Inequalities for the $\pi\pi$ Partial Waves from the Properties of a Class of Orthogonal Polynomials*

A. P. Balachandran
Physics Department, Syracuse University, Syracuse, New York 13210†
and Tata Institute of Fundamental Research, Bombay, India

and

Maurice L. Blackmon
Physics Department, Syracuse University, Syracuse, New York 13210
(Received 30 December 1970)

The $\pi\pi$ partial-wave amplitudes are the projections of the corresponding scattering amplitudes with respect to the Legendre polynomials with appropriate arguments. In the past few years, several inequalities for the $\pi\pi$ partial-wave amplitudes have been obtained by using analyticity and unitarity. We show that many of these inequalities may be generalized into inequalities for the projections of the $\pi\pi$ scattering amplitudes with respect to suitable classes of orthogonal polynomials. The generalization yields several new inequalities for the $\pi\pi$ partial waves in the region $0 \le s \le 4m_\pi^2$. In particular, we derive such inequalities involving only s, p, and d waves.

I. INTRODUCTION

In previous papers,[1] the Martin inequalities[2] on the partial-wave amplitudes of the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ were utilized in conjunction with the crossing symmetry of the $\pi\pi$ scattering amplitudes to derive an infinite number of inequalities involving the s- and p-wave $\pi\pi$ partial waves. In the present work, we extend the results of Refs. 1 and 2 by introducing certain classes of functions which are non-negative on the Mandelstam triangle and associated classes of orthogonal polynomials. The inequalities given in Refs. 1 and 2 may then be generalized in a simple way into inequalities for the projections of the $\pi\pi$ scattering amplitudes with respect to these orthogonal polynomials. The striking correspondence between the properties of the usual partial waves and these "generalized partial waves" owes its origin to the fact that the classes of orthogonal polynomials we consider and their functions of the second kind have positivity properties which are very similar to those of the Legendre polynomials and their functions of the second kind. The existence of such close analogies between the nature of the ordinary partial waves and of the generalized partial waves suggests that the latter may be useful in other contexts in scattering theory. The inequalities on the generalized partial waves may be rewritten as inequalities on the canonical partial waves. The latter lead to new results when they form an extension of the inequalities derived in Ref. 1. However, the inequalities which come from the extension of the Martin inequalities are implied by the work of Yndurain,[3] and at least partially implied by the work of Common[4] as well.

Section II generalizes Martin's results while Sec. IV generalizes our previous results using methods developed by Balachandran *et al.*[5] Unlike the inequalities in Sec. IV, those in Sec. II do not involve $\pi\pi$ s waves. In Sec. III, we establish that at least a subset of the inequalities in Sec. II are implied by a representation theorem on $\pi\pi$ partial waves proved independently by Yndurain,[3] by Common,[4] and by Froissart.[6] (Unlike Common and Yndurain, Froissart does not consider the positivity properties of the weight function in the representation nor its consequences on the nature of the partial waves.) We conclude the text of the paper with some general remarks in Sec. V. Appendix A contains a discussion of the properties of orthogonal polynomials. Appendix B describes some new inequalities involving the (usual) $\pi\pi$ s, p, and d waves.

II. INEQUALITIES FOR $l \geq 2$ PARTIAL WAVES

We will denote by $A^{(0)}(s,t)$ and $A^{(c)}(s,t)$ the scattering amplitudes which in the s channel describe the reactions $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$, respectively. The pion mass will be taken to be $\frac{1}{2}$ so that $s+t+u=1$. The partial-wave expansion of $A^{(i)}$ in the s and t channels are

$$A^{(i)}(s,t) = \sum_{l=0}^{\infty}(2l+1)a_l^{(i)}(s)P_l(z_s) = \sum_{l=0}^{\infty}(2l+1)b_l^{(i)}(t)P_l(z_t), \qquad (2.1)$$

where

$$z_s = 1+2t/(s-1), \quad z_t = 1+2s/(t-1).$$

Also, $a_l^{(i)}(s)=0$ if l is odd and for $i=0$, $a_l^{(0)}(s)$

$= b_l^{(0)}(s)$.

Let

$$\Delta = \{x | x \in [-1, +1]\} \otimes \{\tau | \tau \in [0, 1]\} . \quad (2.2)$$

We shall denote by $\mathcal{C}$ the class of functions $\omega(x,\tau)$ which are defined on Δ and fulfill the following properties on Δ:

$$\text{(a)}\ \ \omega(x,\tau) \geq 0 , \quad (2.3)$$

$$\text{(b)}\ \ \int_{-1}^{1} dx\ \omega(x,\tau)x^{\nu} < \infty, \quad \nu = 0, 1, 2, \ldots . \quad (2.4)$$

The subset of functions $\omega(x,\tau)$ of class $\mathcal{C}$ which are invariant under $x \to -x$ will be denoted by $\mathcal{C}_s$. Thus

$$\text{(c)}\ \ \omega(-x,\tau) = \omega(x,\tau) \quad (2.5)$$

if $\omega \in \mathcal{C}_s$. The orthogonal polynomial $P_l^{\omega}(x)$ relative to the weight $\omega \in \mathcal{C}$ is by definition a polynomial of precise degree l in x for each τ in the interval $[0, 1]$ for which orthogonality relations

$$\int_{-1}^{1} dx\ \omega(x,\tau)P_l^{\omega}(x)x^{\nu} = 0,$$

$$\nu = 0, 1, 2, \ldots, l-1;\ l = 1, 2, 3, \ldots \quad (2.6)$$

are valid for τ in the same interval. The dependence of P_l^{ω} on τ has been suppressed. We will further assume that P_l^{ω} is so normalized that

$$P_l^{\omega}(1) > 0, \quad \tau \in [0, 1]. \quad (2.7)$$

The method of construction of P_l^{ω} is well known in the literature. [See, for example, Refs. 7 or 8, or Appendix A of Ref. 5(b).] When $\omega(x,\tau)$ is a polynomial in x for fixed τ, ωP_l^{ω} can be expressed in a particularly simple form in terms of Legendre polynomials. Appendix A contains this formula. It is also useful to have a systematic method for the construction of non-negative functions on Δ. Representation theorems for non-negative functions are available in the literature[9] and may be adapted for this purpose.

With each P_l^{ω}, we can associate a "function of the second kind" Q_l^{ω} by the formula

$$Q_l^{\omega}(x) = \frac{1}{2}\int_{-1}^{1} dy\ \omega(y,\tau)\frac{P_l^{\omega}(y)}{x-y},$$

$$x \notin [-1, +1];\quad \tau \in [0, 1]. \quad (2.8)$$

We have adopted a definition of Q_l^{ω} which differs from the definition conventionally used in the literature essentially be a factor of ω. In Appendix A, we will prove that for $0 \leq \tau \leq 1$ and for any $\omega \in \mathcal{C}$,

$$Q_l^{\omega}(x) \geq 0,\ \ l = 0, 1, 2, \ldots;\ x > 1 \quad (2.9)$$

$$Q_l^{\omega}(x) \leq 0,\ \ l = 0, 2, 4, \ldots;\ x < -1 \quad (2.10)$$

$$Q_l^{\omega}(x) \geq 0,\ \ l = 1, 3, 5, \ldots;\ x < -1 . \quad (2.11)$$

We wish to show the following: Let

$$a_l^{(i)}(s,\omega) = \tfrac{1}{2}\int_{-1}^{1} dz_s \omega(z_s, s)P_l^{\omega}(z_s)A^{(i)}(s,t),$$

$$i = 0, c;\ \omega \in \mathcal{C};\ s \in [0, 1]. \quad (2.12)$$

Then

$$a_l^{(i)}(s,\omega) \geq 0 \text{ for } l = 2, 4, 6, \ldots \text{ and } s \in [0, 1). \quad (2.13)$$

Therefore, if

$$\omega(z_s, s)P_l^{\omega}(z_s) = \sum_{\nu=0}^{\infty} \omega_{\nu l}(s)P_{\nu}(z_s), \quad (2.14)$$

where $P_{\nu}(z_s) \equiv P_{\nu}^{\omega=1}(z_s)$ are the Legendre polynomials,

$$\sum_{\nu=0} \omega_{\nu l}(s)a_{\nu}^{(i)}(s) \geq 0$$

$$\text{for } i = 0, c;\ l = 2, 4, 6 \ldots;\ s \in [0, 1). \quad (2.15)$$

Because of (2.6), $\omega_{\nu l}(s) = 0$ if $\nu < l$, and therefore the sum on ν in (2.14) or (2.15) can be restricted to $\nu \geq l$.

The inequalities (2.13) are extensions of the Martin inequalities[2]

$$a_l^{(i)}(s) \geq 0 \text{ for } i = 0, c;\ l = 2, 4, 6 \ldots;\ s \in [0, 1) \quad (2.16)$$

to a wide class of "generalized partial waves." Their derivation is also very similar to that of Martin. Thus, $A^{(i)}(s,t)$ has a fixed-s dispersion relation with two subtractions when $0 \leq s < 1$.[10] By projecting out $a_l^{(i)}(s,\omega)$ from this representation, we obtain the generalized Froissart-Gribov formula

$$a_l^{(i)}(s,\omega) = \frac{2}{\pi(1-s)}\int_1^{\infty} dt' A_t^{(i)}(s,t')$$

$$\times \left[Q_l^{\omega}\left(\frac{2t'}{1-s} - 1\right) - Q_l^{\omega}\left(1 - \frac{2t'}{1-s}\right)\right],$$

$$i = 0, c;\ l = 2, 3, 4, \ldots;\ s \in [0, 1). \quad (2.17)$$

The absorptive part in the t channel of $A^{(i)}(s,t)$ has been denoted by $A_t^{(i)}(s,t)$. The t channels of $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ refer to the elastic processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^- \to \pi^0\pi^-$, respectively, and therefore,[11]

$$A_t^{(i)}(s,t') \geq 0 \text{ for } i = 0, c;\ t' \geq 1;\ s \in [0, 1). \quad (2.18)$$

The result (2.13) follows from (2.17) when we use (2.9), (2.10), and (2.18).

We illustrate (2.13) and its consequence (2.15) by a simple example. Let

$$\omega(z_s, s) = (1 - z_s^2). \quad (2.19)$$

The polynomials P_l^{ω} may then be identified with the associated Legendre polynomials $P_l^{(1,1)}$. It follows from Christoffel's formula that

$$(1-z_s^2)P_l^\omega(z_s)=c_l[P_l(z_s)-P_{l+2}(z_s)], \quad c_l=\text{const}>0. \tag{2.20}$$

[See, for example, Appendix A.] The content of (2.13) and (2.15) is thus

$$a_l^{(i)}(s)-a_{l+2}^{(i)}(s)\geq 0 \quad \text{for } i=0,c;\ l=2,4,6,\ldots;\ s\in[0,1). \tag{2.21}$$

A refinement of the above result becomes possible when we notice that the argument of the first Q_l^ω in (2.17), for example, ranges over the interval $[x_0(s),\infty)$, where

$$x_0(s)=(1+s)/(1-s). \tag{2.22}$$

Thus, for the first Q_l^ω, we have not yet utilized the interval $[1, x_0(s))$ $[0<s<1]$ where, too, the positivity of Q_l^ω persists by (2.9). To remedy this defect, let us define

$$\gamma(s,\rho)=(1-\rho)+\rho x_0(s), \tag{2.23}$$

where ρ is a constant. Then, for $\omega\in\mathcal{C}$,

$$\frac{1}{2}\int_{-\gamma(s,\rho)}^{\gamma(s,\rho)} dz_s\,\omega\left(\frac{z_s}{\gamma(s,\rho)},s\right)P_l^\omega\left(\frac{z_s}{\gamma(s,\rho)}\right)A^{(i)}(s,t)$$
$$=\frac{2}{\pi(1-s)}\int_1^\infty dt' A_t^{(i)}(s,t')\left[Q_l^\omega\left(\frac{1}{\gamma(s,\rho)}\left(\frac{2t'}{1-s}-1\right)\right)-Q_l^\omega\left(\frac{1}{\gamma(s,\rho)}\left(1-\frac{2t'}{1-s}\right)\right)\right]$$
$$\geq 0 \quad \text{for } i=0,c;\ l=2,4,6,\ldots;\quad \rho\in[0,1];\ s\in[0,1). \tag{2.24}$$

In terms of the partial waves $a_l^{(i)}(s)$, Eq. (2.24) is equivalent to the following: *The series*

$$a_l^{(i)}(s,\omega,\rho)=\sum_{L\geq 0}(2L+1)a_L^{(i)}(s)\times\tfrac{1}{2}\int_{-1}^{1}dz_s\,\omega(z_s,s)\,P_l^\omega(z_s)P_L(\gamma(s,\rho)z_s),$$
$$i=0,c;\ l=2,4,6,\ldots,;\ \omega\in\mathcal{C} \tag{2.25}$$

is non-negative when $s\in[0,1)$ *and* $\rho\in[0,1]$.

III. CONNECTION WITH THE COMMON-YNDURAIN INEQUALITIES

Common[4] and Yndurain[3] (see also Ref. 6) have shown that a necessary condition for the existence of the Froissart-Gribov representation for $a_L^{(i)}(s)$ for $s\in[0,1)$ and $L=2,4,6,\ldots$ is that $a_L^{(i)}(s)$ fulfill certain inequalities which are necessary and sufficient for them to be the moments of a non-negative function $\phi^{(i)}(s,\xi)$ over an appropriate interval, that is,

$$a_L^{(i)}(s)=\int_0^{1/r(s)}d\xi\,\phi^{(i)}(s,\xi)\xi^L,\quad i=0,c;\quad L=2,4,6,\ldots;\ s\in[0,1) \tag{3.1a}$$

where

$$r(s)=x_0(s)+[x_0(s)^2-1]^{1/2} \tag{3.1b}$$

and

$$\phi^{(i)}(s,\xi)\geq 0 \quad \text{when } s\in[0,1) \text{ and } \xi\in[0,1/r(s)]. \tag{3.1c}$$

Here (3.1c) is a consequence of the positivity property (2.18) of the absorptive parts $A_t^{(i)}$. We will now show that the non-negativity of the series (2.25) [which for $\rho=0$ reduces to (2.15)] is implied by the above representation provided $\omega\in\mathcal{C}_s$.

On inserting (3.1a) in (2.25) and using

$$S(\gamma(s,\rho)z_s,\xi)\equiv\sum_{L=0}^{\infty}(2L+1)\xi^L P_L(\gamma(s,\rho)z_s)=\frac{1-\xi^2}{[1-2\gamma(s,\rho)z_s\xi+\xi^2]^{3/2}}, \tag{3.2}$$

we find

$$a_l^{(i)}(s,\omega,\rho)=\int_0^{1/r(s)}d\xi\,\phi^{(i)}(s,\xi)\times\tfrac{1}{2}\int_{-1}^{1}dz_s\,\omega(z_s,s)P_l^\omega(z_s)S(\gamma(s,\rho)z_s,\xi),$$
$$l=2,4,6,\ldots;\ s\in[0,1);\ \rho\in[0,1]. \tag{3.3}$$

[The identity (3.2) follows easily from the expression for the generating function for Legendre polynomials in terms of Legendre polynomials.] We will prove in Appendix A that

$$\int_{-1}^{1}dz_s\,\omega(z_s,s)P_l^\omega(z_s)z_s^{\nu}\geq 0,\quad l=0,1,2,\ldots;\ \nu=0,1,2,\ldots;\ s\in[0,1] \tag{3.4}$$

provided $\omega\in\mathcal{C}_s$. In (3.3), let us therefore write

$$S(\gamma(s,\rho)z_s,\xi)=\sum_{\nu=0}^{\infty}\frac{c_\nu}{\nu!}z_s^{\nu}, \tag{3.5}$$

where

$$c_\nu = (1-\xi^2)\frac{[3\gamma(s,\rho)\xi][5\gamma(s,\rho)\xi]\cdots[(2\nu+1)\gamma(s,\rho)\xi]}{(1+\xi^2)^{(2\nu+3)/2}}. \tag{3.6}$$

Since $c_\nu \geq 0$ in the region of interest in (3.3), the result $a_l^{(i)}(s,\omega,\rho) \geq 0$ for the specified restrictions on s, ω, and ρ follows from (3.4).

We have not succeeded in showing that (3.1) implies (2.25) when $\omega(-z_s, s) \neq \omega(z_s, s)$, nor do we know whether (2.25) implies (3.1).

We refer the reader to Yndurain[3] for the statement of the necessary and sufficient conditions on $a_L^{(i)}(s)$ $[L=2,4,6,\ldots,\ s\in[0,1)]$ for the existence of their Froissart-Gribov representations with absorptive parts which fulfill (2.18). These conditions of course imply the inequalities (2.25).

IV. INEQUALITIES WHICH INVOLVE S WAVES AND HIGHER WAVES

In this section we will investigate the consequences of the preceding results on the partial-wave amplitudes $a_l^{(i)}(s)$ and $b_l^{(i)}(t)$ when the crossing symmetry of $A^{(i)}(s,t)$ is imposed. Since we have found no good way of evaluating the impact of the inequalities (2.25) on the nature of these partial waves except when $\rho=0$ and $\omega\in\mathcal{C}_s$, we shall hereafter concentrate exclusively on the inequalities (2.13) with $\omega\in\mathcal{C}_s$.

In previous research,[1] a number of inequalities have been obtained involving the s and p waves $a_0^{(i)}(s)$, $b_0^{(i)}(t)$, and $b_1^{(i)}(t)$ starting from the Martin inequalities (2.16) and using crossing. A main point of this section is that when $\omega\in\mathcal{C}_s$, those inequalities may be generalized into ones involving $a_0^{(i)}(s,\omega)$, $b_0^{(i)}(t,\omega)$, and $b_1^{(i)}(t,\omega)$ provided we use (2.13) instead of (2.15). The definition of $b_l^{(i)}(t,\omega)$ for $\omega\in\mathcal{C}$ is

$$b_l^{(i)}(t,\omega) = \tfrac{1}{2}\int_{-1}^{1} dz_t\,\omega(z_t,t)P_l^\omega(z_t)A^{(i)}(s,t),\quad i=0,c;\ t\in[0,1]. \tag{4.1}$$

We have already seen in Sec. III that $a_l^{(i)}(s,\omega)$ and $a_l^{(i)}(s)$ enjoy analogous positivity properties. The accumulation of such common properties between $a_l^{(i)}(s,\omega)$, $b_l^{(i)}(t,\omega)$, and $a_l^{(i)}(s)$, $b_l^{(i)}(t)$ tends to suggest that the former set of generalized partial waves may have a significance in scattering theory which is not yet properly understood. Of more immediate concern for us is the fact that when the relevant $\omega(z_s,s)$ are low-order polynomials in s and t, the inequalities of this section can be readily converted into ones involving the first few partial waves. Such inequalities involving s, p, d waves are studied in Appendix B.

The following results may be proved: *Let* $\mathcal{H}^{(0)}(s,t)$ *be any antisymmetric function of s and t such that, for* z_s, $s\in\Delta$, *and for some* $\omega\in\mathcal{C}_s$,

$$\mathcal{H}^{(0)}(s,t) = \omega(z_s,s)\sum_{l=0}^{\infty} h_l^{(0)}(s,\omega)P_l^\omega(z_s), \tag{4.2}$$

where

$$h_l^{(0)}(s,\omega) \geq 0,\quad l=2,4,6,\ldots;\ s\in[0,1]. \tag{4.3}$$

Then

$$-\int_0^1 ds(1-s)h_0^{(0)}(s,\omega)a_0^{(0)}(s,\omega) \geq 0. \tag{4.4}$$

For the proof we refer the reader to paper I, where a similar result for $\omega=1$ is derived. Note that due to (A34) in Appendix A, $a_l^{(i)}(s,\omega)=0$ if $\omega\in\mathcal{C}_s$ and l is odd. The expansion of $a_0^{(0)}(s,\omega)$ in (4.4) in terms of $a_l^{(0)}(s)$ may be achieved, if desired, by expressing ωP_0^ω as a series in Legendre polynomials as in the transition from (2.13) to (2.15).

A somewhat different statement which is equally valid for $i=0$ and c is the following:

Let

$$\mathcal{H}(s,t) = \omega_1(z_s,s)\sum_{l=0}^{\infty} h_l(s,\omega_1)P_l^{\omega_1}(z_s) \tag{4.5}$$

$$= \omega_2(z_t,t)\sum_{l=0}^{1} g_l(t,\omega_2)P_l^{\omega_2}(z_t), \tag{4.6}$$

where $\omega_1\in\mathcal{C}_s$, $\omega_2\in\mathcal{C}$, *and*

$$h_l(s,\omega_1) \geq 0,\quad l=2,4,6,\ldots;\ s\in[0,1]. \tag{4.7}$$

Then

$$\sum_{l=0}^{1}\int_0^1 ds(1-s)[g_l(s,\omega_2)b_l^{(i)}(s,\omega_2) - h_l(s,\omega_1)a_l^{(i)}(s,\omega_1)] \geq 0,\quad i=0,c. \tag{4.8}$$

The proof is once more patterned on the work of paper I.

In previous papers we constructed a class of $\mathcal{H}^{(0)}$ and $\mathcal{H}$ which fulfilled the preceding requirements for the choice $\omega=\omega_1=\omega_2=1$. (The notation there was $\mathcal{H}^{(0)}=H^{(0)}$, $\mathcal{H}=H^{(c)}$.) The constructions were essentially based on the observations that

$$\tfrac{1}{2}\int_{-1}^{1} dz_s P_l(z_s)t^\lambda \geq 0,\quad l=2,4,6,\ldots;\ -1<\lambda\leq 0 \text{ or } 2m+1\leq\lambda\leq 2m+2\ (m=0,1,2,\ldots);\ s\in[0,1) \tag{4.9}$$

$$\tfrac{1}{2}\int_{-1}^{1} dz_s P_l(z_s)t^\lambda \leq 0,\quad l=2,4,6,\ldots;\ 0\leq\lambda\leq 1;\ s\in[0,1) \tag{4.10}$$

$$\tfrac{1}{2}\int_{-1}^{1} dz_s P_l(z_s)(1-2t)^n \geq 0,$$

$$l=2,4,6,\ldots;\ n=0,1,2,\ldots;\ s\in[0,1]. \qquad (4.11)$$

Now it will be shown in Appendix A that the following analogs of (4.9)–(4.11) are valid for projections with respect to ωP_l^ω when $\omega\in\mathcal{C}_s$:

$$\tfrac{1}{2}\int_{-1}^{1} dz_s \omega(z_s,s)P_l^\omega(z_s)t^\lambda \geq 0,$$

$$l=2,4,6,\ldots;\ -1<\lambda\leq 0$$

$$\text{or } 2m+1\leq\lambda\leq 2m+2\ (m=0,1,2\ldots);\ s\in[0,1) \qquad (4.12)$$

$$\tfrac{1}{2}\int_{-1}^{1} dz_s \omega(z_s,s)P_l^\omega(z_s)t^\lambda \leq 0,$$

$$l=2,4,6,\ldots;\ 0\leq\lambda\leq 1;\ s\in[0,1) \qquad (4.13)$$

$$\tfrac{1}{2}\int_{-1}^{1} dz_s \omega(z_s,s)P_l^\omega(z_s)(1-2t)^n \geq 0,$$

$$l=2,4,6,\ldots;\ n=0,1,2;\ s\in[0,1]. \qquad (4.14)$$

We will denote by $H^{(0)}$ and H the functions $\mathcal{H}^{(0)}$ and $\mathcal{H}$ which refer to the case $\omega=\omega_1=\omega_2=1$. Then, because of (4.12)–(4.14), several sets of allowed $\mathcal{H}^{(0)}$ and $\mathcal{H}$ may be constructed by suitably choosing $H^{(0)}$ and H from paper I and multiplying them with appropriate weight functions. We have the following results: *Let $H^{(0)}$ and H be defined in any one of the following ways:*

$$H^{(0)}(s,t)=t^\lambda s - ts^\lambda,\quad H(s,t)=t^\lambda s,$$

$$-1<\lambda\leq 0 \text{ or } 2m+1\leq\lambda\leq 2m+2;\ m=0,1,2,\ldots \qquad (4.15)$$

$$H^{(0)}(s,t)=-t^\lambda s+ts^\lambda,\quad H(s,t)=-t^\lambda s,$$

$$0\leq\lambda\leq 1 \qquad (4.16)$$

$$H^{(0)}(s,t)=t^\lambda(1-s)-(1-t)s^\lambda,\quad H(s,t)=t^\lambda(1-s),$$

$$-1<\lambda\leq 0 \text{ or } 2m+1\leq\lambda\leq 2m+2;\ m=0,1,2,\ldots \qquad (4.17)$$

$$H^{(0)}(s,t)=-t^\lambda(1-s)+(1-t)s^\lambda,\quad H(s,t)=-t^\lambda(1-s),$$

$$0\leq\lambda\leq 1 \qquad (4.18)$$

$$H^{(0)}(s,t)=(1-2t)^n s - t(1-2s)^n,\quad H(s,t)=(1-2t)^n s,$$

$$n=0,1,2,\ldots \qquad (4.19)$$

$$H^{(0)}(s,t)=(1-2t)^n(1-s)-(1-t)(1-2s)^n,$$

$$H(s,t)=(1-2t)^n(1-s),$$

$$n=0,1,2,\ldots \qquad (4.20)$$

$$H^{(0)}(s,t)=M_{11}+M_{1s}s+M_{1(1-2s)}(1-2s)+\cdots+M_{t1}t$$

$$+M_{ts}ts+M_{t(1-2s)}t(1-2s)+\cdots$$

$$=\sum_{\alpha(t),\beta(s)} M_{\alpha(t)\beta(s)}\alpha(t)\beta(s), \qquad (4.21)$$

where $\alpha(x)$, $\beta(x)$ take on the values 1, x, $1-2x$, x^2, $(1-2x)^2$, . . . and where the matrix M with nuerical entries satisfies

$$\text{(i)}\quad M_{\alpha(t)\beta(s)}=-M_{\beta(t)\alpha(s)}, \qquad (4.22)$$

$$\text{(ii)}\quad \sum_{\beta(s)} M_{\alpha(t)\beta(s)}\beta(s)\geq 0$$

for $s\in[0,1]$ and $\alpha(t)=t^2,\ (1-2t)^2,\ t^3,\ (1-2t)^3,\ldots.$ (4.23)

Then we may set

$$\text{(a)}\quad \mathcal{H}^{(0)}(s,t)=\omega(z_s,s)H^{(0)}(s,t), \qquad (4.24)$$

where $\omega(z_s,s)\in\mathcal{C}_s$ and is furthermore completely symmetric under interchanges of s, t, u, and

$$\text{(b)}\quad \mathcal{H}(s,t)=\omega_1(z_s,s)H(s,t), \qquad (4.25)$$

where $\omega_1(z_s,s)\in\mathcal{C}_s$.

The weights ω in (4.2) and (4.24) are the same. So are the weights ω_1 in (4.5) and (4.25). *The weight ω_2 in (4.6) is to be obtained by a change of variable from the corresponding weight ω_1:*

$$\omega_2(z_t,t)=\omega_1[z_s(z_t,t),\ s(z_t,t)], \qquad (4.26)$$

where

$$z_s(z_t,t)=1+\frac{4t}{(1-t)(1-z_t)-2}, \qquad (4.27)$$

$$s(z_t,t)=\tfrac{1}{2}(1-t)(1-z_t). \qquad (4.28)$$

The proof of these results again uses the methods of paper I and may be omitted. Simple ways of constructing the matrix M are also described in that paper. Examples of ω for (4.24) are stu and $st+tu+us$ and the square of the modulus of any polynomial in these variables and examples of ω_1 for (4.25) are $t+u$ and tu.

We finally observe that the construction of $H^{(0)}$ in (4.21) and hence that of $\mathcal{H}^{(0)}$ by the formula (4.24) may be generalized to involve fractional powers of s and t as was described in paper I.

V. FINAL REMARKS

We have attempted to formulate the results of this paper in a form which closely resembles the results in Refs. 1 and 2 and have placed special emphasis on displaying the nature of the generalized partial waves. Thus, for example, Eqs. (4.4) and (4.8) could easily have been reformulated so as to contain more generalized partial waves. Again the existence of ω_2 is not really used in Eqs. (4.5)–(4.8), so that the inequality (4.8) could have been written without introducing ω_2 at all by replacing the integral over $\sum g_l b_l^{(i)}$ by the integral

$$\int_0^1 dt\,(1-t)\times\tfrac{1}{2}\int_{-1}^{1} dz_t\,\mathcal{K}(s,t)A^{(i)}(s,t)\,.$$

It should also be noted that all the inequalities in the text can be refined and made to involve $\pi\pi$ total cross sections and other experimentally accessible data as was done in paper I. Some theorems on the allowed crossing properties of partial-wave sums over subsets of partial waves can also be proved following paper I. We have omitted such details however in the belief that they can be filled in by the interested reader.

ACKNOWLEDGMENTS

One of us (A.P.B.) wishes to thank S. M. Roy for suggestions on the paper and the Tata Institute of Fundamental Research for its hospitality.

APPENDIX A

Properties of Orthogonal Polynomials

The standard reference for orthogonal polynomials is Szegö's book[7]; see also Ref. 8 and Appendix A of Ref. 5(b). Only such results are described here as are of some relevance to the text of the paper or to the literature in the field. The discussion given here regarding the positivity properties of P_l^ω and Q_l^ω may in parts be new, at least in its methodology of proofs.

For convenience of notation, we will suppress the dependence on τ of the weight function $\omega(x,\tau)$ and denote it simply by $\omega(x)$. The translation of the results to the weight function $\omega(x,\tau)$ is of course immediate. Thus if $\omega(x)\in\mathcal{C}$,

$$\text{(a)}\quad \omega(x)\geqslant 0 \quad\text{for } x\in[-1,+1]\,, \qquad \text{(A1)}$$

$$\text{(b)}\quad \int_{-1}^{1} dx\,\omega(x)x^\nu<\infty\,,\quad \nu=0,1,2,\ldots\,. \qquad \text{(A2)}$$

If $\omega(x)\in\mathcal{C}_s$, then besides (A1) and (A2), it enjoys the symmetry property

$$\text{(c)}\quad \omega(-x)=\omega(x) \text{ for } x\in[-1,+1]\,. \qquad \text{(A3)}$$

It will further be assumed that the weight functions we deal with are nonvanishing almost everywhere on $[-1,+1]$. The orthogonal polynomial $P_l^\omega(x)$ with respect to the weight $\omega(x)\in\mathcal{C}$ is as usual defined to be a polynomial of degree l with the following properties:

$$\text{(a)}\quad P_l^\omega(x)=\sum_{\nu=0}^{l}\alpha_{l\nu}x^\nu\,,\quad\text{where } \alpha_{ll}\neq 0\,, \qquad \text{(A4)}$$

$$\text{(b)}\quad \int_{-1}^{1} dx\,\omega(x)P_l^\omega(x)x^\nu=0\,,$$

$$\nu=0,1,2,\ldots,l-1;\; l=1,2,3,\ldots \qquad \text{(A5)}$$

$$\text{(c)}\quad P_l^\omega(1)>0\,. \qquad \text{(A6)}$$

[$P_l^\omega(x)$ cannot vanish at $x=1$; see below.] Along with P_l^ω, we will also be interested in the functions Q_l^ω of the second kind defined by

$$Q_l^\omega(x)=\tfrac{1}{2}\int_{-1}^{1} dy\,\omega(y)\frac{P_l^\omega(y)}{x-y}\,,\quad x\notin[-1,+1]\,. \qquad \text{(A7)}$$

This Appendix consists of three parts. In the first part, we study P_l^ω and Q_l^ω when $\omega\in\mathcal{C}$, while in the second part ω is further restricted to be in $\mathcal{C}_s$. In the final part, we illustrate with two examples what appears to be a relatively simple method for obtaining inequalities for P_l^ω and Q_l^ω when $\omega\in\mathcal{C}$ or $\mathcal{C}_s$.

The Case where $\omega\in\mathcal{C}$

The Zeros of $P_l^\omega(x)$

It is well known[12] that all the zeros of $P_l^\omega(x)$ are simple and are located in the interval $-1<x<1$.

Some immediate consequences of this theorem are the following: Since $P_l^\omega(1)>0$ by the convention (A6), we may conclude that $P_l^\omega(x)>0$ for *all* $x\geqslant 1$, i.e.,

$$P_l^\omega(x)>0,\quad x\geqslant 1;\; l=0,1,2,\ldots\,. \qquad \text{(A8)}$$

As $x\to\infty$, (A4) and (A8) yield $\alpha_{ll}>0$. Therefore, as $x\to-\infty$, $P_l^\omega(x)>0$ for l even and $P_l^\omega(x)<0$ for l odd. Thus by the theorem,

$$P_l^\omega(x)>0,\quad x\leqslant -1;\; l=0,2,4,\ldots \qquad \text{(A9)}$$

$$P_l^\omega(x)<0,\quad x\leqslant -1;\; l=1,3,5,\ldots\,. \qquad \text{(A10)}$$

Another result, which has been extensively used in the literature, in particular when the orthogonal polynomial in question is the Legendre polynomial, states that if $P_l^\omega(x)$ is a constant multiple of one of the classical orthogonal polynomials,[13] then its first l derivatives are positive for $x\geqslant 1$. This result may also be proved quite simply using the above theorem. For, in this event, $d^\nu P_l^\omega(x)/dx^\nu$ ($\nu=1,2,3,\ldots,l$) is itself a nonzero constant times one of the classical orthogonal polynomials[13] and therefore cannot vanish for $x\geqslant 1$. But for $x\to\infty$, these derivatives are positive since α_{ll} is positive; hence the result.

Recurrence Formulas

Any three consecutive orthogonal polynomials are connected by the relation[14]

$$xP_l^\omega(x)=A_{l+1}P_{l+1}^\omega(x)+B_lP_l^\omega(x)+C_{l-1}P_{l-1}^\omega(x),$$

$$l=1,2,3,\ldots \qquad \text{(A11)}$$

where A_{l+1}, B_l, and C_{l-1} are constants and

$$A_{l+1}>0,\quad C_{l-1}>0\,. \qquad \text{(A12)}$$

Suppose that $q(x)$ is a polynomial of degree n not exceeding l. From

$$q(y)=\sum_{\nu=0}^{n}\frac{1}{\nu!}(y-x)^{\nu}\frac{d^{\nu}q(x)}{dx^{\nu}} \quad (A13)$$

and (A5), we find

$$\tfrac{1}{2}\int_{-1}^{1}dy\ \omega(y)\frac{q(y)P_l^{\omega}(y)}{x-y}=q(x)Q_l^{\omega}(x). \quad (A14)$$

Thus if $q(x)$ is a polynomial of degree $\leq l$ and

$$q(x)P_l^{\omega}(x)=\sum_{\rho=0}^{n+l}\eta_{\rho}P_{\rho}^{\omega}(x), \quad (A15)$$

where the η_{ρ}'s are constants, then

$$q(x)Q_l^{\omega}(x)=\sum_{\rho=0}^{n+l}\eta_{\rho}Q_{\rho}^{\omega}(x). \quad (A16)$$

Christoffel's Formula

Let the weight function $\omega(x)\in\mathfrak{C}$ be a polynomial of degree n with zeros at $x_1, x_2, \ldots, x_n$. Thus

$$\omega(x)=\mu\prod_{i=1}^{n}(x-x_i), \quad \mu\neq 0. \quad (A17)$$

Then if $x_i\neq x_j$ whenever $i\neq j$,

$$\omega(x)P_l^{\omega}(x)=\lambda_l\det\begin{vmatrix} P_l(x) & P_{l+1}(x) & \cdots & P_{l+n}(x)\\ P_l(x_1) & P_{l+1}(x_1) & \cdots & P_{l+n}(x_1)\\ & \cdots & & \\ P_l(x_n) & P_{l+1}(x_n) & \cdots & P_{l+n}(x_n)\end{vmatrix}, \quad (A18)$$

where the constant λ_l is subject to the constraint (A6) and P_l denotes the Legendre polynomial. In the case of a zero x_k of multiplicity $m>1$, the corresponding rows of the determinant in (A18) are to be replaced by the rows

$$\begin{matrix} P_l(x_k) & P_{l+1}(x_k) & \cdots & P_{l+n}(x_k)\\ P_l^{(1)}(x_k) & P_{l+1}^{(1)}(x_k) & \cdots & P_{l+n}^{(1)}(x_k)\\ & \cdots & & \\ P_l^{(m-1)}(x_k) & P_{l+1}^{(m-1)}(x_k) & \cdots & P_{l+n}^{(m-1)}(x_k),\end{matrix} \quad (A19)$$

where $P_l^{(\nu)}(x_k)=d^{\nu}P_l(x_k)/dx_k^{\nu}$.

For a proof of this formula, see Ref. 15.

Positivity Properties

We will prove the following results:

(a) $Q_l^{\omega}(x)>0, \quad x>1; \quad l=0,1,2,\ldots$ (A20)

(b) $Q_l^{\omega}(x)<0, \quad x<-1; \quad l=0,2,4\ldots$ (A21)

(c) $Q_l^{\omega}(x)>0, \quad x<-1; \quad l=1,3,5,\ldots.$ (A22)

Let

$$I(\lambda,l)=\tfrac{1}{2}\int_{-1}^{1}dx\ \omega(x)P_l^{\omega}(x)(1-x)^{\lambda}, \quad \lambda>-1 \quad (A23)$$

where $\omega(x)$ is sufficiently smooth to guarantee the existence of the integral for $\lambda>-1$. Clearly $I(\lambda,0)>0$ for all $\lambda>-1$, but besides this, we have the additional inequalities

(d) $I(\lambda,l)\geq 0, \quad -1<\lambda\leq 0; \quad l=1,2,3,\ldots$ (A24)

(e) $I(\lambda,l)\leq 0, \quad 2m\leq\lambda\leq 2m+1;$

$l=2m+1, 2m+2, 2m+3,\ldots; \quad m=0,1,2,\ldots$ (A25)

(f) $I(\lambda,l)\geq 0, \quad 2m+1\leq\lambda\leq 2m+2;$

$l=2m+2, 2m+3, 2m+4,\ldots; \quad m=0,1,2,\ldots.$ (A26)

To prove (A20), we note that if

$$J(x)=\tfrac{1}{2}\int_{-1}^{1}dy\ \omega(y)\frac{[P_l^{\omega}(y)]^2}{x-y}, \quad (A27)$$

then

$$J(x)>0 \text{ for } x>1. \quad (A28)$$

The result (A20) follows from (A14) and (A8). Similarly, (A21) and (A22) follow from $J(x)<0$ for $x<-1$ on using (A14), (A9), and (A10).[16]

To prove (A24), we start from the dispersion relation

$$(1-x)^{\lambda}=-\frac{\sin\pi\lambda}{\pi}\int_{-\infty}^{0}dx'\frac{|x'|^{\lambda}}{1-x'-x}, \quad -1<\lambda<0 \quad (A29)$$

and find

$$I(\lambda,l)=-\frac{\sin\lambda\pi}{\pi}\int_{-\infty}^{0}dx'\,|x'|^{\lambda}Q_l^{\omega}(1-x'), \quad -1<\lambda<0. \quad (A30)$$

This representation yields (A24) for $-1<\lambda<0$ due to (A20). It is established for $\lambda=0$ as well by letting $\lambda\to 0$ in the relation $I(\lambda,l)\geq 0$ and using continuity or by a direct calculation.

It remains to show (A25) and (A26). Let $\lambda=\lambda_0+n$, where $-1<\lambda_0<0$ and $n=1,2,3,\ldots.$ Then from (A29),

$$I(\lambda_0+n,l)=-\frac{\sin\pi\lambda_0}{\pi}\int_{-\infty}^{0}dx'\,|x'|^{\lambda_0}\times\tfrac{1}{2}\int_{-1}^{1}dx\ \omega(x)\frac{(1-x)^nP_l^{\omega}(x)}{1-x'-x}. \quad (A31)$$

So if $n\leq l$, by (A14),

$$I(\lambda_0+n,l)=-\frac{\sin\pi\lambda_0}{\pi}\int_{-\infty}^{0}dx'\,|x'|^{\lambda_0}x'^nQ_l^{\omega}(1-x') \quad (A32)$$

$$\begin{matrix} >0 & \text{if } n=2,4,6,\ldots \\ \text{and} & \\ <0 & \text{if } n=1,3,5,\ldots. \end{matrix} \quad (A33)$$

This proves (A25) and (A26) for $\lambda\neq$ integer and by continuity for $\lambda=$integer as well.

The Case where $\omega \in \mathfrak{C}_S$

Symmetry Properties

It may be shown from $\omega(-x)=\omega(x)$ that[17]

$$P_l^\omega(-x)=(-1)^l P_l^\omega(x)\,, \qquad \text{(A34)}$$

$$Q_l^\omega(-x)=(-1)^{l+1} Q_l^\omega(x)\,. \qquad \text{(A35)}$$

As a consequence, the recurrence relation for P_l^ω (and hence for Q_l^ω) simplifies to

$$xP_l^\omega(x)=A_{l+1}P_{l+1}^\omega(x)+C_{l-1}P_{l-1}^\omega(x)\,, \qquad \text{(A36)}$$

where

$$A_{l+1}>0, \quad C_{l-1}>0\,. \qquad \text{(A37)}$$

Positivity Properties

In addition to the previous inequalities, we are able to show that

$$\text{(a)} \quad \int_{-1}^{1} dx\,\omega(x)P_l^\omega(x)x^\nu \geq 0\,,$$

$$l=0,1,2,\ldots;\ \nu=0,1,2,\ldots\,. \qquad \text{(A38)}$$

With the help of (A38), the inequalities for

$$I(\lambda,l)=\tfrac{1}{2}\int_{-1}^{1} dx\,\omega(x)P_l^\omega(x)(1-x)^\lambda, \quad \lambda>-1$$

can be extended. Written out in full, they read as follows:

(b) If $-1<\lambda\leq 0$, (A39)

then

$I(\lambda,l)\geq 0$ for $l=0,1,2,\ldots$. (A40)

(c) If $2m\leq\lambda\leq 2m+1;\ m=0,1,2,\ldots$ (A41)

then

$I(\lambda,0)>0$

and

$I(\lambda,l)\leq 0$ for $l=2m+1, 2m+2, 2m+3,\ldots$. (A42)

(d) If $2m+1\leq\lambda\leq 2m+2,\ m=0,1,2,\ldots$ (A43)

then

$I(\lambda,l)\geq 0$ for $l=0,2,4,\ldots$ (A44)

and for $l=2m+3, 2m+5, 2m+7,\ldots$. (A45)

The results (4.12)–(4.14) are implied by the above. [Use $1-2t=s+(1-s)z_s$ and expand $(1-2t)^n$ in a power series in z_s for (4.14).]

The proof of (A38) is based on (A36)–(A37). Repeated application of the latter yields the expansion

$$x^nP_l^\omega(x)=\sum_{\rho=0}^{l+n}\beta_\rho P_\rho^\omega(x)\,, \quad \beta_\rho\geq 0;\ n=0,1,2,\ldots \qquad \text{(A46)}$$

and hence the equation

$$\int_{-1}^{1} dx\,\omega(x)x^nP_l^\omega(x)=\beta_0P_0^\omega(x)\int_{-1}^{1}dx\,\omega(x)\,, \qquad \text{(A47)}$$

where $P_0^\omega(x)$ is a constant which is positive by convention. Thus (A38) is proved.

We have already derived (A40), (A42), and (A45). It remains to discuss (A44). We may restrict ourselves to nonintegral values of λ during the demonstration since integer values of λ may be recovered by a limiting procedure. For $\lambda>0$ and $\lambda\neq$ integer, we find from

$$(1-x)^\lambda=\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)\Gamma(\nu+1)}x^\nu \qquad \text{(A48)}$$

that

$$I(\lambda,l)=\sum_{\substack{\nu\geq l\\ \nu=\text{even}}}\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)\Gamma(\nu+1)}\times\tfrac{1}{2}\int_{-1}^{1}dx\,\omega(x)P_l^\omega(x)x^\nu \qquad \text{(A49)}$$

for even l. The odd powers of x do not contribute, because of the symmetry property (A34) of $P_l^\omega(x)$. Since

$$\Gamma(z)>0 \text{ if } z>0 \text{ or if } -(2m+2)<z<-(2m+1);$$

$$m=0,1,2,\ldots \qquad \text{(A50)}$$

(A44) follows from (A38).

Inequalities Involving P_l^ω and Q_l^ω

Given two weight functions ω_1 and ω_2, $P_l^{\omega_2}$ and $Q_l^{\omega_2}$ can sometimes be expressed in a relatively simple way in terms of $P_l^{\omega_1}$ and $Q_l^{\omega_1}$. Christoffel's formula, which was described above, provides such an example. The inequalities we have obtained with the choice $\omega=\omega_2$ may thus be reformulated as inequalities for $P_l^{\omega_1}$ and $Q_l^{\omega_1}$. We have found this method to be a fairly simple and powerful way of deriving inequalities involving $P_l^{\omega_1}$ and $Q_l^{\omega_1}$. As illustrations of this method, we will now show that

$$\frac{Q_{l-1}^\omega(x_0)}{Q_l^\omega(x_0)}>\frac{P_{l-1}^\omega(x)}{P_l^\omega(x)}\,,$$

$$l=1,2,3,\ldots;\ x_0>1;\ x\geq 1;\ \omega\in\mathfrak{C} \qquad \text{(A51)}$$

and

$$\frac{Q_{l-1}^\omega(x_0)}{Q_l^\omega(x_0)}>\frac{Q_{l-1}^\omega(x)}{Q_l^\omega(x)}\,,$$

$$l=1,2,3,\ldots;\ x_0>x>1;\ \omega\in\mathfrak{C}\,. \qquad \text{(A52)}$$

For the choice $\omega=1$, Martin[18] has proved (A52) while (A51) can be derived from (A52).[19] For on letting $x\to 1$ and recalling that the divergent term in $Q_l(x)$ as $x\to 1$ is independent of l, we find from (A52) that

$$\frac{Q_{l-1}(x_0)}{Q_l(x_0)}>1, \quad l=1,2,3,\ldots;\ x_0>1\,. \qquad \text{(A53)}$$

Since $P_l(x)$ increases when l increases and $x>1$ and x is fixed,[20] and since $P_l(1)=1$, Eq. (A51) follows from Eq. (A53).

The derivation of (A51) and (A52) is as follows: For the weight

$$\omega'(x) = \omega(x)/(x_0 - x), \quad x_0 > 1 \tag{A54}$$

the orthogonal polynomials $P_l^{\omega'}(x)$ for $l \geq 1$ are given by

$$P_l^{\omega'}(x) = \alpha_l [Q_{l-1}(x_0) P_l^{\omega}(x) - Q_l^{\omega}(x_0) P_{l-1}^{\omega}(x)], \tag{A55}$$

where α_l is any constant, $\alpha_l > 0$. [See Appendix B of Ref. 5(b).] As $x \to \infty$, the first term dominates and therefore $P_l^{\omega'}(x) > 0$ because of (A20) and (A8). By the theorem on zeros we are assured first that $P_l^{\omega'}(x)$ in (A55) fulfills the convention $P_l(1) > 0$ and second that

$$P_l^{\omega'}(x) > 0 \quad \text{for } x \geq 1, \tag{A56}$$

which is the inequality (A51).

The formula

$$Q_l^{\omega'}(x) = \tfrac{1}{2}\int_{-1}^{1} dy \frac{\omega(y)}{(x_0 - y)} \frac{P_l^{\omega'}(y)}{(x - y)} \tag{A57}$$

for $Q_l^{\omega'}(x)$ becomes on using (A55) and writing $[(x_0 - y)(x - y)]^{-1}$ in partial fractions,

$$Q_l^{\omega'}(x) = \frac{\alpha_l}{x_0 - x}[Q_{l-1}^{\omega}(x_0) Q_l^{\omega}(x) - Q_l^{\omega}(x_0) Q_{l-1}^{\omega}(x)], \quad l = 1, 2, 3, \ldots . \tag{A58}$$

The result (A52) now follows from (A20).

It is a simple exercise to generalize (A51) and (A52) to negative values of x_0 and/or x.

APPENDIX B

Inequalities for the $l \leq 2$ $\pi\pi$ Partial Waves

In this Appendix we list some expressions for ω and ω_1 and some corresponding expressions for $H^{(0)}$ and H which lead to inequalities involving the (canonical) s, p, and d $\pi\pi$ partial waves on using (4.4) and (4.8). In view of the elementary nature of the computations involved, the inequalities themselves are not explicitly tabulated. We shall, however, illustrate the method in an example. In the equations below, the weight functions in (B1) are completely symmetric in s, t, and u, while those in (B2) and (B3) are symmetric only in t and u.

(a) $\omega(z_s, s) = stu, \quad st + tu + us\,.$ (B1)

For either of these ω's, $H^{(0)}$ may be identified with any one of the $H^{(0)}$'s listed in (4.15)–(4.23).

(b) $\omega_1(z_s, s) = s, \ t + u, \ tu\,.$ (B2)

Here H may be chosen to be any one of the H's tabulated in (4.15)–(4.20).

(c) $\omega_1(z_s, s) = s(t + u)\,.$ (B3)

The following H's are acceptable for these ω_1's:

$$H(s,t) = t^{\lambda}, \quad -1 < \lambda \leq 0 \text{ or } 2m + 1 \leq \lambda \leq 2m + 2; \quad m = 0, 1, 2, \ldots \tag{B4}$$

$$H(s,t) = -t^{\lambda}, \quad 0 \leq \lambda \leq 1 \tag{B5}$$

$$H(s,t) = (1 - 2t)^n, \quad n = 0, 1, 2, \ldots . \tag{B6}$$

The expressions for H in (B4) are obtained by adding the expressions for H in (4.15) and (4.17). This was done in order to suppress the linear term in s in (4.15) and (4.17) which would induce an f-wave term in the inequalities. [The $\omega_2(z_t, t)$ which arises from $s(t+u) = s(1-s)$ involves a quadratic term in z_t. Therefore, $b_1^{(i)}(s, \omega_2)$ in (4.8) contains $b_3^{(i)}(s)$. We choose such H's in (B4) for which $g_1(s, \omega_2) = 0$.] The construction of (B5) and (B6) is similar.

We will illustrate the explicit computation of the inequalities for the case $\omega = st + tu + us$ and $H^{(0)} = t^{\lambda}s - ts^{\lambda}$, where λ is subject to the usual restrictions. Since

$$\omega(z_s, s) = s(1 - s) + [\tfrac{1}{2}(1 - s)]^2(1 - z_s^{\,2}), \tag{B7}$$

we find with $P_0^{\omega}(z_s) = 1$,

$$a_0^{(0)}(s, \omega) = \tfrac{1}{6}[(1 - s)(1 + 5s)a_0^{(0)}(s) - (1 - s)^2 a_2^{(0)}(s)]. \tag{B8}$$

Further, from

$$\omega(z_s, s)\sum_{l=0}^{\infty} h_l^{(0)}(s, \omega) P_l^{\omega}(z_s) = \omega(z_s, s)(t^{\lambda}s - ts^{\lambda}), \tag{B9}$$

we can compute $h_0^{(0)}(s, \omega)$. Let

$$K_\lambda(s) = \int_{-1}^{1} dz_s \omega(z_s, s) t^{\lambda} = 2(1 - s)^{\lambda+1}\left[\frac{s}{\lambda + 1} + \frac{1 - s}{(\lambda + 2)(\lambda + 3)}\right]. \tag{B10}$$

[We have evaluated the integral by expressing it in terms of beta functions by a linear change of variables.]

On using the orthogonality properties of P_l^{ω}, (B9) gives

$$h_0^{(0)}(s, \omega)\int_{-1}^{1} dz_s \omega(z_s, s) = \int_{-1}^{1} dz_s \omega(z_s, s)[t^{\lambda}s - ts^{\lambda}]. \tag{B11}$$

Thus

$$h_0^{(0)}(s, \omega) = \frac{1}{K_0(s)}[sK_\lambda(s) - s^{\lambda}K_1(s)] \tag{B12}$$

and the integral

$$F(\lambda) = -\int_0^1 ds\,(1 - s)\frac{sK_\lambda(s) - s^{\lambda}K_1(s)}{K_0(s)} \times [(1 - s)(1 + 5s)a_0^{(0)}(s) - (1 - s)^2 a_2^{(0)}(s)] \tag{B13}$$

fulfills the inequalities

$$F(\lambda) \geq 0, \quad -1 < \lambda \leq 0; \; 2m+1 \leq \lambda \leq 2m+2; \quad m=0,1,2,\ldots \tag{B14}$$

$$F(\lambda) \leq 0, \quad 0 \leq \lambda \leq 1. \tag{B15}$$

The functions which weight $a_0^{(0)}(s)$ and $a_2^{(0)}(s)$ in (B13) are, in general, rational in s for λ = integer. This property is easily traced to the fact that $st+tu+us$ is not factorizable into the form $\psi_1(z_s)\psi_2(s)$. It may be noted that in the inequalities derived in Ref. 1 such weighting factors are always polynomials in s whenever $H^{(0)}(s,t)$ or $H(s,t)$ is a polynomial in s and t.

---

*Work supported in part by the U. S. Atomic Energy Commission.

†Permanent address.

[1]A. P. Balachandran and M. L. Blackmon, (a) Phys. Letters 31B, 655 (1970), and (b) preceding paper, Phys. Rev. D 3, 3133 (1971), hereafter referred to as paper I. For closely related and partially overlapping work, see O. Piguet and G. Wanders, Phys. Letters 30B, 418 (1969); R. Roskies, J. Math. Phys. 11, 2913 (1970); M. R. Pennington, Nucl. Phys. 24, 317 (1970); see also W. Case, Phys. Rev. D 3, 2472 (1971).

[2]A. Martin, Nuovo Cimento 47, 265 (1967); S. M. Roy, Phys. Rev. Letters 20, 1016 (1968).

[3]F. J. Yndurain, Nuovo Cimento 64, 225 (1969).

[4]A. K. Common, Nuovo Cimento 63, 863 (1969); 65, 581 (1970).

[5](a) A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968); (b) A. P. Balachandran, W. J. Meggs, P. Ramond, and J. Nuyts, *ibid.* 187, 2080 (1969); (c) A. P. Balachandran, W. Case, and M. Modjtehedzadeh, Phys. Rev. D 1, 1773 (1970); (d) M. Modjtehedzadeh, Phys. Rev. D (to be published) and further references contained in these papers.

[6]M. Froissart, Phys. Rev. 123, 1053 (1961). See in particular Eq. (3).

[7]G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939), p. 27.

[8]The Bateman Manuscript Project, *Higher Transcendental Functions,* edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. II, p. 158.

[9]N. I. Akhiezer, *The Classical Moment Problem* (Oliver and Boyd, London, 1965), pp. 74, 77, and 203 ff; Ref. 7, p. 4; D. V. Widder, *The Laplace Transform* (Princeton Univ. Press, Princeton, 1946), p. 152; Ref. 1(b), footnote 12.

[10]Y. S. Jin and A. Martin, Phys. Rev. 135B, 1375 (1964).

[11]V. N. Gribov and I. Ya. Pomeranchuk, Zh. Eksperim. i. Teor. Fiz. 43, 308 (1962) [Soviet Phys. JETP 16, 220 (1963)]; A. Martin, Nuovo Cimento 42, 930 (1966).

[12]Reference 7, p. 44; Ref. 8, p. 158.

[13]Reference 8, p. 163.

[14]Reference 7, p. 42; Ref. 8, 158.

[15]Reference 7, p. 29, Ref. 8, p. 159.

[16]See in this connection the Appendix of the paper of W. Case, Ref. 1.

[17]Reference 7, p. 29.

[18]A. Martin, Nuovo Cimento 61, 56 (1969), Appendix.

[19]We are grateful to S. M. Roy for a discussion on this point.

[20]This well-known result may be proved from Christoffel's formula (A18) with $\omega(x)=1-x$ which gives $(1-x)P_l^{\omega}(x)=\lambda_l[P_l(x)-P_{l+1}(x)]$. Since $(1-x)P_l^{\omega}(x)$ cannot vanish for $x>1$ by the theorem on zeros, $P_l(x)-P_{l+1}(x)$ maintains a constant sign for $x>1$, while it is clearly negative as $x\to\infty$.