and A. Sirlin, Phys. Rev. Letters 25, 1231 (1970).

[13]The form factors of Eqs. (20) are analytic functions of the variables q and p. Their singularity structure consists of cuts, as required by unitarity, and also certain particle poles. The singularities of the $W_i^{\mu\nu}$, W_i^{ν}, W_i arising from vacuum-vacuum matrix elements of Θ_0 cancel in the Ward-identity relations and may thus be disregarded throughout the paper.

[14]In writing these Ward identities, we assume that the vertex functions are not divergent. If they were, the identities would have to be suitably modified. For example, see K. G. Wilson, Phys. Rev. 181, 1909 (1969).

[15]Analysis of the equations involving Θ_8 also yields this result.

[16]Although we choose in this paper to emphasize exploring the specific possibility that $SU(2)\times SU(2)$ is an approximate symmetry (c near $-\sqrt{2}$), our basic equations may be of interest for other values of c.

[17]A similar analysis of the A_1A_1 matrix elements merely fixes two of the six parameters, $H_{\epsilon i}$, $H_{\epsilon' i}$.

[18]Particle Data Group, Phys. Letters 33B, 1 (1970).

[19]There are two numerical errors in Ref. 7 which slightly modify the results. We list the corrected values here.

[20]Other related calculations include J. Ellis, Phys. Letters 33B, 591 (1970); R. Jackiw, Phys. Rev. D 3, 1351, 1360 (1971); G. Segrè, *ibid.* 3, 1303 (1971); R. Crewther, Phys. Letters 33B, 305 (1970); J. Ellis, P. H. Weisz, and B. Zumino, *ibid.* 34B, 91 (1971).

[21]J. J. Brehm and E. Golowich, Phys. Rev. D 2, 1668 (1970).

[22]In this regard, note that our use of $SU(3)$ as a good symmetry for the $t=0$ matrix elements of Eq. (29) differs from that of Gell-Mann, Oakes, and Renner (see Ref. 5). We thank R. Crewther for a relevant communication.

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Inequalities on Double Partial-Wave Amplitudes from Positivity, Analyticity, and Crossing Symmetry*

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Using "geometrical" inequalities similar to those of Martin for the partial waves $f_l(s)$ of the π^0-π^0 amplitude in the Mandelstam triangle, we derive sets of inequalities, following from positivity and analyticity alone, and involving only a finite number of the Balachandran double partial-wave amplitudes a_{nl}. Application of crossing symmetry gives further inequalities. Similar inequalities are obtained from the more constraining positivity and analyticity inequalities for $f_l(s)$, given by Yndurain.

I. INTRODUCTION

In this paper we derive a number of exact constraints on the π^0-π^0 scattering amplitude, following from positivity and analyticity and from crossing symmetry. This pursuit has received considerable attention the past few years,[1-3] and results derived by Martin[4] and others have been used to test various model amplitudes for the π-π system. As in the above work, the constraints derived here consist of inequalities on the partial-wave amplitudes $f_l(s)$ and on the "double" partial-wave amplitudes a_{nl} introduced by Balachandran and others.[5] These inequalities are valid within the Mandelstam triangle ($s\geq 0$, $t\geq 0$, and $u\geq 0$), and have implications for the physical amplitude once some form of analytic continuation is assumed.

We start with the work of Martin[4] and derive inequalities following from positivity and analyticity alone, relating only a finite number of partial waves in the s channel. In Sec. II we briefly review the aspect of Martin's work that we require and derive the inequalities given in (2.16). In Sec. III we introduce the amplitudes a_{nl} and derive inequalities relating a finite number of these amplitudes, again a consequence of positivity and analyticity only. In Sec. IV crossing symmetry is incorporated using the crossing matrix derived by Balachandran *et al.*,[3] leading to a further set of inequalities relating only a finite number of a_{nl}'s. In Sec. V, similar inequalities are derived from an alternative constraint, giving a "tighter" inequality; these different forms are then compared. These results and other implications are discussed in Sec. VI.

II. INEQUALITIES ON PARTIAL-WAVE AMPLITUDES

The starting point is the technique developed by Martin[4] to derive crossing-symmetry constraints on the partial-wave amplitudes $f_l(s)$ of the π^0-π^0 scattering amplitude $F(s,t)$ in the Mandelstam triangle $4 \geq s, t, u \geq 0$. (Energy units are such that $m_\pi^2 = 1$.)

Martin observes that $F(s,t)$ obeys a fixed-s dispersion relation with two subtractions[6] ("analyticity"),

$$F(s,t) = C(s) + \frac{t^2}{\pi}\int_4^\infty \frac{A_t(s,t')dt'}{t'^2(t'-t)} + \frac{u^2}{\pi}\int_4^\infty \frac{A_u(s,u')du'}{u'^2(u'-u)} \tag{2.1}$$

for $0 \leq s \leq 4$, and that the t-channel absorptive part $A_t(s,t)$ is non-negative for $t > 4$ and $0 \leq s \leq 4$. [This is "positivity," and follows from the positivity of $\mathrm{Im} f_l(t)$.] Symmetry under u-t interchange implies $A_t(s,t) = A_u(s,t)$. Putting $x = \cos\theta$, $z_0 = (4+s)/(4-s)$, and $A(s,\cos\theta) = A_t(s,t)$, Eq. (2.1) becomes

$$F(s,t) = C'(s) + \frac{1}{\pi}\int_{z_0}^\infty dz\, A(s,z)\frac{x^2}{z^2}\left(\frac{1}{z-x} + \frac{1}{z+x}\right), \tag{2.2}$$

from which follows the Froissart-Gribov definition of partial waves, viz., for even $l \geq 2$ and $0 \leq s \leq 4$,

$$f_l(s) = \frac{2}{\pi}\int_{z_0}^\infty Q_l(z)A(s,z)dz\,. \tag{2.3}$$

Using the Christoffel-Darboux relation

$$\frac{1}{z-x} = \sum_{l=0}^{L-1}(2l+1)P_l(x)Q_l(z) + L\frac{P_L(x)Q_{L-1}(z) - P_{L-1}(x)Q_L(z)}{z-x}, \tag{2.4}$$

we can write $F(s,t)$ as a truncated series $F_L(s,x)$ and a remainder $R_L(s,x)$:

$$F(s,t) = F_L(s,x) + R_L(s,x)\,, \tag{2.5}$$

$$F_L(s,x) = \sum_{l=0}^{L-2}(2l+1)f_l(s)P_l(x)\,, \tag{2.6}$$

$$R_L(s,x) = \frac{2L}{\pi}\int_z^\infty A(s,z) \times \frac{zQ_{L-1}(z)P_L(x) - xP_{L-1}(x)Q_L(z)}{z^2 - x^2}dz\,. \tag{2.7}$$

Because $A(s,x) \geq 0$ and $Q_l(z) > 0$ for $z \geq z_0 > 1$, and because $zQ_{l-1}(z)/Q_l(z)$ is an increasing function of z, there are regions in the triangle in which $R_L(s,x)$ has a definite sign, that of $P_L(x)$. For $1 \geq x \geq 0$, these regions are bounded by the curves

$$P_L(x) = 0\,,$$
$$\varphi_L(x) = \frac{z_0 Q_{L-1}(z_0)}{Q_L(z_0)} - \frac{xP_{L-1}(x)}{P_L(x)} = 0\,, \tag{2.8}$$

as shown in Fig. 1. If we define a function

$$\mathcal{K}_L(x,z) = L\frac{P_L(x)zQ_{L-1}(z)/Q_L(z) - xP_{L-1}(x)}{z^2 - x^2}, \tag{2.9}$$

we then get

$$R_L(s,x) = \frac{1}{\pi}\int_{z_0}^\infty \mathcal{K}_L(x,z)Q_L(z)A(s,z)dz\,. \tag{2.10}$$

If

$$\left.\begin{aligned} H_L(x,s) &= \sup \mathcal{K}_L(x,z) \\ h_L(x,s) &= \inf \mathcal{K}_L(x,z) \end{aligned}\right\} \text{ for } z_0 \leq z < \infty\,, \tag{2.11a}$$

then

$$h_L(x,s)f_L(s) \leq R_L(s,x) \leq H_L(x,s)f_L(s)\,. \tag{2.11b}$$

Since $\mathcal{K}_L(x,z)$ is continuous in z, it is notational-

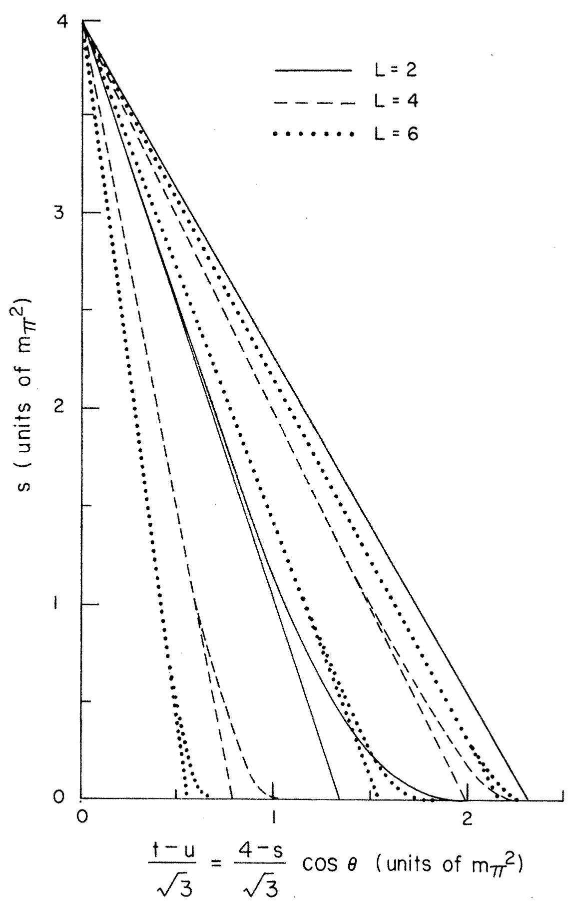

FIG. 1. The Mandelstam triangle, showing the curves $P_L(x) = 0$ (straight lines) and $\varphi_L(x) = 0$ (curved lines) for a number of even L values. Between a straight line and the corresponding curved line that coalesces with it as $s \to 4$, $R_L(s,x)$ changes sign. (Only $\cos\theta \geq 0$ is shown, because of the symmetry of the diagram about the line $u = t$.)

ly convenient to invoke (from the mean-value theorem) the existence of a point z' $(z_0 \leqslant z' < \infty)$, such that

$$R_L(s,x) = \mathcal{H}_L(x,z')f_L(s)\,. \tag{2.11c}$$

Similarly define

$$\mathcal{G}_L(x,z) = (L+1)\,\frac{xP_{L+1}(x) - zQ_{L+1}(z)/Q_L(z)P_L(x)}{z^2 - x^2}\,, \tag{2.12}$$

with

$$\left.\begin{aligned} G_L(x,s) &= \sup \mathcal{G}_L(x,z) \\ g_L(x,s) &= \inf \mathcal{G}_L(x,z) \end{aligned}\right\} \text{ for } z_0 \leqslant z < \infty\,, \tag{2.13a}$$

and a point z'' such that

$$R_L(s,x) = \mathcal{G}_{L-2}(x,z'')f_{L-2}(s)\,. \tag{2.13b}$$

[The recursion relations for the Legendre functions show that $Q_L(z)\mathcal{H}_L(x,z) = \mathcal{G}_{L-2}(x,z)Q_{L-2}(z)$, giving two equivalent forms for $R_L(s,x)$.]

Martin then shows that the uncertainty in $R_L(s,x)$, defined by $(G_L - g_L)/P_L(x)$, is a minimum on the line $\varphi_L(x) = 0$, and by considering the intersection of lines $\varphi_{L_t}(t, x_t)$ and $\varphi_{L_s}(s, x_s)$, one set in the t channel and the other in the s channel, obtains crossing-symmetry inequalities relating $f_l(s)$ and $f_{l'}(t)$.

We will consider truncated expansions $F_L(s,x)$ and $F_J(s,x)$ in the s channel only, and observe that

$$F(s,t) = F_L(s,x) + R_L(s,x) = F_J(s,x) + R_J(s,x),$$

so that we can obtain relations with only a finite number of partial waves:

$$\begin{aligned} F_{J+2} - F_L &= \sum_{l=L}^{J}(2l+1)P_l(x)f_l(s) \\ &= R_L(s,x) - R_{J+2}(s,x)\,. \end{aligned} \tag{2.14}$$

Using (2.11c) and (2.13b) with the "dummy" variables z' and z'', we obtain

$$\begin{aligned} F_J - F_{L+2} &= \sum_{l=L+2}^{J-2}(2l+1)P_l(x)f_l(s) \\ &= \mathcal{G}_L(x,2'')f_L(s) - \mathcal{H}_J(x,2')f_J(s)\,, \end{aligned} \tag{2.15}$$

with $z_0(s) \leqslant z'$, $z'' < \infty$, and $-1 \leqslant x \leqslant 1$. These relations involve partial waves from $f_L(s)$ to $f_J(s)$ only. By replacing $\mathcal{H}_J$ and $\mathcal{G}_L$ in (2.15) by H_J or h_J and G_L or g_L, depending on the signs, we obtain linear inequalities relating a finite number of partial waves, as a consequence of positivity and analyticity.

We commence by examining (2.14), and the simpler case of choosing a region of s and x so that $R_{J+2}(s,x)$ and $R_L(s,x)$ have definite, but opposite, signs. The "tighter" inequalities, following from (2.15) by varying z' and z'', will be discussed in Sec. VI.

Let the sign of R_L in a particular range of x be ϵ_L [the sign of $P_L(x)$]. We then observe from Fig. 1 that for a given pair L, J, with $J \geqslant L$, there exists a region in which $\epsilon_L \cdot \epsilon_{J+2} = -1$ (the intersection of a region in which R_L has a definite sign ϵ_L while R_{J+2} has the opposite sign). These regions may be more readily identified in Fig. 2, where the zeros and signs of $P_L(x)$ and the zeros of $\varphi_L(x)$ are displayed. The "worst" case, that of $s=0$, is plotted; gaps between the bars are the intervals in which $R_L(s,x)$ changes sign, and are largest for $s=0$; horizontal lines indicate $s = 0.2m_\pi^2$. As s approaches 4 the curve $\varphi_L(x)$ moves towards x_L, the fixed zero of $P_L(x)$. The intersection of intervals of definite sign for R_L and R_{J+2}, $a < x < b$,

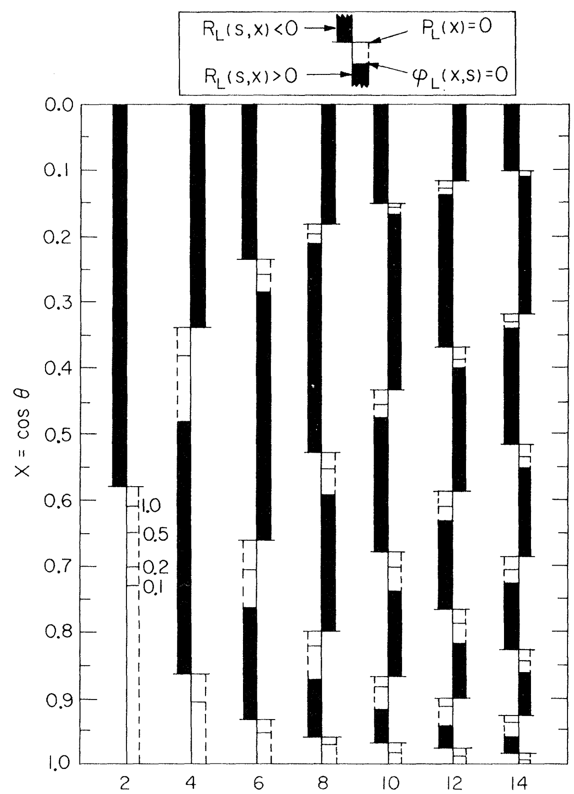

FIG. 2. A bar chart displaying the zeros of $P_L(x)$ and $\varphi_L(x)$, and the sign of $R_L(s,x)$, for $x \geq 0$ and $s=0$. A bar to the left of the central line is $R_L < 0$, and to the right is $R_L > 0$, while in the gaps the sign is uncertain. The lower end (larger x) of each bar is an s-independent zero of $P_L(x)$, x_L, while the upper end, a zero of $\varphi_L(x)$, moves with s towards x_L. A horizontal line in the gap indicates $s = 0.2m_\pi^2$. (More s values are given for $L=2$, showing how the region of uncertain sign decreases with increasing s.)

giving $\epsilon_L \epsilon_{J+2} = -1$ is easily read off, and for an x in this interval,

$$\epsilon_L \sum_{l=L}^{J} (2l+1) f_l(s) P_l(x) = \epsilon_L R_L - \epsilon_L R_{J+2} > 0 . \tag{2.16}$$

For example, in the central region $(-x_L, x_L)$, bounded by the pair of zeros of $P_L(x)$ closest to 0, $\epsilon_L = (-1)^{L/2}$. As L increases, x_L decreases, so that $-x_L < -x_{J+2} < 0 < x_{J+2} < x_L$, and thus both R_L and R_{J+2} have definite signs in $(-x_{J+2}, x_{J+2})$.

The simplest version of this is for $L=J$: $\epsilon_L (2L+1) P_L(x) f_L(s) > 0$. But $\epsilon_L P_L(x) > 0$, so that $f_L(x) > 0$, as anticipated.

If $J = L+4$, $\epsilon_{L+6} = -\epsilon_L$ in the central region $(-x_{L+6}, x_{L+6})$, and we have three adjacent partial waves related:

$$(2L+1)|P_L(x)| f_L(s) - (2L+5)|P_{L+2}(x)| f_{L+2}(s) + (2L+9)|P_{L+4}(x)| f_{L+4}(s) > 0 . \tag{2.17}$$

In order to relate two adjacent partial waves, we go to an off-central region. With $L=2$ and $J=4$ we see that $R_2 < 0$ for $x < 0.577$, while $R_6 > 0$ for $0.285 < x < 0.661$ (which is true for all $s > 0$; x goes down to 0.238 if $s=4$). So for x in (0.285, 0.577) [or (0.238, 0.577) if $s=4$],

$$|5P_2(x)| f_2(s) - 9P_4(x) f_4(s) > 0 . \tag{2.18}$$

This is nontrivial if $P_4(x) > 0$, i.e., $x < 0.33$. Then $0 < f_4/f_2 < |5P_2/9P_4| = c$, which is most constraining as x approaches $\varphi_6(x) = 0$, the lower end of the interval. For $s=0$, $c=2.28$ while for $s=4$, $c=1.35$.

This is to be compared with the result[4]

$$\frac{f_4(s)}{f_2(s)} \leqslant \frac{Q_4(z_0)}{Q_2(z_0)} = \begin{cases} 1 & \text{if } s=0 \\ \sim 0.19/z_0^2 & \text{as } s \to 4 \end{cases} \tag{2.19}$$

or the weaker results,[4] also derived by Yndurain[7] and Common,[8]

$$\frac{f_4(s)}{f_2(s)} \leqslant \frac{1}{u_0^2} = \begin{cases} 1 & \text{if } s=0 \\ \sim 0.25/z_0^2 & \text{as } s \to 4 ; \end{cases} \tag{2.20}$$

where $u_0 = z_0 + (z_0^2 - 1)^{1/2}$. So at best, our inequality is too "slack" by a factor of about $2\frac{1}{3}$. On using (2.15) we improve the result considerably. We need to account for the threshold zero of $f_l(s)$ as s approaches 4, which is done in (2.19) and (2.20). Similar results hold true for the inequalities involving more l values; they are not as tight a test of analyticity and positivity as are those given by Yndurain[7] and are clearly only a necessary condition for positivity. Their importance is that they are linear constraints, and that the coefficients of $f_l(s)$ only involve s in a simple "geometrical" way, which enables us to write a relation valid for all s, with s-independent coefficients. It is, in particular, this latter fact that allows us to derive inequalities relating a finite number of a_{nl}'s.

III. INEQUALITIES ON THE DOUBLE PARTIAL-WAVE AMPLITUDES

We now explicitly display the s dependence of the partial-wave amplitudes by expanding the scattering amplitude in a double partial-wave series introduced by Balchandran and collaborators.[5] We will content ourselves with the spinless equal-mass case, although the formalism has been extended to include spin and unequal mass.[9] A combination of helicity amplitudes having positivity properties analogous to those of the π^0-π^0 amplitude has been exhibited by Martin and Mahoux,[10] and Case[11] has used the result to derive inequalities for the general case, similar to those below. His inequalities, however, use no more than the equivalent of $f_l(s) \geqslant 0$; the generalization of our "higher" inequalities will be extended to this case of unequal mass and spin in a later publication. Other inequalities for the spinless case, expressed in terms of integrals over $f_l(s)$,[12] also use only $f_l(s) \geqslant 0$.

In our energy units, the s-channel basis functions which have simple crossing properties and are orthogonal over the Mandelstam triangle are[5]

$$S_n^l(s,t) = \left(\frac{4-s}{4}\right)^l P_n^{(2l+1,0)}\left(\frac{s-2}{2}\right) P_l(x_s) . \tag{3.1}$$

The crossing-invariant inner product for two functions $g(s,t)$ and $h(s,t)$ is

$$\begin{aligned}(g(s,t), h(s,t)) &= \int_0^4 \frac{ds}{4} \int_0^4 \frac{dt}{4}\, \theta(4-s-t) g^*(s,t) h(s,t) \\ &= \int_0^4 \frac{ds}{4}\left(\frac{4-s}{4}\right) \int_{-1}^1 \frac{dx_s}{2}\, g^*(s,x_s) h(s,x_s) ,\end{aligned} \tag{3.2}$$

and the basis functions have the simple orthogonality property

$$(S_n^l(s,t), S_N^L(s,t)) = \frac{\delta_{lL}\delta_{nN}}{2(n+1+l)(2l+1)} . \tag{3.3}$$

We then expand the scattering amplitude as

$$F(s,t) = \sum_{l=0}^{\infty} (2l+1) \sum_{n=0}^{\infty} 2(n+l+1) a_{nl} \times \left(\frac{4-s}{4}\right)^l P_n^{(\alpha,0)}\left(\frac{s-2}{2}\right) P_l(x) , \tag{3.4}$$

$$f_l(s) = \sum_{n=0}^{\infty} 2(n+l+1) a_{nl} \left(\frac{4-s}{4}\right)^l P_n^{(\alpha,0)}\left(\frac{s-2}{2}\right), \tag{3.5}$$

where $\alpha = 2l+1$ for convenience. A similar expansion in the t channel is possible, giving amplitudes b_{nl}. These are related to a_{nl} by a finite-dimensional crossing matrix $C^{\sigma}_{ll'}$, $\sigma = n+l$,

$$a_{\sigma-l,l} = \sum_{l'=0}^{\sigma} C^{\sigma}_{ll'} b_{\sigma-l',l'}, \tag{3.6}$$

where

$$C^{\sigma}_{ll'} = 2(\sigma+1)K^{\sigma}_{ll'}(2l'+1), \tag{3.7}$$

and $K^{\sigma}_{ll'} = (S^l_n(s,t), S^{l'}_{n'}(t,s))$, with $n'+l' = \sigma = n+l$, is given explicitly in Ref. 5. Crossing symmetry, $F(s,t) = F(t,s)$, then implies $a_{nl} = b_{nl}$.

We now insert (3.5) into (2.16), and multiply by a function $[(4-s)/4]^{k+1}$ which is positive in the interval $0 \leqslant s \leqslant 4$. Integrating over s, we obtain

$$\epsilon_L \sum_{l=L}^{J} (2l+1)P_l(x) \sum_{n=0}^{\infty} a_{nl} \times 2(n+l+1) \times \int_0^4 \frac{ds}{4}\left(\frac{4-s}{4}\right)^{l+k+1} P_n^{(\alpha,0)}\left(\frac{s-2}{2}\right). \tag{3.8}$$

The Jacobi polynomials $P_n^{(\alpha,0)}(x)$ are orthogonal over $(-1,1)$ with weight $w_\alpha(x) = (1-x)^\alpha$ so that the integral in (3.8),

$$D^k_{nl} = \frac{2(n+l+1)}{2^{k+l+2}} \int_{-1}^{1} dx (1-x)^{k-l} w_\alpha(x) P_n^{(\alpha,0)}(x), \tag{3.9}$$

is zero for k large enough to make up the weight $w_\alpha(x)$ yet leave a polynomial of degree less than n:

$$D^k_{nl} = 0 \quad \text{if } n > k-l \geqslant 0. \tag{3.10}$$

So if $k \geqslant J$ (the maximum value of l in the sum), the sum on n becomes finite and we finally have

$$\epsilon_L \sum_{l=L}^{J} (2l+1)P_l(x) \sum_{n=0}^{k-l} a_{nl} D^k_{nl} > 0 \tag{3.11}$$

for any $k \geqslant J$ and any $x \in (a,b)$, the interval in which R_L and R_{J+2} have fixed, opposite signs. The value of $\sigma = n+l$ ranges from L to $k \geqslant J$.

The numbers D^k_{nl} are easy to obtain from the explicit forms of the Jacobi polynomials, or numerically via recursion relations. The first few are

$$\begin{aligned} D^k_{0l} &= 2(l+1)/(k+l+2), \\ D^k_{1l} &= 2(l+2)(l-k)/[(k+l+2)(k+l+3)], \\ D^k_{2l} &= 2(l+3)\left[\frac{(l+1)(\alpha+2)}{(k+l+2)} - \frac{2(l+2)(\alpha+2)}{(k+l+3)} + \frac{(l+2)(\alpha+4)}{(l+k+4)}\right]. \end{aligned} \tag{3.12}$$

We thus obtain sets of inequalities on the a_{nl}, following from positivity and analyticity alone, in which we can vary the number of l and n values. Varying x in its allowed range will change the coefficients somewhat, but it is usually evident what value is most "useful," as in Sec. II for Eq. (2.18).

The first few inequalities are simple and interesting. For $L = J$, Eq. (3.2) is

$$\epsilon_L(2L+1)P_L(x) \sum_{n=0}^{k-L} a_{nL} D^k_{nL} > 0. \tag{3.13a}$$

But $\epsilon_L P_L(x) > 0$ in (a,b), so that

$$\sum_{n=0}^{k-L} a_{nL} D^k_{nL} > 0. \tag{3.13b}$$

So for $k = L$, $a_{0L} D^L_{0L} > 0$, and from (3.12), $D^L_{0L} > 0$, giving

$$a_{0L} > 0. \tag{3.13c}$$

For $k = L+1$,

$$a_{0L} D^{L+1}_{0L} + a_{1L} D^{L+1}_{1L} > 0, \tag{3.14a}$$

i.e.,

$$2(L+1)a_{0L} - a_{1L} > 0. \tag{3.14b}$$

For $k = L+2$,

$$a_{0L} D^{L+2}_{0L} + a_{1L} D^{L+2}_{1L} + a_{2L} D^{L+2}_{2L} > 0. \tag{3.14c}$$

Consider now $J = L+2$, $k \geqslant L+2$, and x chosen as for Eq. (2.18); then

$$\epsilon_L(2L+1)P_L(x) \sum_{n=0}^{k-L} a_{nl} D^k_{nl} + \epsilon_L(2L+5)P_{L+2}(x) \sum_{n=0}^{k-L-2} a_{nl} D^k_{nl} > 0. \tag{3.15a}$$

Set $k = L+2$; then recalling $\epsilon_L P_L(x) > 0$ and x chosen so $\epsilon_L P_{L+2} < 0$, we have

$$(2L+1)|P_L(x)|\{a_{0L} D^{L+2}_{0L} + a_{1L} D^{L+2}_{1L} + a_{2L} D^{L+2}_{2L}\} - (2L+5)|P_{L+2}(x)| a_{0L+2} D^{L+2}_{0L+2} > 0. \tag{3.15b}$$

Notice that $\{\cdots\} > 0$ from (3.14c), and also $D^{L+2}_{0L+2} a_{0L+2} > 0$ from (3.13c), so that this inequality is not trivially satisfied.

As pointed out by Case,[11] instead of multiplying by a particular positive polynomial, we can use Lukács theorem[13] to express any polynomial $P_m(s)$ of order m which is positive in the interval $0 \leqslant s \leqslant 4$, as

$$\begin{aligned} &P_m(s) = s[A(s)]^2 + (4-s)[B(s)]^2 \quad (m \text{ odd}) \\ \text{or}\quad & \\ &P_m(s) = [C(s)]^2 + s(4-s)[E(s)]^2 \quad (m \text{ even}), \end{aligned} \tag{3.16}$$

where $A(s)$, $B(s)$, $C(s)$, and $E(s)$ are any real polynomials of s. Then, if a function $F(s)$ is positive, the integral

$$\int_0^4 \frac{ds}{4} F(s) P_m(s) > 0 \,. \tag{3.17}$$

We expand the polynomials $A(s)$, $B(s)$, $C(s)$, and $E(s)$ as polynomials in the variable $y=(4-s)/4$, with coefficients α_i, β_i, γ_i, and δ_i, respectively, as follows:

$$A(s)=\sum_{L=0}^{m_A} \alpha_i y^i, \quad B(s)=\sum_{L=0}^{n_B} \beta_i y^i, \quad \text{etc.} \tag{3.18}$$

These coefficients are arbitrary real numbers. Then if we put

$$w_X^{i,j}=\int_0^4 \frac{ds}{4} X(s) F(s) \left(\frac{4-s}{4}\right)^{i+j} \tag{3.19}$$

and

$$X(s)=1,\ s,\ (4-s)/4,\ \text{or } s(4-s)/4\,, \tag{3.20}$$

the positivity of the quadratic form

$$\sum_{i,j}^{m_x} w_x^{ij} \alpha_i \, \alpha_j$$

for an arbitrary set of numbers $\{\alpha_i\}$ leads to the positivity of all the determinants:

$$D_X(M,N)=\begin{vmatrix} w_X^{M,M} & w_X^{M,M+1} & \cdots & w_X^{M,N+M} \\ & \cdots & & \\ w_X^{M+N,M} & w_X^{M+N,M+1} & \cdots & w_X^{M+N,M+N} \end{vmatrix} . \tag{3.21}$$

In particular, $D_X(M,0)=w_X^{M,M}>0$ for all $M\geqslant 0$.

Now if $F(s)$ is that polynomial leading to Eq. (3.8), we see that instead of D_{nl}^k, we should consider the more general moments

$$\mathfrak{M}_{(X)nl}^{ij}=2(n+l+1)\int_0^4 \frac{ds}{4} X(s) \times\left(\frac{4-s}{4}\right)^{i+j+l} P_n^{(\alpha,0)}\left(\frac{s-2}{2}\right). \tag{3.22}$$

Obviously,

$$\mathfrak{M}_{(1)nl}^{ij}=D_{nl}^{i+j-1}, \tag{3.23a}$$

$$\mathfrak{M}_{(4-s)nl}^{ij}=D_{nl}^{i+j}, \tag{3.23b}$$

and since $s/4=1-(4-s)/4$, we see that

$$\mathfrak{M}_{(s)nl}^{ij}=\mathfrak{M}_{(1)nl}^{ij}-\mathfrak{M}_{(4-s)nl}^{ij}=D_{nl}^{i+j-1}-D_{nl}^{i+j}, \tag{3.24a}$$

while

$$\mathfrak{M}_{(s(4-s)/4)nl}^{i,j}=\mathfrak{M}_{(4-s)nl}^{ij}-\mathfrak{M}_{(1)nl}^{i+1,j+1}=D_{nl}^{i+j}-D_{nl}^{i+j+1}. \tag{3.24b}$$

Apart from the determinantal inequalities which are more general, the simple linear inequality $w_X^{M,M}\geqslant 0$ has a relation of further interest contained in it; Eq. (3.23) duplicates the previous result, Eq. (3.11) ($k=2M-1$ if odd, $k=2M$ if even), but (3.24) leads in a similar fashion to

$$\epsilon_L \sum_{l=L}^{J} (2l+1) P_l(x) \left(\sum_{n=0}^{k-l} a_{nl} D_{nl}^k - \sum_{n=0}^{k+1-l} a_{nl} D_{nl}^{k+1}\right) > 0 \,. \tag{3.25}$$

Combining this with (3.11), we get the additional set of inequalities

$$[k]\equiv \epsilon_L \sum_{l=L}^{J} (2l+1) P_l(x) \sum_{n=0}^{k-l} a_{nl} D_{nl}^k > \epsilon_L \times \sum_{l=L}^{J} (2l+1) P_L(x) \sum_{n=0}^{k+1-l} a_{nl} D_{nl}^{k+1} > 0 \,. \tag{3.26}$$

Thus for $k=J=L$ we have

$$a_{0L} D_{0L}^L > a_{0L} D_{0L}^{L+1} + a_{1L} D_{1L}^{L+1} > 0 \tag{3.27}$$

and for $k=L+1$,

$$a_{0L} D_{0L}^{L+1} + a_{1L} D_{1L}^{L+1} > a_{0L} D_{0L}^{L+2} + a_{1L} D_{1L}^{L+2} + {}_{2L} D_{2L}^{L+2} > 0 \,. \tag{3.28}$$

In fact, we have an infinite chain of inequalities for a given L, J, and x; that is, for any $k \geqslant J$

$$[k]>[k+1]>[k+2]>\cdots>0 \tag{3.29}$$

and each bracket involves only a finite number of a_{nl}'s.

IV. APPLICATION OF CROSSING SYMMETRY

Further inequalities are derived from those of Sec. III by the application of crossing symmetry. Using the crossing matrix, Eq. (3.6), to rewrite the inequalities in terms of the t-channel amplitudes b_{nl}, we then introduce crossing symmetry by the identification $b_{nl}=a_{nl}$.

Equation (3.11) becomes

$$[k]^c \equiv \epsilon_L \sum_{l=L}^{J} (2l+1) P_l(x) \sum_{n=0}^{k-l} D_{nl}^k \sum_{l'=0}^{\sigma} C_{ll'}^{\sigma} \, b_{n'l'} > 0 \,, \tag{4.1}$$

where $\sigma=n+l=n'+l'$ and only even values of l and l' enter. It is important to note that only values of σ from L to k are involved, and of course that the S-wave $l'=0$ enters.

For low values of L, J, and k only a few b_{nl}'s appear, giving fairly simple relations. If $k=L=J$, Eq. (3.26) becomes

$$D_{0L}^L \sum_{l'=0}^{L} C_{Ll'}^L \, b_{L-l',l'} > D_{0L}^{L+1} \sum_{l'=0}^{L} C_{Ll'}^L \, b_{L-l',l'} + D_{1L}^{L+1} \sum_{l'=0}^{L=1} C_{Ll'}^{L+1} \, b_{L+1,l',l'} > \cdots > 0 \,. \tag{4.2}$$

If no subtractions are needed ($l_0=0$), as suggested in Ref. 6, we can have $L=0$ in (4.2):

$$D^0_{00}C^0_{00}b_{00} > D^1_{00}C^0_{00}b_{00} + D^1_{10}C^1_{00}b_{10} > \cdots > 0\,, \qquad (4.3)$$

while the uncrossed (3.11) gives

$$D^0_{00}a_{00} > D^1_{00}a_{00} + D^1_{10}a_{10} > \cdots > 0\,. \qquad (4.4)$$

If we now use crossing symmetry, $b_{nl}=a_{nl}$ and the numerical values of the D's and C's from (3.12) and (3.7), we have two inequalities: from analyticity and positivity,

$$a_{00} > \tfrac{2}{3}a_{00} - \tfrac{1}{3}a_{10} > 0\,, \qquad (4.5a)$$

and from crossing symmetry,

$$a_{00} > \tfrac{2}{3}a_{00} + \tfrac{1}{6}a_{10} > 0\,. \qquad (4.5b)$$

Manipulating, we get

$$a_{00} > -a_{10} > -2a_{00} \qquad (4.6a)$$

and

$$4a_{00} > -a_{10} > -2a_{00}\,. \qquad (4.6b)$$

The tighter constraint is that following from analyticity and positivity alone, (4.6a). The lower bound of $-2a_{00}$ may be increased by including more a_{nl}'s by using larger k values, as in (3.29). If (4.6a) is violated from above, we may be able to conclude that subtractions are necessary or that our amplitude does not have positivity and analyticity properties at all, depending on whether or not inequalities for higher L are violated.

The inequality for $L=J=2$ is valid for $l_0 \leq 2$, and (3.11) is then

$$D_{02}{}^2 a_{02} > D_{02}{}^3 a_{02} + D_{12}{}^3 a_{12} > 0 \qquad (4.7)$$

while (4.2) becomes

$$D_{02}{}^2(\tfrac{1}{3}a_{20} + \tfrac{1}{6}a_{02}) > D_{02}{}^3(\tfrac{1}{3}a_{20} + \tfrac{1}{6}a_{02}) + D_{12}{}^3(\tfrac{3}{4}a_{12} - \tfrac{1}{4}a_{30}) > 0\,. \qquad (4.8)$$

The S-wave parts a_{n0} in (4.8) do not appear in (4.7), and so immediate comparison is impossible. However, if the numerical values of our D-wave parts a_{n2} do satisfy (4.7), we can use them successively to bound the possible S-wave parts using the inequality (4.8).

As we go to higher L, many more inequalities, involving higher values of $\sigma = n+l$, must be simultaneously satisfied, and in each case the use of the crossed inequalities $[k]^c$ enables us to relate the S-wave components to the remainder.

For the special case of $L=J$ we define

$$[k]_L = \sum_{n=0}^{k-L} D^k_{nL}a_{nL}\,, \qquad (4.9)$$

which involves σ from L to k. The crossed bracket $[k]^c_L$ is similarly defined with $C^{\sigma}_{Ll'}$ inserted and summed over l'.

Consider, for example, L and $L+2$. From (3.29), for $L=J$ we have the pair of chains

$$[L]_L > [L+1]_L > [L+2]_L > [L+3]_L > \cdots > 0\,,$$
$$[L+2]_{L+2} > [L+3]_{L+2} > \cdots > 0\,, \qquad (4.10)$$

while for $J=L+2$ we have the coupled inequalities (for appropriate range of x)

$$(2L+1)|P_L(x)|[L+2]_L - (2L+5)|P_L+2x|[L+2]_{L+2} > (2L+1)|P_L(x)|[L+3]_L - (2L+5)|P_{L+2}(x)|[L+3]_{L+2} > \cdots > 0\,. \qquad (4.11)$$

The expressions $[L+2]_L$, $[L+3]_L$, $[L+2]_{L+2}$, and $[L+3]_{L+3}$ involve σ from L to $L+3$, and when crossed add in $a_{L+2,0}$ and $a_{L+3,0}$ (others already appear in lower inequalities).

Combining (4.10) and (4.11) with crossing, we have the nontrivial constraint involving $a_{L+2,0}$ and $a_{L+3,0}$ in terms of the other a_{nl} with $n+l \leq L+3$:

$$(2L+1)|P_L(x)|([L+2]^c_L - [L+3]^c_L) > (2L+5)|P_{L+2}(x)|([L+2]^c_{L+2} - [L+3]^c_{L+2}) > 0. \qquad (4.12)$$

V. TIGHTER INEQUALITIES

As exemplified near the end of Sec. II, the inequalities derived above are only necessary constraints, and are not the "tightest" possible. In this section we obtain similar inequalities for the a_{nl}'s, based on a result of Yndurain,[7] who considered the power-moment problem related to the Froissart-Gribov definition (2.3) of the partial waves

$$f_l(s) = \frac{2}{\pi}\int_{z_0}^{\infty} A(s,z)Q_l(z)dz\,, \qquad (5.1)$$

where $A(s,z)$ is a non-negative function for $z_0(s) \leq z < \infty$ and $0 \leq s \leq 4$, $z_0 = (4+s)/(4-s)$.

Using the representation for $Q_l(z)$ given by

$$Q_l(z) = \int_0^{\infty} d\theta[z + (z^2-1)^{1/2}\cosh\theta]^{-l-1}\,, \qquad (5.2)$$

an associated moment problem may be defined by

$$f_l(s) = \int_{u_0}^{\infty} dz\, B(s,z)z^{-l-1}\,, \qquad (5.3)$$

with $u_0(s) = z_0 + (z_0{}^2 - 1)^{1/2}$ and $B(s,z)$ non-negative. By identifying it as a Hausdorff moment problem,[7] the set of inequalities

$$\delta^m f_l(s) \equiv \sum_{j=0}^{m}\binom{m}{j}(-1)^j f_{l+2j}(s)\left(\frac{1}{u_0}\right)^{2m-2j} \geq 0 \qquad (5.4)$$

are obtained for all $m \geq 0$ and even $l \geq l_0$, necessary and sufficient for non-negative $B(s,z)$.

We will use this expression much in the way (2.16) was used to obtain inequalities on the a_{nl}'s.

Now $u_0(s) = (4+s+4\sqrt{s})/(4-s) \geq 1$, and as in Sec. III, we need the s-dependent coefficients (here $\sim u_0^l$) to be simple enough to make the moment integrals of (3.19) simple enough to cut off with a finite number of a_{nl}'s for each k.

We may replace $u_0(s)$ in (5.4) by any function $u_1(s)$ such that $u_1(s) \leq u_0(s)$ for $0 \leq s \leq 4$. This follows from Eq. (5.3) because $(z^{-2})^{+l+1}(u_1^{-2} - z^{-2})^m$ is positive for $u_1 \leq u_0 \leq z < \infty$. Equation (5.4) now becomes only the necessary condition (with changes in summation indices, and removal of common factors)

$$\sum_{l=L}^{J} \binom{(J-L)/2}{(l-L)/2} (-1)^{(l-L)/2} f_l(s)\, u_1(s)^l \geq 0\,, \qquad (5.5)$$

with even L, $J \geq l_0$.

If we put $u_1(s) = 1$, we have a result very similar to (2.16), with $\epsilon_L(2l+1)P_l(x)$ replaced by

$$\binom{(J-L)/2}{(l-L)/2} (-1)^{(l-L)/2}.$$

Again, we have not canceled the threshold zero, $f_l(s) \sim (4-s)^l$ as $s \to 4$, resulting in "slack" inequalities.

However, if we put $u_1(s) = 4/(4-s) \leq u_0(s)$, we do cancel this zero and obtain a tighter constraint; this form still permits the integration in a simple manner.

With $f_l(s)$ given by (3.5), we multiply (5.5) by $[(4-s)/4]^{M+1}$, and integrate over s:

$$\sum_{l=L}^{J} \binom{(J-L)/2}{(l-L)/2} (-1)^{(l-L)/2} \sum_{n=0}^{\infty} 2(n+l+1) a_{nl} \times \int_0^4 \frac{ds}{4} \left(\frac{4-s}{4}\right)^{M+1} P_n^{(2l+1,0)} \frac{(s-2)}{2} \geq 0\,. \qquad (5.6)$$

If $M \geq 2J$, we can terminate the $n = M - 2l$, as in Sec. III. This integral gives, in fact, D_{nl}^{M-l-1} [Eq. (3.9)]. We then have

$$\{M\} \equiv \sum_{l=L}^{J} \binom{(J-L)/2}{(l-L)/2} (-1)^{(l-L)/2} \sum_{n=0}^{M-2l} D_{nl}^{M-l} a_{nl} \geq 0\,, \qquad (5.7)$$

with $M \geq 2J$, to be compared with (3.11).

By considering the more general moments leading to (3.29), we obtain the chain of inequalities

$$\{M\} \geq \{M+1\} \geq \{M+2\} \geq \cdots \geq 0\,, \qquad (5.8)$$

which hold for a given L, J and $M \geq 2J$.

The principal difference between (5.7) and (3.11) (apart from the obvious absence of x, and its associated range) is that for a given L, J and minimum M or k, about twice as many a_{nl}'s enter into (5.7). In fact, if $M = 2J$, then a_{nl} for $n = 0, \ldots, 2(J-l)$ and all values of σ from L to $2J - L$ are represented in (5.7), while for $k = J$, then a_{nl}, $n = 0, \ldots, J - l$ and all values of σ from L to J appear in (3.11). These are the same for the simplest case of $L = J$. This has the consequence that many more a_{nl}'s are "mixed" in by crossing in the manner of Sec. IV.

If we now define, for $L = J$ and $M \geq 2L$,

$$\{M\}_L \equiv \sum_{n=0}^{M-2L} D_{nL}^{M-L} a_{nL} \geq 0\,, \qquad (5.9)$$

then with $M - L = k \geq L$, we regain (3.13b), with the identification

$$\{k+L\}_L \equiv [k]_L; \quad \text{i.e.,} \quad \{M\}_L = [M-L]_L\,.$$

Then if $J = L + 2$, $M \geq 2J = 2L + 4$, Eq. (5.7) gives $\{M\}_L - \{M\}_{L+2} \geq 0$, which is

$$[M-L]_L - [M-L-2]_{L+2} \geq 0\,. \qquad (5.10)$$

This means that the chain of inequalities

$$[L+4]_L - [L+2]_{L+2} \geq [L+5]_L - [L+3]_{L+2} \geq \cdots \geq 0 \qquad (5.11)$$

is to be compared with (4.11).

If we pick the closer approximation,

$$u_1(s) = (4+s)/(4-s) = z_0(s) \geq 4/(4-s)\,,$$

(5.6) becomes

$$\sum_{l=L}^{J} \binom{(J-L)/2}{(l-L)/2} (-1)^{(l-L)/2} \sum_{n=0}^{\infty} 2(n+l+1) a_{nl} \times \int_0^4 \frac{ds}{4} \left(\frac{4-s}{4}\right)^{M+1} \left(\frac{4+s}{4}\right)^l P_n^{(2l+1,0)} \left(\frac{s-2}{2}\right) \geq 0\,. \qquad (5.12)$$

Now we expand

$$\left(\frac{4+s}{4}\right)^l = \left(2 - \frac{4-s}{4}\right)^l = \sum_{r=0}^{l} \binom{l}{r} 2^{l-r} \left(\frac{4-s}{4}\right)^r (-1)^r \qquad (5.13)$$

and thus obtain the improved constraint inequality

$$\sum_{l=L}^{J} \binom{(J-L)/2}{(l-L)/2} (-1)^{(l-L)/2} 2^l \sum_{r=0}^{l} (-2)^r \binom{l}{r} \times \sum_{n=0}^{M+r-l} D_{nl}^{M+r-l} a_{nl} \geq 0\,, \qquad (5.14)$$

with $M \geq 2J$ to ensure that the sum is finite. Again, the application of crossing using (3.7) will "mix" in all a_{nl} values for $n = 0$ to $M - l$, and σ now ranges from L to $M \geq 2J$, more than in the previous two cases.

Approximating the numerator of $u_0(s)$ by some better rational function of s will give an even better inequality, with a larger number of a_{nl}'s.

VI. CONCLUSIONS AND DISCUSSION

We have presented a number of inequalities, based on successively tighter positivity and analyticity constraints, which involve only a finite number of a_{nl}'s; this number increases with the tightness of the constraint. Applying crossing symmetry via the crossing matrix leads to further inequalities which may be used either to test deviation of a given model from crossing symmetry, or to successively determine approximate values of some a_{nl}'s given others, in particular, to determine the S wave if two subtractions are needed, as expected.[4]

The approach presented here will provide a simple generalization of the inequalities for combinations of helicity amplitudes, given by Case,[11] by incorporating more of the constraint due to positivity and analyticity.

Using the a_{nl}'s is really a particularly useful parametrization of the s dependence of the $f_l(s)$'s within the Mandelstam triangle, especially significant when we wish to consider crossing symmetry. The original "geometrical" inequalities given by Martin[4] compare the partial waves at various points within the triangle, and some statement about their s dependence is required. We simply specify how many a_{nl}'s are significant, and this specifies how "smooth" the s dependence is, while still retaining the threshold zero and facilitating crossing. This expansion is also useful outside of the triangle, when combined with partial-wave dispersion relations,[5] and Grassberger[14] uses it to obtain inequalities on the left-hand discontinuity in these relations, from positivity and crossing.[15]

It was noted in Sec. II and Ref. 2 that the "geometrical" Martin inequalities on $f_l(s)$ may be tightened by using the actual bounds on the remainders $R_L(s,x)$, rather than only their signs. In fact, one is able to replace (2.18) by the stronger form [from (2.15) with $J=L+2$]

$$\mathcal{G}_L(x,z'')f_L(s)-\mathcal{K}_{L+2}(x,z')f_{L+2}(s)=0\,, \qquad (6.1)$$

giving the ratio

$$\frac{f_{L+2}(s)}{f_L(s)}=\frac{\mathcal{G}_L(x,z'')}{\mathcal{K}_{L+2}(x,z')}\,, \qquad (6.2)$$

with

$$z_0(s)\le z',\ z''<\infty\,.$$

Using the result presented after (3.13b) we obtain

$$\frac{f_{L+2}(s)}{f_L(s)}=\frac{Q_{L+2}(z'')}{Q_L(z'')}\frac{\mathcal{K}_{L+2}(x,z'')}{\mathcal{K}_{L+2}(x,z')}. \qquad (6.3)$$

We are now to vary z' and z'' to maximize the right-hand side, giving an upper bound, and to choose x so that the ratio is in fact positive and is the least upper bound. It turns out that this condition occurs when both z' and $z''\to z_0(s)$, giving again the result (2.19).

We may similarly consider $J>L+2$, and from (2.15) obtain a result involving $Q_J(z')/Q_L(z'')$ which will go as $z_0^{-(J-L)}$ as z' and $z''\to z_0$; however we will not obtain the correct powers z_0^{l-L} in each coefficient to compensate for the threshold zero at $s=4$, only one over-all factor. This again indicates that the "geometrical" inequalities, at best, are not as constraining a test of positivity and analyticity as the ("almost" sufficient) conditions of Yndurain.[7]

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[15]After this work was completed I came upon a paper by R. Roskies [Yale Report No. 2726-562, 1970 (unpublished)] in which similar results are obtained. His positivity constraint, expressed on the a_{nl}'s as $\sum_{l,n}\eta_{nl}a_{nl} \geq 0$, is similar to Eqs. (3.11), (5.7), or (5.14) above, but the method used in Secs. III and V above to derive explicit sets of η_{nl} seems somewhat more compact. His analysis of the crossing content, using a previously derived parametrization [R. Roskies, J. Math. Phys. 11, 482 (1970)], combines the constraint of several inequalities.

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Inequalities for the s- and p-Wave $\pi\pi$ Partial-Wave Amplitudes*

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An infinite number of inequalities are derived for integrals over the s- and p-wave $\pi\pi$ amplitudes in the interval $0 \leq s \leq 4m_\pi^2$ in terms of the $\pi\pi$ total cross sections and other experimentally accessible data. The main ingredients in the derivations are crossing symmetry, the positivity of the even $l \geq 2$ partial waves of the reactions $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ in the interval $0 \leq s \leq 4m_\pi^2$, and some known bounds on the crossed-channel absorptive parts of these reactions. It is shown that if the partial-wave sum over any subset of $\pi^0\pi^0 \to \pi^0\pi^0$ partial waves is itself invariant under permutations of s, t, and u, and this subset contains the s wave, then the entire $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude has to vanish identically. (Actually, a somewhat stronger result is proved for the amplitudes of both the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ or for any linear combination of these amplitudes with positive coefficients.)

I. INTRODUCTION

Some years ago, Martin[1] proved that the partial-wave amplitudes with angular momenta $l \geq 2$ of the processes $\pi^0\pi^0 \to \pi^0\pi^0$ and $\pi^0\pi^0 \to \pi^+\pi^-$ are non-negative when the square of the center-of-mass energy s is restricted to be in the region $0 \leq s \leq 1$. (We take the pion mass m_π to be $\frac{1}{2}$ and denote the Mandelstam variables by s, t, and u.) Later work by Common and by Yndurain[2] extended Martin's results and revealed a more refined set of inequalities for these partial waves. General methods for studying the crossing properties of partial waves have been developed by Balachandran *et al.* and by Modjtehedzadeh.[3] In this and subsequent papers, we will use the positivity properties of the partial waves due to Martin, Common, and Yndurain, in conjunction with the crossing properties of the partial waves of four-body processes studied by Balachandran *et al.* and some other known properties of scattering amplitudes, to derive an infinite number of integral inequalities for the $\pi\pi$ partial waves. The emphasis in the present work will be on stating *simple* algorithms for writing down inequalities which involve only the s and p waves. Further, the Common-Yndurain refinement of the Martin inequalities will be completely ignored here. For these reasons, the results will not be exhaustive. (A preliminary account of this research has been reported elsewhere.[4]) In a second paper, we will develop suitable elementary (and therefore incomplete) algorithms for deriving partial-wave inequalities, taking advantage of the work of Common and Yndurain, while in a third paper, an attempt will be made to state systematically all such inequalities which follow from crossing symmetry and the Martin-Common-Yndurain positivity properties of the partial waves. For similar and occasionally overlapping research, we refer the reader to Piguet and Wanders, to Roskies, and, most recently, to Pennington.[5]

Some unanticipated insights provided by these inequalities refer to the allowed crossing properties of partial-wave sums over subsets of partial waves. They are partially described below and merit attention since they indicate some possible difficulties in enforcing crossing symmetry and unitarity in any model.