[13]G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, Providence, R. I., 1959), Vol. 23, p. 4.

[14]P. Grassberger, Z. Physik 236, 410 (1970).

[15]After this work was completed I came upon a paper by R. Roskies [Yale Report No. 2726-562, 1970 (unpublished)] in which similar results are obtained. His positivity constraint, expressed on the a_{nl}'s as $\sum_{l,n} \eta_{nl} a_{nl} \geq 0$, is similar to Eqs. (3.11), (5.7), or (5.14) above, but the method used in Secs. III and V above to derive explicit sets of η_{nl} seems somewhat more compact. His analysis of the crossing content, using a previously derived parametrization [R. Roskies, J. Math. Phys. 11, 482 (1970)], combines the constraint of several inequalities.

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Inequalities for the s- and p-Wave $\pi\pi$ Partial-Wave Amplitudes*

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An infinite number of inequalities are derived for integrals over the s- and p-wave $\pi\pi$ amplitudes in the interval $0 \leq s \leq 4m_\pi^2$ in terms of the $\pi\pi$ total cross sections and other experimentally accessible data. The main ingredients in the derivations are crossing symmetry, the positivity of the even $l \geq 2$ partial waves of the reactions $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ in the interval $0 \leq s \leq 4m_\pi^2$, and some known bounds on the crossed-channel absorptive parts of these reactions. It is shown that if the partial-wave sum over any subset of $\pi^0\pi^0 \to \pi^0\pi^0$ partial waves is itself invariant under permutations of s, t, and u, and this subset contains the s wave, then the entire $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude has to vanish identically. (Actually, a somewhat stronger result is proved for the amplitudes of both the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ or for any linear combination of these amplitudes with positive coefficients.)

I. INTRODUCTION

Some years ago, Martin[1] proved that the partial-wave amplitudes with angular momenta $l \geq 2$ of the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ are non-negative when the square of the center-of-mass energy s is restricted to be in the region $0 \leq s \leq 1$. (We take the pion mass m_π to be $\frac{1}{2}$ and denote the Mandelstam variables by s, t, and u.) Later work by Common and by Yndurain[2] extended Martin's results and revealed a more refined set of inequalities for these partial waves. General methods for studying the crossing properties of partial waves have been developed by Balachandran *et al.* and by Modjtehedzadeh.[3] In this and subsequent papers, we will use the positivity properties of the partial waves due to Martin, Common, and Yndurain, in conjunction with the crossing properties of the partial waves of four-body processes studied by Balachandran *et al.* and some other known properties of scattering amplitudes, to derive an infinite number of integral inequalities for the $\pi\pi$ partial waves. The emphasis in the present work will be on stating *simple* algorithms for writing down inequalities which involve only the s and p waves. Further, the Common-Yndurain refinement of the Martin inequalities will be completely ignored here. For these reasons, the results will not be exhaustive. (A preliminary account of this research has been reported elsewhere.[4]) In a second paper, we will develop suitable elementary (and therefore incomplete) algorithms for deriving partial-wave inequalities, taking advantage of the work of Common and Yndurain, while in a third paper, an attempt will be made to state systematically all such inequalities which follow from crossing symmetry and the Martin-Common-Yndurain positivity properties of the partial waves. For similar and occasionally overlapping research, we refer the reader to Piguet and Wanders, to Roskies, and, most recently, to Pennington.[5]

Some unanticipated insights provided by these inequalities refer to the allowed crossing properties of partial-wave sums over subsets of partial waves. They are partially described below and merit attention since they indicate some possible difficulties in enforcing crossing symmetry and unitarity in any model.

The construction of the inequalities by our method involves two classes of auxiliary functions, one each for $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$, such that for each such function there is an inequality involving only the s- and p-wave $\pi\pi$ amplitudes. Due to the greater symmetry of the amplitude for the process $\pi^0\pi^0 \to \pi^0\pi^0$, the allowed functions are somewhat more numerous for it as compared to the process $\pi^0\pi^0 \to \pi^+\pi^-$. In Sec. II we state our notation and characterize the auxiliary functions for either of these processes.

In Sec. III we review our previous paper on inequalities,[4] relate it to Sec. II, and then extend it to obtain a continuously infinite number of new inequalities.

Section IV generalizes the work of Sec. III in a new direction and develops simple rules for finding many more bounds.

In Sec. V we refine these inequalities into ones involving integrals over partial waves in the region $0 \le s \le 1$ and integrals over total cross sections and other experimentally accessible data. While this method of refinement is well known to workers in the field, we describe it in view of its physical interest.

In Sec. VI we prove that *if the partial-wave sum over any subset of partial waves of either the process $\pi^0\pi^0 \to \pi^0\pi^0$ or the process $\pi^0\pi^0 \to \pi^+\pi^-$ produces the correct s wave in all channels, then the remaining partial waves of that process are identically zero.* In particular, for the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude, if the partial-wave sum over a subset of partial waves is itself symmetric under permutations, of s, t, and u and if the subset contains the s wave, the remaining $\pi^0\pi^0 \to \pi^0\pi^0$ partial waves are identically zero. It is then an easy consequence of the Froissart-Gribov representation that the corresponding scattering amplitude itself is identically zero. If the absorptive part of the amplitude which vanishes refers to $\pi^0\pi^0 \to \pi^0\pi^0$, the optical theorem will also force the amplitude for $\pi^0\pi^0 \to \pi^+\pi^-$ (and many other amplitudes as well) to vanish identically. Similar remarks may also be made for any linear combination of the amplitudes of the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ with positive coefficients.

In the Appendix we summarize the explicit forms of many of the inequalities and give further examples of acceptable auxiliary functions using the methods of Sec. IV.

This paper concentrates on getting results involving only s and p waves in view of their greater practical interest. The results can, however, be generalized to involve more partial waves.

In a recent paper Case[6] has obtained inequalities involving a finite number of partial waves of the $\pi\pi \to N\bar{N}$ and $\pi N \to \pi N$ systems which are the analogs of the positivity properties of the $\pi\pi$ partial waves.

II. CHARACTERIZATION OF THE AUXILIARY FUNCTIONS

We denote by $A^{(0)}(s,t)$ and $A^{(c)}(s,t)$ the scattering amplitudes which in the s channel describe the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$, respectively. Their partial-wave expansions are

$$A^{(i)}(s,t)=\sum_{l=0}^{\infty}(2l+1)a_l^{(i)}(s)P_l(z_s), \quad i=0,c \tag{2.1}$$

$$=\sum_{l=0}^{\infty}(2l+1)b_l^{(i)}(t)P_l(z_t), \quad i=0,c \tag{2.2}$$

where $a_l^{(i)}(s)=0$ if l is odd and $a_l^{(0)}(s)=b_l^{(0)}(s)$ while $z_s=1+2t/(s-1)$ and $z_t=1+2s/(t-1)$. Martin[1] has shown that[7]

$$a_l^{(i)}(s)\ge 0 \quad \text{for } i=0,c;\ l=2,4,6,\ldots;\ 0\le s\le 1. \tag{2.3}$$

We prove the following theorem regarding the $\pi^0\pi^0 \to \pi^0\pi^0$ s-wave. *Let $H^{(0)}(s,t)$ be a function of s and t which has the following properties*[8]*: (a) $H^{(0)}(s,t)$ is antisymmetric in s and t; (b) the s-channel partial waves $h_l^{(0)}(s)$ of $H^{(0)}(s,t)$ are nonnegative for $0\le s\le 1$ and $l=2,4,6,\ldots$. Thus*

$$H^{(0)}(s,t)=-H^{(0)}(t,s) \tag{2.4}$$

$$=\sum_{l=0}^{\infty}h_l^{(0)}(s)P_l(z_s), \tag{2.5}$$

where

$$h_l^{(0)}(s)\ge 0 \quad \text{for } 0\le s\le 1 \text{ and } l=2,4,6,\ldots. \tag{2.6}$$

Then

$$-\int_0^1 ds(1-s)h_0^{(0)}(s)a_0^{(0)}(s)\ge 0, \tag{2.7}$$

where equality is attained if and only if $h_l^{(0)}(s)\times a_l^{(0)}(s)=0$ for $l=2,4,6,\ldots$ and $0\le s\le 1$.

Note that for convenience we shall use different normalizations for the partial waves of $A^{(0)}$ and of the auxiliary functions. The proof of (2.7) is simple. Let Δ be the Mandelstam triangle $0\le s,t,u\le 1$. As Δ is invariant under the $s \leftrightarrow t$ permutation, while $A^{(0)}$ and $H^{(0)}$ are symmetric and antisymmetric, respectively, under the same operation, we have

$$\int_\Delta\int ds\,dt\,H^{(0)}(s,t)A^{(0)}(s,t)=0. \tag{2.8}$$

Changing variables from s and t to s and z_s and using (2.5), we find that

$$\int_0^1 ds(1-s)\times\tfrac{1}{2}\int_{-1}^{1}dz_s\left[\sum_{l=0}^{\infty}h_l^{(0)}(s)P_l(z_s)A^{(0)}(s,t)\right]=0$$

or that

$$-\int_0^1 ds(1-s)h_0^{(0)}(s)a_0^{(0)}(s) = \sum_{l=2}^{\infty}\int_0^1 ds(1-s)h_l^{(0)}(s)a_l^{(0)}(s)\,. \tag{2.9}$$

Equations (2.3) and (2.6) then lead to the required result.

The preceding method has to be modified for the process $\pi^0\pi^0 \to \pi^+\pi^-$. Let $H^{(c)}(s,t)$ be a function of s and t which has the following properties:
(a) *$H^{(c)}(s,t)$ is at most linear in s for fixed t.*
(b) *The s-channel partial waves $h_l^{(c)}(s)$ of $H^{(c)}(s,t)$ are non-negative for $0 \le s \le 1$ and $l = 2, 4, 6, \ldots$.*
Thus

$$H^{(c)}(s,t) = \sum_{l=0}^{\infty} h_l^{(c)}(s)P_l(z_s) \tag{2.10}$$

$$= \sum_{l=0}^{1} g_l^{(c)}(t)P_l(z_t), \tag{2.11}$$

where

$$h_l^{(c)}(s) \ge 0 \text{ for } 0 \le s \le 1 \text{ and } l = 2, 4, 6, \ldots\,. \tag{2.12}$$

Then

$$\sum_{l=0}^{1}\int_0^1 ds(1-s)\Big[g_l^{(c)}(s)b_l^{(c)}(s) - h_l^{(c)}(s)a_l^{(c)}(s)\Big] \ge 0, \tag{2.13}$$

where equality is attained if and only if $h_l^{(c)}(s)\times a_l^{(c)}(s) = 0$ for $l = 2, 4, 6, \ldots$ and $0 \le s \le 1$.

We have the identities

$$\int_\Delta\int ds\,dt\,H^{(c)}(s,t)A^{(c)}(s,t) = \int_0^1 ds(1-s)\times\tfrac{1}{2}\int_{-1}^1 dz_s\Big[\sum_{l=0}^{\infty}h_l^{(c)}(s)P_l(z_s)\Big]A^{(c)}(s,t)$$

$$= \int_0^1 dt(1-t)\times\tfrac{1}{2}\int_{-1}^1 dz_t\Big[\sum_{l=0}^{1}g_l^{(c)}(t)P_l(z_t)\Big]A^{(c)}(s,t), \tag{2.14}$$

which leads to

$$\sum_{l=0}^{1}\int_0^1 ds(1-s)\Big[g_l^{(c)}(s)b_l^{(c)}(s) - h_l^{(c)}(s)a_l^{(c)}(s)\Big] = \sum_{l=2}^{\infty}\int_0^1 ds(1-s)h_l^{(c)}(s)a_l^{(c)}(s). \tag{2.15}$$

Equations (2.3) and (2.12) then demonstrate the required result.

If equality is attained in (2.7) or (2.13) and a corresponding $h_l^{(i)}(s)$ is not identically zero for $0 \le s \le 1$ and some $l = l_0$, where l_0 is even and ≥ 2, then it is usually possible to infer that $a_{l0}^{(i)}(s) \equiv 0$. Such is in fact the case in all the inequalities we state in this paper. But if $a_{l0}^{(i)}(s) \equiv 0$ for $l_0 \ge 2$ and even, then $A^{(i)}(s,t)$ has to vanish for consistency with the Froissart-Gribov representation. (See, e.g., the end of Sec. VI.) Thus, although we do not exclude the equality symbol in stating our results in what follows, it is allowed only if the appropriate $H^{(i)}(s,t)$ is linear in t.

It should be remarked that for each $H^{(c)}(s,t)$, there is an $H^{(0)}(s,t)$ defined by $H^{(0)}(s,t) = H^{(c)}(s,t) - H^{(c)}(t,s)$ since the s-channel partial waves of $H^{(c)}(s,t)$ fulfill the positivity requirements while $H^{(c)}(t,s)$ does not affect the s-channel partial waves with angular momenta ≥ 2. The correspondence, however, cannot always be inverted.

III. REVIEW OF PREVIOUS WORK AND A GENERALIZATION

The inequalities of Ref. 4 (see also Pennington, Ref. 5) were based on the observation that the even partial waves of $(1-z_s)^n$ and $z_s{}^n$ are non-negative. Thus[9]

$$\tfrac{1}{2}\int_{-1}^1 dz_s\,(1-z_s)^n P_l(z_s) = (-1)^l 2^n \frac{(n!)^2}{\Gamma(n-l+1)(n+l+1)!}\,, \tag{3.1a}$$

$$\tfrac{1}{2}\int_{-1}^1 dz_s\,z_s{}^n P_l(z_s) = 2^l\,\frac{n!\,[\frac{1}{2}(n+l)]!}{\Gamma(\frac{1}{2}(n-l)+1)(n+l+1)!} \quad \text{if } (n-l) \text{ is even}$$

$$= 0 \quad \text{if } (n-l) \text{ is odd}. \tag{3.1b}$$

Since $t = \frac{1}{2}(1-s)(1-z_s)$, all the even s-channel partial waves of t^n are non-negative for $0 \le s \le 1$ due to (3.1a). Since $(1-2t) = s + (1-s)z_s$, $(1-2t)^n$ is a linear combination of powers of z_s with non-negative coefficients for $0 \le s \le 1$, so that all its s-channel partial waves are non-negative for $0 \le s \le 1$. Thus $t^n s$, $t^n(1-s)$, $(1-2t)^n s$, and $(1-2t)^n(1-s)$ are all allowed as $H^{(c)}$. [The inequality due to t^n, for example, can be obtained by adding the inequalities due to $t^n s$ and $t^n(1-s)$ so that the latter constitute a better set than just t^n.] For each of these $H^{(c)}$'s, we may set $H^{(0)}(s,t)=H^{(c)}(s,t)-H^{(c)}(t,s)$. There are thus the following sets of $H^{(c)}$ and $H^{(0)}$:

$$H^{(c)}(s,t)=t^n s,\quad H^{(0)}(s,t)=t^n s - ts^n,\quad n=0,1,2,\ldots \tag{3.2}$$

$$H^{(c)}(s,t)=t^n(1-s),\quad H^{(0)}(s,t)=t^n(1-s)-(1-t)s^n,\quad n=0,1,2,\ldots \tag{3.3}$$

$$H^{(c)}(s,t)=(1-2t)^n s,\quad H^{(0)}(s,t)=(1-2t)^n s - t(1-2s)^n,\quad n=0,1,2,\ldots \tag{3.4}$$

$$H^{(c)}(s,t)=(1-2t)^n(1-s),\quad H^{(0)}(s,t)=(1-2t)^n(1-s)-(1-t)(1-2s)^n,\quad n=0,1,2,\ldots. \tag{3.5}$$

The corresponding inequalities are stated in the Appendix. They were originally derived in Ref. 4. They were also independently found by Pennington.[5]

To generalize these bounds, consider the integral

$$I(\lambda,l)=\tfrac{1}{2}\int_{-1}^{1}dz_s(1-z_s)^\lambda P_l(z_s). \tag{3.6}$$

The integral is analytic and regular in λ for $\lambda>-1$ and for each fixed $l=0,1,2,\ldots$. In this region, we can evaluate the integral as indicated in Ref. 9 to find

$$I(\lambda,l)=(-1)^l\frac{2^\lambda[\Gamma(\lambda+1)]^2}{\Gamma(\lambda-l+1)\Gamma(\lambda+l+2)},\quad \lambda>-1. \tag{3.7}$$

Now $1/\Gamma(z)$ is entire,[10] so that $\Gamma(z)$ has no zeros. Further, the residues at the simple poles of $\Gamma(z)$ at $z=-n$ $(n=0,1,2,\ldots)$ are $(-1)^n/n!$.[10] It follows that $\Gamma(z)>0$ if $z>0$ or if $-(2m+2)<z<-(2m+1)$, $m=0,1,2,\ldots$, and that

$$I(\lambda,l)\geq 0 \tag{3.8a}$$

if

$$l=0,2,4,\ldots \tag{3.8b}$$

and either

$$2m-1\leq\lambda\leq 2m,\quad m=1,2,3,\ldots \tag{3.8c}$$

or

$$-1<\lambda\leq 0. \tag{3.8d}$$

This leads to the following $H^{(c)}$ and $H^{(0)}$:

$$H^{(c)}(s,t)=t^\lambda s,\qquad H^{(0)}(s,t)=t^\lambda s - ts^\lambda, \tag{3.9}$$

$$H^{(c)}(s,t)=t^\lambda(1-s),\qquad H^{(0)}(s,t)=t^\lambda(1-s)-(1-t)s^\lambda. \tag{3.10}$$

It is understood that λ is restricted as in (3.8c) or (3.8d).[11] Then $\lambda=(2m-1)$ or $(2m)$, and (3.9) and (3.10) reduce to (3.2) and (3.3).

When $0\leq\lambda\leq 1$, the preceding argument also shows that $I(\lambda,l)\leq 0$ for $l=2,4,6,\ldots$. Therefore, we may set

$$H^{(c)}(s,t)=-t^\lambda s,\qquad H^{(0)}(s,t)=-t^\lambda s+ts^\lambda, \tag{3.11}$$

$$H^{(c)}(s,t)=-t^\lambda(1-s),\qquad H^{(0)}(s,t)=-t^\lambda(1-s)+(1-t)s^\lambda \tag{3.12}$$

for $0\leq\lambda\leq 1$.

The inequalities due to (3.9)–(3.12) are given in the Appendix.

IV. A GENERALIZATION OF THE INEQUALITIES OF SECTION III

There is an elementary way of obtaining many more inequalities by using the observations of Secs. II and III. We prove the following theorem. *Let*

$$\begin{aligned}H^{(0)}(s,t)&=M_{11}+M_{1s}s+M_{1(1-2s)}(1-2s)+\cdots\\&\quad+M_{t1}t+M_{ts}ts+M_{t(1-2s)}t(1-2s)+\ldots\\&=\sum_{\alpha(t)\beta(s)}M_{\alpha(t)\beta(s)}\alpha(t)\beta(s),\end{aligned} \tag{4.1}$$

where $\alpha(x)$, $\beta(x)$ take on the values 1, x, $(1-2x)$, x^2, $(1-2x)^2,\ldots,$ *and where the matrix M with numerical entries $M_{\alpha(t)\beta(s)}$ satisfies the following properties:*

(a) $M_{\alpha(t)\beta(s)}=-M_{\beta(t)\alpha(s)}$, (4.2)

(b) $\sum_{\beta(s)}M_{\alpha(t)\beta(s)}\,\beta(s)\geq 0$ for $0\leq s\leq 1$; $\alpha(t)=t^2,\ (1-2t)^2,\ t^3,\ (1-2t)^3,\ldots.$ (4.3)

Then if

$$H^{(0)}(s,t)=\sum_{l=0}^{\infty}h_l^{(0)}(s)P_l(z_s),$$

we have

$$-\int_0^1 ds(1-s)h_0^{(0)}(s)a_0^{(0)}(s)\geq 0,$$

where equality is attained if and only if $h_l^{(0)}(s)\times a_l^{(0)}(s)=0$ for $0\leq s\leq 1$ and $l=2,4,6\ldots.$

Equation (4.2) ensures that $H^{(0)}(s,t)=-H^{(0)}(t,s)$, while (4.3) ensures that $h_2(s),\ h_4(s),\ h_6(s),\ldots\geq 0$ for $0\leq s\leq 1$, owing to the properties of the partial waves of t^n and $(1-2t)^n$ which we already proved in Sec. III. Note that if the rows of M are labeled by $1, t, (1-2t),\ldots,$ while its columns are labeled by $1, s, (1-2s), \ldots$, then M is an antisymmetric matrix.

We illustrate the above by the following example:

$$M = \begin{array}{c} 1 \\ t \\ 1-2t \\ t^2 \\ (1-2t)^2 \\ t^3 \end{array} \overset{\begin{array}{cccccc} 1 & s & (1-2s) & s^2 & (1-2s)^2 & s^3 \end{array}}{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}} \tag{4.4}$$

This M leads to

$$H^{(0)}(s,t) = t^3(1-2s)^2 - (1-2t)^2 s^3 + (1-2t)^2 s - t(1-2s)^2 \tag{4.5}$$

and a corresponding inequality for $a_0^{(0)}$, which is easy to compute using (3.1). We have adopted the convention in (4.4) that the omitted columns of M labeled by $(1-2s)^3$, $s^4, (1-2s)^4, \ldots$ and omitted rows of M labeled by $(1-2t)^3, t^4, (1-2t)^4, \ldots$ are identically zero. To specify the elements of M, we start with an entry $+1$ for $M_{t^3(1-2s)^2}$. The corresponding term, $t^3(1-2s)^2$, in $H^{(0)}$ has partial waves of the correct sign in the s channel. Antisymmetry requires a -1 for $M_{(1-2t)^2 s^3}$ and the term $-(1-2t)^2 s^3$ it generates in $H^{(0)}$ has partial waves of the wrong sign in this channel. To correct for this, after noting that $s \geq s^3$ for $0 \leq s \leq 1$, we assign a $+1$ for $M_{(1-2t)^2 s}$ since it leads to a term $s(1-2t)^2$ in $H^{(0)}$. Finally, antisymmetry requires $M_{t(1-2s)^2}$ to be -1. The term $-t(1-2s)^2$ does not influence d waves and higher waves in the s channel, so that (4.5) is the final form for $H^{(0)}$.

The trick of constructing permissible M's by suitably inserting ± 1 as its entries is readily generalized when the dimension of M is larger. More sophisticated forms of M may also be constructed by using representation theorems for non-negative functions on the interval $[0, 1]$.[12]

Continuous analogs of the above result may be obtained using the results of Sec. III. It is enough to give an example, since the general statement is along lines similar to the one above. Consider

$$H^{(0)}(s,t) = t^\lambda s^\mu - t^\mu s^\lambda + t^\mu - s^\mu, \tag{4.6}$$

where λ and μ are any two numbers restricted by $(2m-1) < \lambda$, $\mu < 2m$, $m = 1, 2, 3, \ldots$. This $H^{(0)}$ is antisymmetric. The s-channel partial waves of $t^\lambda s^\mu$ have the correct positivity, while those of $-t^\mu s^\lambda$ do not. But since $0 \leq s^\lambda \leq 1$ when $s \in [0, 1]$, this error of sign is compensated by the partial waves of t^μ. Since $-s^\mu$ contributes only to the s wave in the s channel, (4.6) defines an acceptable $H^{(0)}$.

Owing to the linearity requirement, we have found no simple generalization of the constructions in Sec. III for $H^{(c)}$.

V. s- AND p-WAVE INEQUALITIES INVOLVING TOTAL CROSS SECTIONS AND OTHER MEASURABLE DATA

In this section we improve the previous inequalities by finding lower bounds for the right-hand sides of (2.9) and (2.15) in terms of total cross-sections and other observable quantities. The method is well known to workers in the field, but the results are derived here in view of their physical interest.

We briefly recall the steps which lead to the Martin result (2.3). The imaginary parts of the partial waves in the t channels of $A^{(i)}(s,t)$ $(i=0,c)$ are non-negative, since these channels refer to the elastic processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^+ \to \pi^0\pi^+$, respectively. As a consequence,[13]

$$\frac{\partial^n}{\partial s^n} A_t^{(i)}(s,t) \geq 0 \quad n = 0, 1, 2, \ldots; \quad 0 \leq s \leq 1 \tag{5.1}$$

where $A_t^{(i)}(s,t)$ denotes the absorptive part in t of $A^{(i)}(s,t)$. The s-channel partial waves $a_l^{(i)}(s)$ for $0 \leq s \leq 1$ and $l = 2, 4, 6, \ldots$ satisfy the Froissart-Gribov representation[14]

$$a_l^{(i)}(s) = \frac{4}{\pi(1-s)} \int_1^\infty dt\, Q_l\left(\frac{2t}{1-s} - 1\right) A_t^{(i)}(s,t). \tag{5.2}$$

It is also known that

$$Q_l(x) \geq 0 \quad \text{for } x \geq 1. \tag{5.3}$$

Therefore, since $2t/(1-s) - 1 \geq 1$ for $t \geq 1$ and $0 \leq s \leq 1$, Eq. (2.3) follows from using (5.1) with $n=0$ and (5.3) in (5.2).

The inequalities (2.3) can be improved by finding suitable lower bounds for $A_t^{(i)}(s,t)$ in (5.2) in terms of experimentally accessible functions. There are several such bounds of which we describe a few. The simplest of these follows from (5.1):

$$A_t^{(i)}(s,t) \geq A_t^{(i)}(0,t), \quad 0 \leq s \leq 1. \tag{5.4}$$

Here $A_t^{(i)}(0,t)$ is proportional to the $\pi^0\pi^0$ total cross-section for $i=0$ and to the $\pi^0\pi^+$ total cross section for $i=c$. There are actually better bounds than (5.4) for $A_t^{(i)}(s,t)$ in terms of $A_t^{(i)}(0,t)$, due to Martin and to Singh and Roy.[15] Although the Singh-Roy bound is superior to Martin's, we give the latter in view of its greater simplicity:

$$A_t^{(i)}(s,t) \geq 2\left(\frac{t}{t-1}\right)^{1/2} \left[P'_{N+1}\left(1 + \frac{2s}{t-1}\right) + P'_N\left(1 + \frac{2s}{t-1}\right)\right], \quad 0 \leq s \leq 1 \tag{5.5a}$$

provided

$$\tfrac{1}{2}\left(\frac{t-1}{t}\right)^{1/2} A_t^{(i)}(0,t) \geq 1 \tag{5.5b}$$

and

$$A_t^{(i)}(s,t) \geq A_t^{(i)}(0,t), \quad 0 \leq s \leq 1 \tag{5.5c}$$

otherwise. Here N is the largest non-negative integer which fulfills the inequality

$$(N+1)^2 \leq \tfrac{1}{2}\left(\frac{t-1}{t}\right)^{1/2} A_t^{(i)}(0,t). \tag{5.5d}$$

Let us write (5.4) and (5.5) collectively in the form

$$A_t^{(i)}(s,t) \geq \alpha^{(i)}(s,t), \quad 0 \leq s \leq 1 \tag{5.6}$$

where $\alpha^{(i)}(s,t)$ denotes either of the bounding functions. If follows from (5.2) that

$$a_l^{(i)}(s) \geq \frac{4}{\pi(1-s)} \int_1^\infty dt\, Q_l\left(\frac{2t}{1-s}-1\right)\alpha^{(i)}(s,t), \quad l=2,4,6,\ldots;\ 0 \leq s \leq 1 \tag{5.7}$$

which, if we remember (2.9), improves (2.7) to

$$-\int_0^1 ds(1-s)h_0^{(0)}(s)a_0^{(0)}(s) \geq \frac{4}{\pi}\int_0^1 ds \int_1^\infty dt \left[\sum_{\substack{l\geq 2\\ l \text{ even}}} h_l^{(0)}(s) Q_l\left(\frac{2t}{1-s}-1\right)\right]\alpha^{(0)}(s,t). \tag{5.8}$$

A generalization of (5.4) also implied by (5.1) is

$$A_t^{(i)}(s,t) \geq \sum_{n=0}^{N} \frac{s^n}{n!}\left[\frac{\partial^n}{\partial s^n} A_t^{(i)}(s,t)\right]_{s=0}, \quad 0 \leq s \leq 1;\ N=0,1,2,\ldots\ . \tag{5.9}$$

The functions

$$\left[\frac{\partial^n}{\partial s^n} A_t^{(i)}(s,t)\right]_{s=0}$$

can be evaluated for example by a phase-shift analysis. The analog of (5.8) for the bounds (5.9) is

$$-\int_0^1 ds(1-s)h_0^{(0)}(s)a_0^{(0)}(s) \geq \frac{4}{\pi}\int_0^1 ds \int_1^\infty dt \left[\sum_{\substack{l\geq 2\\ l \text{ even}}} h_l^{(0)}(s) Q_l\left(\frac{2t}{1-s}-1\right)\right]\left\{\sum_{n=0}^{N}\frac{s^n}{n!}\left[\frac{\partial^n}{\partial s^n}A_t^{(i)}(s,t)\right]_{s=0}\right\}, \quad N=0,1,2,\ldots\ . \tag{5.10}$$

In the same way for $i=c$,

$$\sum_{l=0}^{1}\int_0^1 ds(1-s)\left[g_l^{(c)}(s)b_l^{(c)}(s) - h_l^{(c)}(s)a_l^{(c)}(s)\right] \geq \frac{4}{\pi}\int_0^1 ds \int_1^\infty dt \left[\sum_{\substack{l\geq 2\\ l \text{ even}}} h_l^{(0)}(s) Q_l\left(\frac{2t}{1-s}-1\right)\right]\alpha^{(c)}(s,t), \tag{5.11}$$

and another inequality where $\alpha^{(c)}(s,t)$ is replaced by the right-hand side of (5.9) with $i=c$.

The sum on l in the preceding results can be evaluated by noting that

$$Q_l\left(\frac{2t}{1-s}-1\right) = \tfrac{1}{2}\int_{-1}^{1} dx \frac{P_l(x)}{[2t/(1-s)-1]-x} \tag{5.12}$$

and that

$$\sum_{\substack{l\geq 2\\ l \text{ even}}} h_l^{(i)}(s)P_l(x) = \tfrac{1}{2}[H^{(i)}(s,\tfrac{1}{2}(1-s)(1-x)) + H^{(i)}(s,\tfrac{1}{2}(1-s)(1+x))] - h_0^{(i)}(s). \tag{5.13}$$

The result is

$$\sum_{\substack{l\geq 2\\ l \text{ even}}} h_l^{(i)}(s)Q_l\left(\frac{2t}{1-s}-1\right) = \tfrac{1}{2}\left(\frac{2t}{1-s}-1\right)\int_{-1}^{1} dy \frac{H^{(i)}(s,\tfrac{1}{2}(1-s)(1-y)) - h_0^{(i)}(s)}{[2t/(1-s)-1]^2 - y^2}. \tag{5.14}$$

VI. A REMARK ON THE CROSSING PROPERTIES OF INCOMPLETE PARTIAL-WAVE SUMS

Let $\{a_{l_\nu}^{(i)}(s)\}$ denote a subset of s-channel partial waves of $A^{(i)}(s,t)$ such that $a_0^{(i)}(s) \in \{a_{l_\nu}^{(i)}(s)\}$. Suppose that the sum

$$\hat{A}^{(i)}(s,t) = \sum_{l_\nu} (2l_\nu+1)a_{l_\nu}^{(i)}(s)P_{l_\nu}(z_s) \tag{6.1}$$

has the property that its t-channel s-wave is $b_0^{(i)}(t)$. Then we can show that $a_l^{(i)}(s)=0$ if $l\notin\{l_\nu\}$. Since the s-channel partial waves for $i=0$ or c are zero if angular momentum is odd, $A^{(i)}(s,t)$ is symmetric under $t\longleftrightarrow u$ exchange, so that its u-channel s wave is necessarily equal to $b_0^{(i)}(u)$. Thus we can state the result as follows: *If the partial-wave sum over a subset of partial waves of either of the processes $\pi^0\pi^0\to\pi^0\pi^0$, $\pi^0\pi^0\to\pi^+\pi^-$ produces the correct s wave in all the channels, the remaining partial waves of that process are identically zero.*

The proof is as follows. We can write

$$A^{(i)}(s,t)=\hat{A}^{(i)}(s,t)+\tilde{A}^{(i)}(s,t), \tag{6.2}$$

where

$$\tilde{A}^{(i)}(s,t)=\sum_{l\notin\{l_\nu\}}(2l+1)a_l^{(i)}(s)P_l(z_s). \tag{6.3}$$

Since $A^{(i)}$ and $\hat{A}^{(i)}$ have the same s waves in both s and t channels, the s wave of $\tilde{A}^{(i)}$ in either of these channels is zero:

$$\tfrac{1}{2}\int_{-1}^{1}dz_s\tilde{A}^{(i)}(s,t)=\tfrac{1}{2}\int_{-1}^{1}dz_t\tilde{A}^{(i)}(s,t)=0. \tag{6.4}$$

As a consequence, every $a_l^{(i)}$ in (6.3) fulfills the Martin positivity (2.3):

$$a_l^{(i)}(s)\geq 0 \quad\text{for } l\notin\{l_\nu\} \text{ and } 0\leq s\leq 1. \tag{6.5}$$

From the work in Sec. III, we know that the even s-channel partial waves of t^n are non-negative for $0\leq s\leq 1$. Thus

$$t^n=\sum_{l=0}^{\infty}h_l(s)P_l(z_s)=g_0(t),\quad n=0,1,2,\ldots \tag{6.6}$$

where

$$h_l(s)\geq 0 \quad\text{for } l=0,2,4,\ldots \text{ and } 0\leq s\leq 1. \tag{6.7}$$

Here we have suppressed the dependence of h_l and g_0 on n.

Consider the identity

$$\iint_\Delta dt\,t^n\tilde{A}^{(i)}(s,t)=\int_0^1 ds(1-s)\times\tfrac{1}{2}\int_{-1}^{1}dz_s\left[\sum_{l=0}^{\infty}h_l(s)P_l(z_s)\right]\tilde{A}^{(i)}(s,t)$$

$$=\int_0^1 dt(1-t)\times\tfrac{1}{2}\int_{-1}^{1}dz_t g_0(t)\tilde{A}^{(i)}(s,t). \tag{6.8}$$

Due to (6.4), we find

$$\sum_{l\notin\{l_\nu\}}\int_0^1 ds(1-s)h_l(s)a_l^{(i)}(s)=0. \tag{6.9}$$

Therefore, since $h_l(s)a_l^{(i)}(s)\geq 0$ for $l\notin\{l_\nu\}$ and $0\leq s\leq 1$,

$$h_l(s)a_l^{(i)}(s)=0 \quad\text{for } l\notin\{l_\nu\} \text{ and } 0\leq s\leq 1. \tag{6.10}$$

Given an l, there is always an n such that $h_l(s)$ is not identically zero. The result $a_l^{(i)}(s)\equiv 0$ for $l\notin\{l_\nu\}$ and all s follows by analytic-continuation arguments. Hence, $\tilde{A}^{(i)}(s,t)\equiv 0$.

From the Froissart-Gribov representation (5.2), we can next infer that $A_t^{(i)}(s,t)=0$. By the optical theorem, it will then follow that the amplitude $A^{(i)}(s,t)$ itself has to vanish identically.

A corollary to the above is that for the $\pi^0\pi^0\to\pi^0\pi^0$ amplitude, if a subset of partial-wave amplitudes adds up to an amplitude $\hat{A}^{(0)}(s,t)$, which is invariant under permutations of s, t, and u, and this subset contains the s wave, then the partial waves which do not belong to this subset are identically zero. As a consequence, the amplitude $A^{(0)}(s,t)$ itself must vanish for consistency with the Froissart-Gribov representation.

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APPENDIX

Several acceptable forms of $H^{(0)}$ and $H^{(c)}$ are tabulated in Eqs. (3.2)–(3.5), (3.9), and (3.10). The s waves of these functions can be projected out by using (3.1a), (3.1b), and (3.7). Equations (2.7) and (2.13) then lead to the following bounds:

Set 1: $H^{(c)}(s,t)=t^\lambda s,\quad H^{(0)}(s,t)=t^\lambda s-ts^\lambda;$

$$\int_0^1 ds(1-s)\left\{\tfrac{1}{2}s^\lambda(1-s)\left[b_0^{(i)}(s)-b_1^{(i)}(s)\right]-\frac{s(1-s)^\lambda}{\lambda+1}a_0^{(i)}(s)\right\}\geq 0. \tag{A1}$$

Set 2: $H^{(c)}(s,t)=t^\lambda(1-s),\quad H^{(0)}(s,t)=t^\lambda(1-s)-(1-t)s^\lambda;$

$$\int_0^1 ds(1-s)\left\{s^\lambda\left[\tfrac{1}{2}(1+s)b_0^{(i)}(s)+\tfrac{1}{2}(1-s)b_1^{(i)}(s)\right]-\frac{(1-s)^{\lambda+1}}{\lambda+1}a_0^{(i)}(s)\right\}\geq 0. \tag{A2}$$

Set 3: $H^{(c)}(s,t)=(1-2t)^n s,\quad H^{(0)}(s,t)=(1-2t)^n s-t(1-2s)^n;$

$$\int_0^1 ds(1-s)\left\{(1-2s)^n\tfrac{1}{2}(1-s)\left[b_0^{(i)}(s)-b_1^{(i)}(s)\right]-\frac{s}{2(1-s)}\left[\frac{1-(2s-1)^{n+1}}{n+1}a_0^{(i)}(s)\right]\right\}\geq 0. \tag{A3}$$

Set 4: $H^{(c)}(s,t)=(1-2t)^n(1-s),\quad H^{(0)}(s,t)=(1-2t)^n(1-s)-(1-t)(1-2s)^n;$

$$\int_0^1 ds(1-s)\left\{(1-2s)^n\left[\tfrac{1}{2}(1+s)b_0^{(i)}(s)+\tfrac{1}{2}(1-s)b_1^{(i)}(s)\right]-\frac{1-(2s-1)^{n+1}}{2(n+1)}a_0^{(i)}(s)\right\}\geq 0. \tag{A4}$$

Here, $i=0,\ c,\ n=0,1,2,\ldots,$ and λ is a number restricted by either

$$2m-1\leq\lambda\leq 2m,\quad m=1,2,3\ldots \tag{A5a}$$

or[11]

$$-1<\lambda\leq 0. \tag{A5b}$$

When $0\leq\lambda\leq 1$, (A1) and (A2) are still valid provided the symbol $\geq$ is replaced by the symbol $\leq$ in each inequality.

Note that if $\lambda=0$ or 1 in (A1) or (A2) or $n=0$ or 1 in (A3) or (A4), the inequality can be replaced by an equality since $h_l^{(i)}(s)a_l^{(i)}(s)\equiv 0$ for $l=2,4,6,\ldots$. The sum rules are then the ones whose derivation was first given by Balachandran and Nuyts.[3] Note also that (A1)–(A4) acquire the form (2.7) for $i=0$ on using $a_0^{(0)}=b_0^{(0)},\ b_1^{(0)}=0$.

The inequalities (A1)–(A4) with $\lambda=0,1,2,\ldots$ were already stated in Ref. 4. For completeness, we reproduce the four inequalities given towards the end of that paper. Proofs are deferred to a later publication.

$$\int_0^1 ds(1-s)(18s^4-32s^3+18s^2-3s)a_0^{(0)}(s)\geq 0, \tag{A6}$$

$$\int_0^1 ds(1-s)(-s^4-s^3+3s^2-s)a_0^{(0)}(s)\geq 0, \tag{A7}$$

$$A\equiv\int_0^1 ds(1-s)(2s^2-s)a_0^{(0)}(s)\geq 0, \tag{A8}$$

$$2AC\geq B^2, \tag{A9}$$

where

$$B=\int_0^1 ds(1-s)(5s^3-3s^2)a_0^{(0)}(s), \tag{A10}$$

$$C=\int_0^1 ds(1-s)(s^4+6s^3-6s^2+s)a_0^{(0)}(s). \tag{A11}$$

Finally, we give five examples of $H^{(0)}$ constructed using the prescription of Sec. IV. The derivation of the corresponding inequalities is simple if one uses (3.1a), (3.1b), and (3.7), and it is therefore omitted.

$$H^{(0)}(s,t)=t^5s^4-t^4s^5+t^4s^3-t^3s^4+t^3s^2-t^2s^3+t^2s-ts^2, \tag{A12}$$

$$H^{(0)}(s,t)=t^5s^4-t^4s^5+t^4s^2-t^2s^4+t^2s^3-t^3s^2+t^3s-ts^3, \tag{A13}$$

$$H^{(0)}(s,t)=t^4s^5-t^5s^4+t^5s^3-t^3s^5+t^3s^4-t^4s^3+t^4s^2-t^2s^4+t^2s^3-t^3s^2+t^3s-ts^3, \tag{A14}$$

$$H^{(0)}(s,t)=t^4s^5-t^5s^4+t^5s^2-t^2s^5+t^2s^4-t^4s^2+t^4s-ts^4, \qquad \text{(A15)}$$

$$H^{(0)}(s,t)=t^{3.5}s^{3.6}-t^{3.6}s^{3.5}+t^{3.6}s^{3.4}-t^{3.4}s^{3.6}+t^{3.4}s^{3.5}-t^{3.5}s^{3.4}+t^{3.5}s-ts^{3.5}. \qquad \text{(A16)}$$

In these constructions every term of the form $-t^\rho s^\sigma$ ($\rho>1$) is immediately followed by a term of the form $+t^\rho s^\lambda$, $\lambda<\sigma$, which compensates for the wrong signs of the s-channel partial waves of $-t^\rho s^\sigma$.

It is perhaps worth remarking that any positive linear combination of these H's is an acceptable auxilary function H.

All the bounds indicated above can be improved by the method described in Sec. V.

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[1]A. Martin, Nuovo Cimento 47A, 265 (1967); see also S. M. Roy, Phys. Rev. Letters 20, 1016 (1968).

[2]A. K. Common, Nuovo Cimento 63A, 863 (1969); 65A, 581 (1970); F. J. Yndurain, *ibid.* 64A, 225 (1969).

[3]A. P. Balachandran and J. Nuyts, Phys. Rev. 172, 1821 (1968); A. P. Balachandran, W. J. Meggs, P. Ramond, and J. Nuyts, *ibid.* 187, 2080 (1969); A. P. Balachandran, W. Case, and M. Modjtehedzadeh, Phys. Rev. D 1, 1773 (1970); M. Modjtehedzadeh, this issue, Phys. Rev. D (to be published); and further references contained in these papers.

[4]A. P. Balachandran and M. L. Blackmon, Phys. Letters 31B, 655 (1970).

[5]O. Piguet and G. Wanders, Phys. Letters 30B, 418 (1969); R. Roskies, J. Math. Phys. 11, 2913 (1970); M. R. Pennington, Nucl. Phys. B24, 317 (1970). See also A. Martin, Nuovo Cimento 63, 167 (1969); G. Auberson, G. Mahoux, O. Brander, and A. Martin, *ibid.* 65, 743 (1970).

[6]W. Case, Phys. Rev. D 3, 2472 (1971).

[7]It is easy to show that every $\pi\pi\to\pi\pi$ partial wave for which the Martin method gives such a positivity condition is of the form $\Sigma_0 a_l^{(0)}(s)+\Sigma_c a_l^{(c)}(s)$, where Σ_0 and Σ_c are non-negative constants. (Cf. Ref. 1 or Ref. 2.)

[8]To be precise, we should impose suitable conditions on $A^{(i)}$ and $H^{(i)}$ to ensure that the various mathematical manipulations are legitimate. We will not be unduly concerned with such formal problems in the paper.

[9]The integral (3.1a) is evaluated in Appendix I of Balachandran and Nuyts (Ref. 3); see Eq. (A11) of that paper. A simple way of evaluating this integral or the integral (3.7) is to use the Rodrigues formula for $P_l(z_s)$ and carry out l partial integrations. The resultant integral is readily identified as a multiple of a beta function after a linear change of variables whose expression in terms of gamma functions is known. We thank Professor P. Johnson for pointing out this method to us. The integral (3.1b) is evaluated, for instance, in E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge U. P., London, 1963), p. 310.

[10]E. T. Whittaker and G. N. Watson, Ref. 9, Chap. XII.

[11]If it is required that

$$\int_\Delta\int ds\,dt\,|H^{(i)}(s,t)|^2<\infty,$$

then (3.8d) must be changed to $-\frac{1}{2}<\lambda\le 0$.

[12]We know of two results of importance in this connection. The first states that a polynomial $T_n(s)$ of degree n is non-negative in the interval $0\le s\le 1$ if and only if it has the representation $T_n(s)=s[A_m(s)]^2+(1-s)[B_m(s)]^2$ if $n=2m+1$ $(m=0,1,2,\ldots)$, and the representation $T_n(s)=[C_m(s)]^2+s(1-s)[D_{m-1}(s)]^2$ if $n=2m$ $(m=0,1,2,\ldots)$. Here, $A_m(s)$, $B_m(s)$, $C_m(s)$, and $D_m(s)$ are polynomials of degree m with real coefficients. The second result states (leaving aside convergence questions) that a function $f(s)$ continuous in the interval $0\le s\le 1$ is non-negative in that interval if and only if it has the expansion

$$f(s)=\sum_{m,n=0}^{\infty}\alpha_{mn}s^m(1-s)^n,$$

where α_{mn} are non-negative constants. For a discussion of these results, see, for example, N. I. Akhiezer, *The Classical Moment Problem* (Oliver and Boyd, London, 1965), pp. 74, 77, and 203 ff.

[13]V. N. Gribov and I. Ya. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. 43, 308 (1962) [Soviet Phys. JETP 16, 220 (1963)]; A. Martin, Nuovo Cimento 42A, 930 (1966).

[14]Y. S. Jin and A. Martin, Phys. Rev. 135, B1375 (1964).

[15]A. Martin, Phys. Rev. 129, 1432 (1963); Virendra Singh and S. M. Roy, Phys. Rev. D 1, 2638 (1970). We thank S. M. Roy for bringing these references to our attention.