+) \approx m^2(A_1). \tag{2.39}$$

When the magnitude of $SU(3)$ mass splitting is considered, the agreement of this result with the observed meson spectrum is reasonable. However, it turns out that Eq. (2.38) is strongly affected by the inclusion of scale-invariance-breaking effects, and the results of A_1 and f dominance are not consistent.[27]

In contrast with theories lacking a dilaton,[28] no difficulty has been experienced in our discussion of the limit of scale invariance, and we conclude this section with some further remarks about this limit. Let us write the PCDC hypothesis in the form[29]

$$\Theta_{\mu\nu} = t_{\mu\nu} + \frac{1}{3}\frac{\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2}{m_\sigma^2 + \partial^2} t_\alpha^{\ \alpha}, \tag{2.40}$$

where $\langle t_{\mu\nu}\rangle$ has no σ pole; the trace of Eq. (2.40) is

$$\Theta_\mu^{\ \mu} = \frac{m_\sigma^2}{m_\sigma^2 + \partial^2} t_\alpha^{\ \alpha}. \tag{2.41}$$

The Poincaré generators are well defined for $\Theta_\mu^{\ \mu} \to 0$ if the σ-pole term in $\Theta_{\mu\nu}$ is not included in the usual definitions.[30] The expressions

$$P_\mu = \int d^3x\, t_{0\mu}, \tag{2.42a}$$

$$M_{\mu\nu} = \int d^3x (x_\mu t_{0\nu} - x_\nu t_{0\mu}) \tag{2.42b}$$

are valid for any value of m_σ. As $\Theta_\mu^{\ \mu}$ vanishes, the unique physical vacuum becomes a member of a continuum of degenerate vacua, but it may still be distinguished by the properties[31]

$$P_\mu|0\rangle = 0, \quad M_{\mu\nu}|0\rangle = 0. \tag{2.43}$$

Paradoxes are avoided if the effects of the dilaton pole are properly included. For simplicity, suppose that apart from a c-number term, the scale-invariance-violating term w in Θ_{00} has unique dimension l, so we have the relation

$$e^{i\alpha D} P_0 e^{-i\alpha D} = e^\alpha P_0 + (e^{-(l+3)\alpha} - e^\alpha)\dot{D}(x_0)/(l+4). \tag{2.44}$$

In the limit $\dot{D} \to 0$, it appears that $e^{-i\alpha D}$ transforms states of mass M_N into states with mass $M_N e^\alpha$, so that transformations with $\alpha < 0$ are forbidden.[13] This anomaly arises from an illegal interchange of limits in the equation

$$\lim_{\Theta_\mu{}^\mu \to 0}\ \lim_{k^2 \to 0} \langle N(p+k)|\Theta_\mu^{\ \mu}|N(p)\rangle = M_N. \tag{2.45}$$

In fact, if we sandwich Eq. (2.44) between nucleon states at rest, the requirement

$$\langle N|e^{i\alpha D} P_0 e^{-i\alpha D}|N\rangle \geq M_N \langle N|N\rangle \tag{2.46}$$

implies the inequality

$$(l+3)e^\alpha + e^{-(l+3)\alpha} \geq l+4, \tag{2.47}$$

which does not depend on the magnitude of symmetry violation. If α is to be unrestricted, Eq. (2.47) implies the constraints $l \geq -3$ or $l < -4$. Making use of the usual constraints on l,[32] we find

$$-3 \leq l \leq -1. \tag{2.48}$$

III. EFFECTS OF MIXING

The application of conformal-invariant results to amplitudes in the real world requires some care:

(i) Conformal symmetry is badly broken, with no candidate for the dilaton below 0.5 GeV. We assume that the mass of the dilaton is significantly less than 1 GeV ($500 \lesssim m_\sigma \lesssim 800$ MeV).

(ii) This symmetry violation may be accompanied by mixing of the dilaton with other $(J^P, I^G) = (0^+, 0^+)$ mesons.

A phenomenological analysis of (ii) has been performed by Carruthers.[23]

There is fairly strong evidence[14,33] for the presence of some members, $\delta(960)$ and $\epsilon'(1060)$ (also called η_{0^+} or S^*), of a possible nonet of scalar mesons. Evidence for the corresponding strange particles is obscure. The ninth member could be the $\epsilon(700)$ meson, but its parameters depend on indirect and controversial analyses of data.[33,34] That only a nonet should be present results from theoretical prejudice; the number of $(0^+, 0^+)$ resonances in the region of interest (below 2 GeV) is not even vaguely suggested by experiment. However, we will usually assume that the nonet picture of two $(0^+, 0^+)$ mesons, ϵ' and the dilaton σ, is correct.

A good illustration of the effect of mixing is provided by applying the PCDC hypothesis to the matrix element[35] $\langle\phi|\Theta_\mu^{\ \mu}|\phi\rangle$:

$$F_\sigma G_{\sigma\phi\phi} + F_{\epsilon'} G_{\epsilon'\phi\phi} = 2m_\phi^2 + O(m_\sigma^2). \tag{3.1}$$

Suppose that the scalar nonet is a set of $^3P_0$ states in the quark model. The apparent suppression of the $\epsilon' \to \pi\pi$ mode relative to $\epsilon' \to K\bar{K}$ suggests that ϵ' contains only strange quarks. Ellis[13] has pointed out that this standard picture implies

$$G_{\sigma\phi\phi} = 0, \tag{3.2}$$

because the corresponding quark diagram is disconnected. Thus Eq. (3.1) shows that the ϵ' pole provides an essential contribution to the unsubtracted dispersion relation for $\langle\phi|\Theta_\mu{}^\mu|\phi\rangle$. As the limit of conformal invariance is approached, the ϵ' state loses its dilaton quality to the σ state, and Eq. (3.1) becomes

$$F_\sigma G_{\sigma\phi\phi} = 2m_\phi{}^2. \tag{3.3}$$

Another obvious example is the sum rule[36]

$$-i\langle 0|[D,\Theta_\mu{}^\mu]|0\rangle = i\int d^4x\,\Theta(x_0)\langle 0|[\Theta_\mu{}^\mu(x),\Theta_\nu{}^\nu(0)]|0\rangle$$
$$\simeq m_\sigma{}^2F_\sigma{}^2 + m_{\epsilon'}{}^2F_{\epsilon'}{}^2. \tag{3.4}$$

Assuming that $\epsilon(700)$ is the dilaton, Carruthers's analysis gives

$$F_\sigma = 102 \text{ MeV}, \quad F_{\epsilon'} = 68 \text{ MeV}. \tag{3.5}$$

Numerically, we observe $m_\sigma{}^2F_\sigma{}^2 \simeq m_{\epsilon'}{}^2F_{\epsilon'}{}^2$, so the contribution of the ϵ' pole to the sum rule is significant. If δ is a c-number, Eqs. (2.5), (3.4), and (3.5) and the value of $\langle 0|u|0\rangle$ given by Gell-Mann, Oakes, and Renner[16] lead to the crude estimate $-l_u(l_u+4) \approx 5$, where l_u is the dimension of u. Since Carruthers's estimates depend on PCDC for baryons, we think that the uncertainty is sufficient to allow $-3 \lesssim l_u \lesssim -1$. Similar account of the ϵ' term should be taken in the standard soft-dilaton theorems

$$F_\sigma\langle\sigma|u_0|0\rangle + F_{\epsilon'}\langle\epsilon'|u_0|0\rangle \simeq l_u\langle 0|u_0|0\rangle, \tag{3.6}$$

$$F_\sigma\langle\sigma|u_8|0\rangle + F_{\epsilon'}\langle\epsilon'|u_8|0\rangle \simeq l_u\langle 0|u_8|0\rangle \simeq 0. \tag{3.7}$$

In some cases, it is expected that the ϵ' contribution is negligible compared with that of the dilaton. Consider the usual method of estimating $G_{\sigma\pi\pi}$ from broken scale invariance.[7,37] Neglecting the mass of the pion, we combine an unsubtracted dispersion relation for $F_2(t)$, Eq. (2.11), and the decomposition

$$\text{Im}F_2(t) = -\tfrac{1}{3}\pi\delta(t-m_\sigma{}^2)F_\sigma G_{\sigma\pi\pi} - \tfrac{1}{3}\pi\delta(t-m_{\epsilon'}{}^2)F_{\epsilon'}G_{\epsilon'\pi\pi} + f(t), \tag{3.8}$$

obtaining

$$1 = \frac{F_\sigma G_{\sigma\pi\pi}}{m_\sigma{}^2} + \frac{F_{\epsilon'}G_{\epsilon'\pi\pi}}{m_{\epsilon'}{}^2} - \frac{3}{\pi}\int dt\,\frac{f(t)}{t}. \tag{3.9}$$

The contribution of the ϵ' term is small, so

$$F_{\epsilon'}F_{\epsilon'\pi}(0) \ll F_\sigma F_{\sigma\pi}(0) \tag{3.10}$$

follows from Eq. (2.23), where the numerical difference is roughly a factor of 10. If the continuum integral in Eq. (3.9) is also insignificant, Eq. (2.28) is obtained, and the prediction for the width of the dilaton is $\Gamma_{\sigma\to\pi\pi} \approx 1200$ MeV. An optimistic view of the accuracy of this result might lead to the conclusion that it violates the Adler-Weisberger sum rule for $\pi\pi$ scattering.[38] However, Sec. IV is devoted to estimating $G_{\sigma\pi\pi}$ according to the method of collinear dispersion relations,[39] and there we conclude that our theory of broken scale invariance is not in conflict with the $\pi\pi$ sum rule. First, a preliminary calculation is necessary to establish an inequality similar to Eq. (3.10).

We apply the method of collinear dispersion relations to evaluate the equal-time commutator

$$\langle\sigma|[F_5^3, \mathfrak{F}_{5\nu}^3]|0\rangle = 0, \tag{3.11}$$

where σ here refers to any $(0^+,0^+)$ meson. The details will be suppressed because our analysis is analogous to the treatment of K_{l3} decay given by Ademollo, Denardo, and Furlan.[40]

According to Eq. (3.11) and a standard Ward identity, the retarded commutator

$$R_\nu(k,q) = (2f_\pi/m_\pi{}^2)i\int d^4x\,e^{-iq\cdot x}\Theta(x_0)\langle\sigma(k)|[\partial^\mu\mathfrak{F}_{5\mu}{}^3(x),\mathfrak{F}_{5\nu}{}^3(0)]|0\rangle$$
$$= ik_\nu V_1 + iq_\nu V_2 \tag{3.12}$$

satisfies the constraint

$$R_\nu(k,0) = 0. \tag{3.13}$$

The usual analysis of the large-q_0 behavior of R_ν leads to the unsubtracted dispersion relation

$$0 = \int dy\,\frac{\text{Im}V_1(y)}{y} \tag{3.14}$$

together with the superconvergence relation

$$0 = \int dy\,\text{Im}V_2(y), \tag{3.15}$$

in the collinear frame $k=(m_\sigma, \vec{0})$, $q=(ym_\sigma, \vec{0})$.

Separation of the pion poles at $y=\pm m_\pi/m_\sigma$, $1\pm m_\pi/m_\sigma$ from the imaginary part of R_ν yields

$$\begin{aligned}\operatorname{Im}V_1(y) &= \pi\epsilon(y-1)\delta((y-1)^2-m_\pi^2/m_\sigma^2)D_{\sigma\pi}(y^2m_\sigma^2)/(m_\pi m_\sigma)^2+\pi\epsilon(y)\delta(y^2-m_\pi^2/m_\sigma^2)\\ &\quad\times[-F_{\sigma\pi}((1-y)^2m_\sigma^2)+G_{\sigma\pi}((1-y)^2m_\sigma^2)]/m_\sigma^2+v_1(y),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\operatorname{Im}V_2(y) &= -\pi\epsilon(y-1)\delta((y-1)^2-m_\pi^2/m_\sigma^2)D_{\sigma\pi}(y^2m_\sigma^2)/(m_\pi m_\sigma)^2-\pi\epsilon(y)\delta(y^2-m_\pi^2/m_\sigma^2)\\ &\quad\times[F_{\sigma\pi}((1-y)^2m_\sigma^2)+G_{\sigma\pi}((1-y)^2m_\sigma^2)]/m_\sigma^2+\nu_2(y),\end{aligned} \tag{3.17}$$

so Eqs. (3.14) and (3.15) become[41]

$$m_\sigma[G_{\sigma\pi}(+)-G_{\sigma\pi}(-)]=(m_\sigma-2m_\pi)F_{\sigma\pi}(+)-(m_\sigma+2m_\pi)F_{\sigma\pi}(-)+\frac{2m_\pi^3}{\pi}\int dy\frac{v_1(y)}{y}, \tag{3.18}$$

$$\frac{D_{\sigma\pi}(+)-D_{\sigma\pi}(-)}{m_\pi^2}=F_{\sigma\pi}(+)+G_{\sigma\pi}(+)-F_{\sigma\pi}(-)-G_{\sigma\pi}(-)+\frac{2m_\pi m_\sigma}{\pi}\int dy\, v_2(y), \tag{3.19}$$

where Eq. (3.18) has been simplified by using Eq. (2.18), the definition of $D_{\sigma\pi}(t)$. In Eq. (3.18), examination of the leading power in m_π gives a result consistent with Eqs. (2.20) and (2.21).

In the collinear frame, only 0^- intermediate states contribute to $\operatorname{Im}R_\nu$. The coupling of the $A_1(1070)$ meson to the axial-vector current, $\langle A_1|\mathfrak{F}_{5\lambda}(0)|0\rangle=\epsilon_\lambda g_{A_1}$, is implicitly contained in the form factors

$$F_{\sigma\pi}(t)=F_{\sigma\pi}(0)-\frac{tg_{A_1}g_{A_1\sigma\pi}}{2m_{A_1}^2(m_{A_1}^2-t)}+\frac{t}{\pi}\int dt'\frac{\operatorname{Im}\overline{F}_{\sigma\pi}(t')}{t'(t'-t)}, \tag{3.20}$$

$$G_{\sigma\pi}(t)=-G_{\sigma\pi\pi}(2f_\pi)^{-1}(m_\pi^2-t)^{-1}-\frac{g_{A_1}g_{A_1\sigma\pi}(m_\sigma^2-m_\pi^2)}{2m_{A_1}^2(m_{A_1}^2-t)}+\frac{1}{\pi}\int dt'\frac{\operatorname{Im}\overline{G}_{\sigma\pi}(t')}{t'-t}, \tag{3.21}$$

where $-\frac{1}{2}i(\sigma+\pi)_\lambda g_{A_1\sigma\pi}$ is the $A_1\sigma\pi$ coupling. The inequality needed in Sec. IV is

$$F_{\epsilon'\pi}(\pm),\quad G_{\epsilon'\pi}(\pm)\ll F_{\sigma\pi}(m_\sigma^2),\quad G_{\sigma\pi}(m_\sigma^2); \tag{3.22}$$

to make it plausible, we must show that the effect of the $A_1\epsilon'\pi$ coupling is negligible.

When ϵ' is the $(0^+,0^+)$ meson involved in Eqs. (3.18) and (3.19), the points at which $F_{\epsilon'\pi}(t)$, $G_{\epsilon'\pi}(t)$, and $D_{\epsilon'\pi}(t)$ are evaluated straddle the point $t=m_{A_1}^2$. The near degeneracy of $\epsilon'(1060)$ and $A_1(1070)$ gives

$$[F_{\epsilon'\pi}(+)+F_{\epsilon'\pi}(-)]_{A_1\text{ pole}}\simeq 0, \tag{3.23a}$$

$$[G_{\epsilon'\pi}(+)+G_{\epsilon'\pi}(-)]_{A_1\text{ pole}}\simeq 0, \tag{3.23b}$$

whereas $D_{\epsilon'\pi}(t)$ does not have a pole near $t=m_{A_1}^2$, so Eq. (3.19) becomes

$$2[F_{\epsilon'\pi}(-)+G_{\epsilon'\pi}(-)]_{A_1\text{ pole}}\simeq[D_{\epsilon'\pi}(+)-D_{\epsilon'\pi}(-)]/m_\pi^2+\text{continuum integrals}. \tag{3.24}$$

From partial conservation of the axial-vector current, $D_{\epsilon'\pi}(t)$ and $D_{\sigma\pi}(t)$ satisfy unsubtracted dispersion relations; while we do not expect that the pion pole at $t=m_\pi^2$ dominates dispersion integrals for $D_{\epsilon'\pi}$ and $D_{\sigma\pi}$ at $t=m_{\epsilon'}^2$ or m_σ^2, the respective π-pole terms should indicate the correct orders of magnitude, i.e.,

$$D_{\epsilon'\pi}(\pm)\ll D_{\sigma\pi}(0),\quad D_{\sigma\pi}(m_\sigma^2). \tag{3.25}$$

Therefore, $g_{A_1\epsilon'\pi}$ is very small and the A_1-pole terms in $F_{\epsilon'\pi}(t)$ and $G_{\epsilon'\pi}(t)$ do not affect the validity of Eq. (3.22).

IV. EFFECTS OF SYMMETRY VIOLATION

Collinear dispersion relations[39] have provided a framework within which soft-pion and soft-kaon results may be corrected for violation of chiral symmetry. In this section, we use this formalism to estimate corrections to Eqs. (2.26) and (2.38) for violation of conformal symmetry.

We assume the standard equal-time commutation relations[26]

$$\langle 0|i[D(0),\mathfrak{F}_{50}(0)]|\pi(q)\rangle=3\langle 0|\mathfrak{F}_{50}(0)|\pi(q)\rangle+O(\vec{q}\,), \tag{4.1}$$

$$\langle 0|i[K_\mu(0),\mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle=0. \tag{4.2}$$

Equation (4.1) is equivalent to the assumption that $S_{\mu\nu}$ is not present [see Eq. (2.3)]. Equation (4.2) is a stronger condition which holds if there are no Schwinger terms more singular than $\vec{\partial}\delta^3(\vec{x})$ in the equal-time commutator

$$E_{\mu\nu\lambda}(x)=[\Theta_{\mu\nu}(0,\vec{x}),\mathfrak{F}_{5\lambda}(0)]. \tag{4.3}$$

The retarded commutators

$$T_\lambda(k,q)=i\int d^4x\, e^{ik\cdot x}\Theta(x_0)\langle 0|[\Theta_\mu{}^\mu(x),\mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle, \tag{4.4}$$

$$D_{\mu\lambda}(k,q)=i\int d^4x\, e^{ik\cdot x}\Theta(x_0)\langle 0|[\mathfrak{D}_\mu(x),\mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle, \tag{4.5}$$

$$K_{\mu\nu\lambda}(k,q)=i\int d^4x\, e^{ik\cdot x}\Theta(x_0)\langle 0|[\mathcal{K}_{\mu\nu}(x),\mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle \tag{4.6}$$

(where $\mathfrak{D}_\mu(x)=x^\nu\Theta_{\mu\nu}$ is the dilation current) are related to the equal-time commutators in Eqs. (4.1) and (4.2) by the identities

$$-ik^\mu D_{\mu\lambda}(k,q)=i\langle 0|[D(0),\mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle+T_\lambda(k,q)+O(k), \tag{4.7}$$

$$-ik^\nu K_{\mu\nu\lambda}(k,q)=i\langle 0|[K_\mu(0),\mathfrak{F}_{5\lambda}(0)]|\pi(q)\rangle-2i\frac{\partial T_\lambda}{\partial k^\mu}+O(k). \tag{4.8}$$

The singularities in k of $D_{\mu\lambda}$ and $K_{\mu\nu\lambda}$ are given by derivatives of the pion-pole term

$$\overline{P}_{\alpha\nu\lambda}(k,q)=-i(k-q)_\lambda(2f_\pi)^{-1}\times\{[2(q-\tfrac{1}{2}k)_\alpha(q-\tfrac{1}{2}k)_\nu-\tfrac{1}{6}(k_\alpha k_\nu-g_{\alpha\nu}k^2)]\times F_1(k^2)+(k_\alpha k_\nu-g_{\alpha\nu}k^2)F_2(k^2)\}/(2q\cdot k-k^2) \tag{4.9}$$

with respect to k, so we obtain

$$-ik^\mu D_{\mu\lambda}(k,q)=-k^\mu\frac{\partial}{\partial k_\nu}\overline{P}_{\mu\nu\lambda}+O(k), \tag{4.10}$$

$$-ik^\nu K_{\mu\nu\lambda}(k,q)=ik^\nu\left(2\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k_\alpha}-\delta^\alpha_\mu\frac{\partial}{\partial k^\beta}\frac{\partial}{\partial k_\beta}\right)\times\overline{P}_{\alpha\nu\lambda}+O(k). \tag{4.11}$$

The substitution of Eq. (4.9) in Eqs. (4.10) and (4.11) is simplified if the identities (2.35) and (2.36) are first applied. The results are

$$-ik^\mu D_{\mu\lambda}=\frac{i(q-k)_\lambda 2m_\pi^2}{(2q\cdot k-k^2)2f_\pi}+\frac{iq_\lambda}{f_\pi}+O(k), \tag{4.12}$$

$$-ik^\nu K_{\mu\nu\lambda}=2\frac{\partial}{\partial k^\mu}\left\{\frac{[2m_\pi^2F_1(k^2)-3k^2F_2(k^2)](q-k)_\lambda}{(2q\cdot k-k^2)2f_\pi}\right\}+(2f_\pi)^{-1}[-2g_{\mu\lambda}+4q_\mu q_\lambda F_1{}'(0)]. \tag{4.13}$$

When $\vec{k}$ vanishes, we define a variable z:

$$k=(zm_\pi,\vec{0}). \tag{4.14}$$

In terms of the equal-time commutator

$$C_\lambda=\int d^4x\, e^{im_\pi zx_0}\delta(x_0)\langle 0|[\Theta_\mu{}^\mu(x),\mathfrak{F}_{5\lambda}{}^3(0)]|\pi(q)\rangle, \tag{4.15}$$

T_λ has the following asymptotic behavior in z[42]:

$$T_\lambda=-C_\lambda/m_\pi z+O(z^{-2}). \tag{4.16}$$

With the expansion[43]

$$T_\lambda=iq_\lambda X_1+ik_\lambda X_2, \tag{4.17}$$

the assumption that C_0 is finite and independent of z implies an unsubtracted dispersion relation for $X_1(z)$,

$$X_1(z)=\frac{1}{\pi}\int dz'\frac{\mathrm{Im}X_1(z')}{z'-z}, \tag{4.18}$$

and a superconvergence relation for $X_2(z)$,

$$0=\int dz\,\mathrm{Im}X_2(z). \tag{4.19}$$

Equations (4.18) and (4.19) will be considered only in the collinear frame, which is defined by the constraints

$$\vec{k}=\vec{q}=\vec{0}. \tag{4.20}$$

The imaginary part of T_λ is given by the formula

$$\mathrm{Im}T_\lambda=iq_\lambda\mathrm{Im}X_1+ik_\lambda\mathrm{Im}X_2=\tfrac{1}{2}(2\pi)^4\sum_I[\langle 0|\Theta_\mu{}^\mu|I\rangle\langle I|\mathfrak{F}_{5\lambda}|\pi(q)\rangle\delta^4(k-P_I)-\langle 0|\mathfrak{F}_{5\lambda}|I\rangle\langle I|\Theta_\mu{}^\mu|\pi(q)\rangle\delta^4(q-k-P_I)]. \tag{4.21}$$

Let us isolate the contribution of the pion pole at $z=0$ in Eq. (4.21):

$$\mathrm{Im}X_1(z)=-\pi\delta(z)(2f_\pi)^{-1}+\mathrm{Im}\overline{X}_1(z), \tag{4.22}$$

$$\mathrm{Im}X_2(z)=\pi\delta(z)(2f_\pi)^{-1}+\mathrm{Im}\overline{X}_2(z). \tag{4.23}$$

Then Eqs. (4.18) and (4.19) may be written

$$X_1(z)=(2f_\pi z)^{-1}+\frac{1}{\pi}\int dz'\frac{\mathrm{Im}\overline{X}_1(z')}{z'-z}, \tag{4.24}$$

$$0=(2f_\pi)^{-1}+\frac{1}{\pi}\int dz\,\mathrm{Im}\overline{X}_2(z). \tag{4.25}$$

Equations (4.1), (4.2), (4.7), (4.8), (4.12), and (4.13) determine the low-z behavior of $X_1(z)$:

$$2f_\pi X_1(z)=z^{-1}-\tfrac{1}{2}+O(z), \tag{4.26}$$

$$2f_\pi\frac{dX_1(z)}{dz}=-z^{-2}-\tfrac{3}{2}F_2(0)+3m_\pi^2F_1{}'(0)+\tfrac{1}{4}+O(z); \tag{4.27}$$

therefore, Eqs. (4.24), (4.26), and (4.27) imply the sum rules

$$\frac{2f_\pi}{\pi}\int dz\frac{\mathrm{Im}\overline{X}_1(z)}{z}=-\tfrac{1}{2}, \tag{4.28}$$

$$\frac{2f_\pi}{\pi}\int dz\frac{\mathrm{Im}\overline{X}_1(z)}{z^2}=-\tfrac{3}{2}F_2(0)+3m_\pi^2F_1{}'(0)+\tfrac{1}{4}. \tag{4.29}$$

In the collinear frame, only $0^\pm$ states contribute to the sum over a complete set of states $|I\rangle$ in Eq. (4.21). We expect that the pion poles at $z=0$, 2 and σ poles at $z=\pm m_\sigma/m_\pi$ will dominate, so these contributions are explicitly displayed:

$$\mathrm{Im}\overline{X}_1(z) = \pi\delta(z-2)[F_1(4m_\pi^2) - 6F_2(4m_\pi^2)]/2f_\pi$$
$$- \pi\epsilon(z)\delta(z^2 - m_\sigma^2/m_\pi^2)F_\sigma[F_{\sigma\pi}((1-z)^2m_\pi^2) + G_{\sigma\pi}((1-z)^2m_\pi^2)]m_\sigma^2/m_\pi^2 + x_1(z), \tag{4.30}$$

$$\mathrm{Im}\overline{X}_2(z) = -\pi\delta(z-2)[F_1(4m_\pi^2) - 6F_2(4m_\pi^2)]/2f_\pi$$
$$+ \pi\epsilon(z)\delta(z^2 - m_\sigma^2/m_\pi^2)F_\sigma[-F_{\sigma\pi}((1-z)^2m_\pi^2) + G_{\sigma\pi}((1-z)^2m_\pi^2)]m_\sigma^2/m_\pi^2 + x_2(z) \tag{4.31}$$

[the $z=0$ contribution appears in Eqs. (4.22) and (4.23)]. Summation over σ is understood if more than one scalar meson contributes. Diagrams which correspond to the pole terms are displayed in Fig. 2.

Substituting Eqs. (4.30) and (4.31), the sum rules (4.19), (4.28), and (4.29) become

$$(2f_\pi)^{-1}[F_1(4m_\pi^2) - 1 - 6F_2(4m_\pi^2)] = \frac{m_\sigma F_\sigma}{2m_\pi}[F_{\sigma\pi}(+) - G_{\sigma\pi}(+) - F_{\sigma\pi}(-) + G_{\sigma\pi}(-)] + \frac{1}{\pi}\int dz\, x_2(z) \tag{4.32}$$

$$(2f_\pi)^{-1}[\tfrac{1}{2} + \tfrac{1}{2}F_1(4m_\pi^2) - 3F_2(4m_\pi^2)] = \tfrac{1}{2}F_\sigma[F_{\sigma\pi}(+) + F_{\sigma\pi}(-) + G_{\sigma\pi}(+) + G_{\sigma\pi}(-)] - \frac{1}{\pi}\int dz\, \frac{x_1(z)}{z}, \tag{4.33}$$

$$(2f_\pi)^{-1}[\tfrac{1}{4} - \tfrac{1}{4}F_1(4m_\pi^2) + 3m_\pi^2F_1'(0) + \tfrac{3}{2}F_2(4m_\pi^2) - \tfrac{3}{2}F_2(0)] = \frac{m_\pi F_\sigma}{2m_\sigma}[F_{\sigma\pi}(+) + G_{\sigma\pi}(+) - F_{\sigma\pi}(-) - G_{\sigma\pi}(-)] + \frac{1}{\pi}\int dz\, \frac{x_1(z)}{z^2}, \tag{4.34}$$

using a notation established in Sec. III.[41]

The interpretation of Eqs. (4.32)–(4.34) is simplified by changing the variable of integration, z. Consider the parametrization

$$x_i = x_i(s,t) = \mathrm{Im}X_i - \mathrm{Im}(\pi \text{ pole} + \sigma \text{ pole})_i, \quad i = 1, 2 \tag{4.35}$$

where s and t are given in terms of the momenta q, k of Eq. (4.4):

$$t = (q-k)^2, \quad s = k^2 > 0. \tag{4.36}$$

In the collinear frame, we obtain

$$s = m_\pi^2 z^2, \quad t = m_\pi^2(1-z)^2. \tag{4.37}$$

In order to understand the magnitude of the continuum integrals and their behavior as the limit $m_\pi \to 0$ is taken, an integration variable such as s or t should be used instead of z. Keeping only the terms of lowest order in m_π^2, we have

$$\int dz\, x_2(z) = -2\int ds\, \frac{\partial}{\partial t}x_2(s,s) + O(m_\pi^2), \tag{4.38}$$

$$\int dz\, \frac{x_1(z)}{z} = \int ds\, \frac{x_1(s,s)}{s} + O(m_\pi^2), \tag{4.39}$$

$$\int dz\, \frac{x_1(z)}{z^2} = -2m_\pi^2\int ds\, \frac{\partial}{\partial t}\frac{x_1(s,s)}{s} + O(m_\pi^4). \tag{4.40}$$

Nothing unexpected happens in Eqs. (4.38) and (4.39), but Eq. (4.40) shows that there is a hidden factor m_π^2 in Eq. (4.34) [corresponding to the factor $q_\lambda q_\mu$ in Eq. (2.37)], which should be removed before symmetry limits are considered or continuum integrals are neglected.

In the limit of scale invariance, it follows from

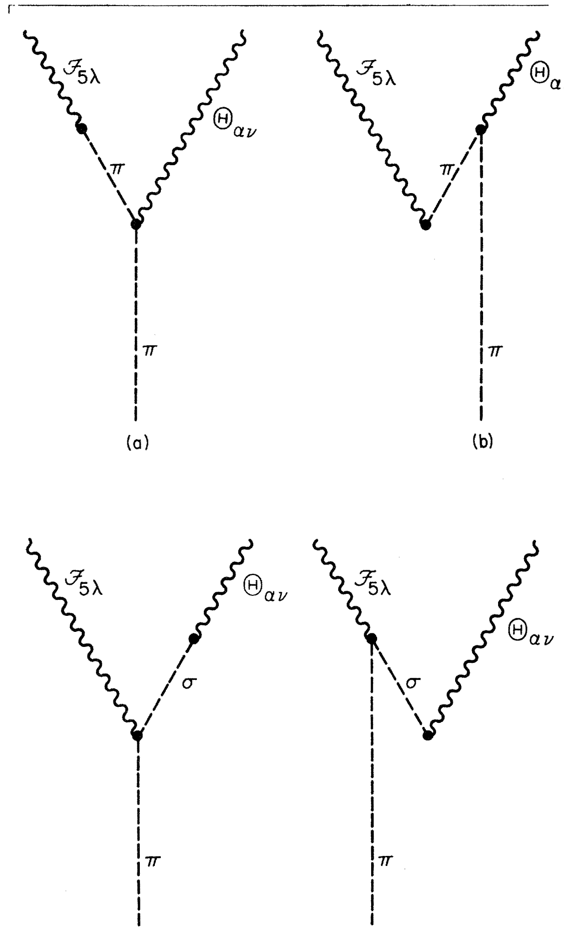

FIG. 2. Diagrams representing poles at (a) $z=0$, (b) $z=2$, (c) $z = m_\sigma/m_\pi$, (d) $z = -m_\sigma/m_\pi$; see Eqs. (4.22), (4.23), (4.30), and (4.31).

Eqs. (2.9) and (2.24) that both sides of the superconvergence relation (4.32) vanish, while Eq. (4.33) reduces to Eq. (2.26). When the artificial factor m_π^2 is removed, Eq. (4.34) becomes

$$(2f_\pi)^{-1}F_1'(0)=F_\sigma F_{\sigma\pi}'(0), \tag{4.41}$$

from which Eq. (2.38) is easily obtained.

Consistent with the result (3.22) of the analysis of Sec. III, we neglect the contribution of the $\epsilon'(1060)$ meson pole to Eqs. (4.32)–(4.34). The importance of showing that the $\epsilon' A_1 \pi$ coupling is small is evident – in these sum rules, the axial-vector form factors for the ϵ' are evaluated at points very close to the position of the A_1 pole. To simplify Eqs. (4.32)–(4.34), we neglect the mass of the pion, a sufficiently accurate approximation for our purposes. Making use of Eqs. (4.38)–(4.40), the relations

$$(2f_\pi)^{-1}\simeq F_\sigma F_{\sigma\pi}(m_\sigma^2)-\frac{1}{\pi}\int ds\,\frac{\partial}{\partial t}\,x_2(s,s), \tag{4.42}$$

$$(2f_\pi)^{-1}\simeq F_\sigma F_{\sigma\pi}(m_\sigma^2)-\frac{1}{2\pi}\int ds\,\frac{x_1(s,s)}{s}, \tag{4.43}$$

$$(2f_\pi)^{-1}[F_1'(0)+3F_2'(0)]\simeq F_\sigma[F_{\sigma\pi}'(m_\sigma^2)+G_{\sigma\pi}'(m_\sigma^2)] -\frac{1}{\pi}\int ds\,\frac{\partial}{\partial t}\,\frac{x_1(s,s)}{s}, \tag{4.44}$$

follow.

Various pole-dominance approximations will now be considered. Let us compare Eqs. (3.9) and (4.43), noting that, with the aid of Eq. (2.23) and the usual neglect of the ϵ' term, Eq. (3.9) may be written

$$(2f_\pi)^{-1}\simeq F_\sigma F_{\sigma\pi}(0)-\frac{3}{2f_\pi\pi}\int ds'\,\frac{f(s')}{s'}. \tag{4.45}$$

Both continuum terms involve semiinfinite integration paths with apparently similar rates of convergence and similar sets of contributing diagrams. So at this level, we are unable to determine that one continuum integral is significantly smaller than the other.

However, numerical agreement of the results of ignoring the continuum integrals of both Eq. (4.43) and Eq. (4.45) would be surprising. According to the usual estimates[44, 45] of the $A_1\sigma\pi$ coupling, the A_1 pole should cause considerable variation in $F_{\sigma\pi}(t)$ between $t=0$ and $t=m_\sigma^2$. In fact, Carruthers[23] has shown that these estimates agree with the result of assuming A_1-pole dominance of $F_{\sigma\pi}(t)$, in which case we have

$$F_{\sigma\pi}(0)/F_{\sigma\pi}(m_\sigma^2)\simeq 1-m_\sigma^2/m_{A_1}^2. \tag{4.46}$$

When continuum integrals are neglected, Eq. (4.45) yields the usual formula

$$F_\sigma G_{\sigma\pi\pi}\simeq m_\sigma^2, \tag{4.47}$$

while Eqs. (4.43) and (4.46) require

$$F_\sigma G_{\sigma\pi\pi}\simeq m_\sigma^2(1-m_\sigma^2/m_{A_1}^2). \tag{4.48}$$

From the point of view of broken scale and chiral invariance, Eqs. (4.47) and (4.48) cannot be distinguished, because only terms $O(m_\sigma^2)$ are determined by symmetry arguments. Numerically, the discrepancy between Eqs. (4.47) and (4.48) amounts to a factor of almost 2.

Of course, this numerical difference could be removed by suitably weakening the assumptions – for example, a tenth scalar meson could be introduced. The author does not believe that such a procedure is necessary at present.[46] By saturating two different dispersion integrals, two estimates of $G_{\sigma\pi\pi}$ have resulted; the difference between these estimates is a measure of the uncertainty involved in predicting $G_{\sigma\pi\pi}$ by arguments based on broken scale invariance. Arguments outside the theory of broken scale invariance allow further progress. The most important theoretical constraint not derived from broken scale invariance is provided by the Adler-Weisberger sum rule for $\pi\pi$ scattering,[38] which requires[44]

$$|F_{\sigma\pi}(0)|\lesssim 1/\sqrt{2}. \tag{4.49}$$

Then most phenomenological estimates[23, 47] of dilaton-baryon couplings favor Eq. (4.48). Having observed the numerical failure of PCDC for $\langle\pi|\Theta_\mu{}^\mu|\pi\rangle$, it is less surprising that σ-pole dominance of $F_2(t)$ is such a crude approximation.

Consider the approximation in which the continuum integral of Eq. (4.44) is neglected. According to Eqs. (2.20) and (2.21), we have

$$F_1'(0)\simeq -3F_2'(0)+2f_\pi F_\sigma F_{\sigma\pi}(m_\sigma^2) \times[2F_{\sigma\pi}'(m_\sigma^2)/F_{\sigma\pi}(m_\sigma^2)-m_\sigma^{-2}]. \tag{4.50}$$

We apply Eqs. (4.43) and (4.48), together with the estimates

$$-3F_2'(0)\simeq F_\sigma G_{\sigma\pi\pi}/m_\sigma^4, \tag{4.51}$$

$$F_{\sigma\pi}'(m_\sigma^2)/F_{\sigma\pi}(m_\sigma^2)\simeq (m_{A_1}^2-m_\sigma^2)^{-1}, \tag{4.52}$$

to obtain

$$F_1'(0)\simeq\frac{m_{A_1}^2+m_\sigma^2}{m_{A_1}^2(m_{A_1}^2-m_\sigma^2)}. \tag{4.53}$$

By ignoring the violation of scale invariance, we can recover our previous conclusion that f dominance of $F_1(t)$ is consistent with the predictions of a scale-invariant theory [see Eq. (2.39)]. However, it is obvious in Eq. (4.53) that scale-breaking effects ruin f-dominance arguments. Our conclusion is supported by recent work of Engels and Höhler.[47]

They estimate the fNN coupling constants from backward dispersion relations for πN scattering, and obtain a result which is three times the value predicted by f dominance.

The application of the method of collinear dispersion relations to related equal-time commutators is straightforward. We close this section with the example

$$\langle 0|[D,\Theta_\mu{}^\mu]|\sigma\rangle = il m_\sigma{}^2 F_\sigma, \tag{4.54}$$

assuming that the dimension of $\Theta_\mu{}^\mu$, l, is unique, apart from a c-number term. Neglecting mixing, we obtain

$$\tfrac{4}{3}m_\sigma{}^2(l+\tfrac{1}{2}) \simeq \langle 0|\Theta_\mu{}^\mu|\sigma,\sigma\rangle_{\text{rest}}. \tag{4.55}$$

The crude estimate

$$\langle 0|\Theta_\mu{}^\mu|\sigma,\sigma\rangle_{\text{rest}} \simeq -\tfrac{4}{3}F_\sigma G_{\sigma\sigma\sigma} + \langle\sigma|\Theta_\mu{}^\mu|\sigma\rangle \tag{4.56}$$

leads to the prediction[48]

$$F_\sigma G_{\sigma\sigma\sigma} \approx (1-l)m_\sigma{}^2 \tag{4.57}$$

for the $\sigma\sigma\sigma$ coupling constant. With ideal mixing of σ and ϵ', the $\sigma\epsilon'\epsilon'$ and $\sigma\sigma\epsilon'$ quark graphs are disconnected, so mixing effects may be small. However, only the term determined by symmetry considerations is displayed, and terms of higher order in $m_\sigma{}^2$ may be significant. Of course, PCDC for $\langle\sigma|\Theta_\mu{}^\mu|\sigma\rangle$ does not fix l in Eq. (4.57).

V. MAGNITUDE OF $SU(2)\times SU(2)$ VIOLATION

Basic to our approach to calculation of soft-meson amplitudes has been the idea that the violation of chiral $SU(2)\times SU(2)$ symmetry is much smaller than the breakdown of conformal invariance. For example, we have assumed that, in the real world, the induced scalar form factor $F_2(t)$ is better approximated by Eq. (2.11) than Eq. (2.9). This implies that $G_{\sigma\pi\pi}$ is $O(m_\sigma{}^2)$ rather than $O(m_\sigma{}^4)$ or $O(m_\pi{}^2)$. However, the position of chiral $SU(3)\times SU(3)$ in the hierarchy of symmetries is less clear.

In general, chiral calculations performed in the limit of scale invariance differ from the usual analyses with $\Theta_\mu{}^\mu \neq 0$. With $\Theta_\mu{}^\mu = 0$, extra insertions arise from diagrams in which the axial-vector current can hook on to an external pion and turn it into a dilaton (and vice versa). Consideration of such insertions was essential in the derivation of Eqs. (2.26) and (2.38) in the scale-invariant limit. A formal example is the threshold amplitude $T(0)$ for forward π^+p scattering, where we show only the dependence on t, the square of the momentum transfer. According to the Adler consistency condition,[49] $T(0)$ vanishes in the limit $m_\pi \to 0$ with $\Theta_\mu{}^\mu \neq 0$. In the limit of scale invariance, the extra insertion changes this result; instead, one gets the formula (*not* to be applied to the real world)

$$T(0) = -2f_\pi F_{\sigma\pi}(0) g_{\sigma NN} = -g_{\sigma NN}{}^2/M_N, \tag{5.1}$$

where the last equality follows from Eqs. (1.7) and (2.26). That there is no contradiction can be seen by explicitly displaying the σ pole in the non-scale-invariant π^+p scattering amplitude:

$$T(t) = \frac{m_\sigma{}^2}{m_\sigma{}^2 - t}\,\frac{g_{\sigma NN}{}^2}{M_N} + \overline{T}(t), \tag{5.2}$$

with $\overline{T}(0) = -g_{\sigma NN}{}^2/M_N$ for both $\Theta_\mu{}^\mu \neq 0$ and $\Theta_\mu{}^\mu = 0$. The limits $m_\sigma{}^2 \to 0$, $t \to 0$ are not interchangeable in Eq. (5.2). Since the Adler consistency condition is in excellent agreement with experiment, chiral $SU(2)\times SU(2)$ symmetry provides a much better description of the real world than does conformal symmetry. On the other hand, results based on chiral $SU(3)\times SU(3)$ symmetry show sufficient deviation from experiment to allow the possibility[50] that $SU(3)\times SU(3)$ and conformal symmetry are violated by the same term in the energy density (i.e., δ equals a c-number). Therefore, it is important to note which $SU(3)\times SU(3)$ results may be affected by dilaton poles.[51]

Evidently, this viewpoint requires that $SU(2)\times SU(2)$ be regarded as a much better symmetry than $SU(3)\times SU(3)$, i.e., $c \approx -\sqrt{2}$ in Eq. (2.1). This has been challenged by Gaillard[52] and by Brandt and Preparata,[53] who prefer $-c \ll \sqrt{2}$, a result based mainly on their analyses of K_{l3} decay. Their main assumption is $\xi(m_K{}^2) \approx \xi(0)$; then a small value of c is required if the confused experimental situation[52] is supposed to favor $\xi(0) \approx -1$. However, when the collinear dispersion relations of Banerjee[54] are examined, it is difficult to avoid the conclusion that $-c \ll \sqrt{2}$ implies $\xi(m_K{}^2) \approx -1$ and $\xi(0) \gg 0$; only by having $c \approx -\sqrt{2}$ can a value $\xi(0) \lesssim -0.5$ be obtained. Then the behavior of the form factors is difficult to explain in terms of dispersion theory. To avoid a mysterious dip in the scalar form factor, λ_+ has to be much larger than the value 0.024 implied by $K^*(890)$ dominance of $f_+(t)$.[55] We doubt that data from K_{l3} decays can be reliably interpreted until all parametrizations are avoided and the form factors are plotted as functions of the momentum transfer squared, so we retain the usual estimate[16] $c \simeq -1.25$.

Recently, Cheng and Dashen[15] obtained the result

$$\sigma_{NN} = \langle N|-\tfrac{1}{3}(\sqrt{2}+c)(\sqrt{2}\,u_0 + u_8)|N\rangle \simeq 110 \text{ MeV} \tag{5.3}$$

by using πN phase shifts, a fixed-t dispersion relation, and the low-energy theorem

$$T(0,0,m_\pi{}^2,m_\pi{}^2) = 4f_\pi{}^2\sigma_{NN} + \text{“}O(m_\pi{}^4)\text{”} \tag{5.4}$$

for the amplitude

$$T(\nu, \nu_B, q^2, q'^2)$$
$$= (m_\pi^2 - q^2)(m_\pi^2 - q'^2)\left(\frac{2f_\pi}{m_\pi^2}\right)^2 i\int d^4x\, e^{iq'\cdot x}\Theta(x_0)$$
$$\times \langle N(p')|[\partial^\mu \mathfrak{F}_{5\mu}{}^3(x), \partial^\nu \mathfrak{F}_{5\nu}{}^3(0)]|N(p)\rangle, \qquad (5.5)$$

with $q = p' + q' - p$, $\nu = (p+p')\cdot(q+q')/4M_N$, and $\nu_B = -q\cdot q'/2M_N$. Since 110 MeV is not much smaller than energies associated with $SU(3)$ breaking, they conclude that $SU(2)\times SU(2)$ and $SU(3)$ violations are comparable in magnitude, which is contrary to expectations that c is near $-\sqrt{2}$. In a theory which does not contain dilatons, their conclusion appears to be unavoidable.

However, a different interpretation is available in dilaton theory. As a subgroup of $SU(3)\times SU(3)$, the physical $SU(3)$ group is distinguished from other $SU(3)$ groups in that its elements leave the vacuum invariant.[56] Then physical $SU(3)$ is not spontaneously broken and perturbation theory in the $SU(3)$ violating parameter makes sense:

$$\langle N|cu_8|N\rangle \simeq \tfrac{1}{2}M_\Sigma + \tfrac{1}{2}M_\Lambda - M_N = 215 \text{ MeV}. \qquad (5.6)$$

This means that dilaton state $|\sigma\rangle$ must be invariant under physical $SU(3)$ transformations in the limit of scale invariance. As scale invariance is broken, the dilaton quality is distributed between the $|\sigma\rangle$ and $|\epsilon'\rangle$ states. Poles in $\langle u_8\rangle$ due to the existence of $|\sigma\rangle$ and $|\epsilon'\rangle$ arise from the nondilaton, or octet, quality of these states. On the other hand, matrix elements of u_0 have σ and ϵ' poles due to the dilaton quality in $|\sigma\rangle$ and $|\epsilon'\rangle$. Therefore the magnitude of $\langle N|u_0|N\rangle$ is $O(m_\pi^2 M_N/m_\sigma^2)$, much larger than $\langle N|u_8|N\rangle$. In general, we expect

$$\langle\psi|u_0|\psi\rangle \gg \langle\psi|u_8|\psi\rangle \qquad (5.7)$$

for all one-particle rest states $|\psi\rangle$ except $|\sigma\rangle$ and $|0^-, \underline{8}\rangle$.

Because of these observations, there is no reason to abandon either the $(\underline{3}, \underline{3}^*)\oplus(\underline{3}^*, \underline{3})$ form of the chiral $SU(3)\times SU(3)$ violating term in the energy density or the value -1.25 for c, if there is a dilaton. In order to apply Eq. (5.3) in a dilaton theory, we should first check the validity of neglecting the terms "$O(m_\pi^4)$" in Eq. (5.4), where the next order in m_π^2 is given by

$$\text{“}O(m_\pi^4)\text{”} = m_\pi^4 \frac{\partial^2}{\partial q^2 \partial q'^2} T(0,0,0,0) + \text{“}O(m_\pi^6)\text{”}. \qquad (5.8)$$

The pion poles at $q^2 = q'^2 = m_\pi^2$ do not contribute to Eq. (5.8). Parametrizing the failure of PCAC in terms of a "heavy pion," π^*, the contribution of the dilaton pole is

$$\left(\frac{2f_\pi}{m_\sigma}\right)^2 g_{\sigma NN}\, i\int d^4x\, e^{iq'\cdot x}$$
$$\times \langle\sigma|T(\partial^\mu \mathfrak{F}_{5\mu}(x), \partial^\nu \mathfrak{F}_{5\nu}(0))|0\rangle_{(\text{no }\pi\text{ poles})}$$
$$= O(M_N(2f_\pi)^4 \langle 0|\partial^\mu \mathfrak{F}_{5\mu}|\pi^*\rangle^2/(m_\sigma m_{\pi^*})^2), \qquad (5.9)$$

with $\langle 0|\partial^\mu \mathfrak{F}_{5\mu}|\pi^*\rangle = O(m_\pi^2/2f_\pi)$, so the correction terms have magnitude $O(4f_\pi^2 M_N m_\pi^4/(m_\sigma m_{\pi^*})^2)$. Neglect of such terms appears to be a satisfactory approximation; a corollary is that the corresponding terms for KN scattering should *not* be thrown away: $m_K^4/m_\sigma^2 = O(m_K^2)$. Therefore we use Eqs. (5.3) and (5.6) to obtain $\langle N|u_0|N\rangle = -1280$ MeV and

$$\langle N|u|N\rangle = \frac{3}{\sqrt{2}(\sqrt{2}+c)}\sigma_{NN} + \left(\frac{1}{c\sqrt{2}} - 1\right)\langle N|cu_8|N\rangle$$
$$\simeq 1060 \text{ MeV}. \qquad (5.10)$$

Within the 20% accuracy of Eq. (5.10), we have

$$\langle N|u|N\rangle \simeq M_N, \quad \langle N|\hat{\Theta}_{00}|N\rangle \simeq -\langle N|\delta|N\rangle, \qquad (5.11)$$

so the formula

$$\langle N|(l_\delta + 4)\delta + (l_u + 4)u|N\rangle = \langle N|\Theta_\mu{}^\mu|N\rangle = M_N \quad (5.12)$$

suggests $\delta = c$-number and $l_u = -3$; however, we are unable to exclude the possibility $-\langle N|\delta|N\rangle = O(M_N)$.

If Eq. (2.1) is accepted and $SU(2)\times SU(2)$ is a much better symmetry than $SU(3)\times SU(3)$, the result of Cheng and Dashen is strong evidence that conformal invariance is spontaneously broken.

VI. CONCLUDING REMARKS

We have found that mixing and scale-violating effects can strongly affect the accuracy of soft-dilaton results. The most important case arises in calculating the $\sigma\pi\pi$ coupling constant. Applying the method of collinear dispersion relations, we find the expression

$$\frac{g_{\sigma\pi\pi}}{g_{\sigma NN}} \simeq \frac{m_\sigma^2}{2m_\pi M_N}\left(1 - \frac{m_\sigma^2}{m_{A_1}^2}\right) \qquad (6.1)$$

for the dimensionless constant $g_{\sigma\pi\pi} = (2m_\pi)^{-1}G_{\sigma\pi\pi}$. Equation (6.1) follows from Eq. (4.48) and PCDC for baryons. The original estimate[7,13] of this ratio did not contain the factor $(1 - m_\sigma^2/m_{A_1}^2)$. Symmetry considerations fail to distinguish the new formula (6.1) from the old one, so, as stated previously,[7] only an order-of-magnitude prediction for $\Gamma_{\sigma\to\pi\pi}$ can be obtained. The uncertainty of the prediction is large enough to include most experimental and theoretical estimates, such as those derived from the $\pi\pi$ Adler-Weisberger sum rule, PCDC for baryons, and the meager data available for $g_{\sigma\pi\pi}$ and $g_{\sigma NN}$.

In analogy with our previous discussion[7] of the connection between the scaling behavior of the axial-vector current and the induced scalar form factor $F_2(t)$, the standard assumption (2.29) for the conformal transformation properties of the axial-vector current provides information about the spin-2 form factor $F_1(t)$ of $\langle\pi|\Theta_{\mu\nu}|\pi\rangle$. The scale-invariant prediction is Eq. (2.38), which appears to confirm the validity of f dominance of $F_1(t)$. However, when the effects of scale-invariance breaking are included by saturating the appropriate collinear dispersion relation [Eq. (4.29)], we obtain the prediction (4.53) for $F_1'(0)$. The disagreement with f dominance involves a factor of 3. Since the data for fNN coupling constants[47] display similar disagreement with f dominance, it appears that f mesons do not couple universally, and our prediction is not contradicted.

Because of our assumption that scale invariance is spontaneously violated, we can offer an alternative interpretation of the discovery of Cheng and Dashen[15] that the current-algebraic "σ term" for πN scattering equals 110 MeV. This result may be viewed as confirmation of our approach to broken symmetry: The limits of $SU(3)\times SU(3)$ and scale invariance are accompanied by the appearance of $(0^{--}, \underline{8})$ and $(0^{++}, \underline{1})$ Nambu-Goldstone bosons, and the violation of $SU(2)\times SU(2)$ is small compared with that of $SU(3)\times SU(3)$ and conformal symmetry. Thus, the "σ term" for $\pi\pi$ scattering is given by either PCAC or the variational principle, but the same methods do not apply when K or η mesons are involved.[50] Similarly, the method of Cheng and Dashen applies to πN, not KN or ηN scattering.

In the author's opinion, there is no reason to doubt that the breakdown of conformal invariance is spontaneous.

A brief summary of this paper was presented at the Coral Gables Conference on Fundamental Interactions at High Energy, University of Miami, January 20–22, 1971; there the author learned of a related discussion of tensor-meson dominance due to Raman.[57]

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$$\mathfrak{F}^a_{5\mu} = \mathfrak{G}^a_{5\mu} - (m_a{}^2 + \partial^2)^{-1}\partial_\mu\partial_\nu \mathfrak{G}^{\nu a}_5 ,$$

$$\mathcal{T}^a_{\alpha\beta} = \mathfrak{U}^a_{\alpha\beta} + C^a(\partial_\alpha \mathfrak{F}^a_\beta - \partial_\beta \mathfrak{F}^a_\alpha)$$

$$- (\mu_a + \partial^2)^{-1}(\partial_\alpha\partial^\gamma \mathfrak{U}^a_{\gamma\beta} - \partial_\beta\partial^\gamma \mathfrak{U}^a_{\gamma\alpha}),$$

where $\langle \mathfrak{G}^a_{5\mu}\rangle$, $\langle \mathfrak{U}^a_{\alpha\beta}\rangle$, and $\langle \mathfrak{Q}^a_\beta\rangle$ [defined by $\mathfrak{F}^a_\beta = \mu_a{}^2(\mu_a{}^2 + \partial^2)^{-1}\mathfrak{Q}^a_\beta$] do not have poles at $t = m_a{}^2$, $\mu_a{}^2$, and $\mu_a{}^2$, respectively. Between finite-mass states at infinite momentum, one finds $F^a_5 \to G^a_5 = \int d^3x\, \mathfrak{G}^a_{50}$ and $\int d^3x\, \mathcal{T}^a_{\alpha 0} \to \int d^3x\, \mathfrak{U}^a_{\alpha 0}$. The divergence equations are

$$\partial^\mu \mathfrak{F}^a_{5\mu} = m_a{}^2(m_a{}^2 + \partial^2)^{-1}\partial^\mu \mathfrak{G}^a_{5\mu} ,$$

$$\partial^a \mathcal{T}^a_{\alpha\beta} = \mu_a{}^2(\mu_a{}^2 + \partial^2)^{-1}\partial^\alpha[\mathfrak{U}^a_{\alpha\beta} + C^a(\partial_\alpha \mathfrak{Q}^a_\beta - \partial_\beta \mathfrak{Q}^a_\alpha)] .$$

[30]C. G. Callan and P. Carruthers (unpublished); P. Carruthers, Ref. 1. The σ-pole term cannot be removed from the definitions (1.2a) and (1.2b) of D and K_μ; in the limit of scale invariance, the pole term causes exp $(i\alpha D)$ and exp $(i\beta_\mu K^\mu)$ to generate a continuum of vacua.

[31]The role of the Poincaré group as a subgroup of the conformal group is analogous to that of physical $SU(3)$ as a subgroup of chiral $SU(3)\times SU(3)$. The Poincaré and $SU(3)$ groups provide classifications of particle states because their generators annihilate the vacuum. The other chiral and conformal generators do not have this property, so the corresponding classification schemes for particle states need not appear.

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