Threshold Zeros and the N/D Method in Yukawa-Potential Scattering*

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The various methods which have been used to enforce the existence of the threshold zeros in the solutions of the partial-wave dispersion relations are compared. It is found that these methods are very closely related and are in fact equivalent under the proper conditions. The above are then studied in detail for the case of the potential scattering of equal-mass particles under the influence of a Yukawa interaction.

I. INTRODUCTION

The N/D method was introduced by Chew and Mandelstam[1] in order to linearize the relativistic partial-wave dispersion relations. However, since its inception, practical applications have been plagued with the problem of finding a sufficiently accurate approximation for the discontinuities across the left-hand cuts[1-4] and the related problem of producing amplitudes that have the correct behavior at threshold.[1,5,6]

We study these problems in the case of Yukawa-potential scattering of equal-mass particles in order to have an exactly soluble model which has analytic structure similar to the relativistic problem. Although our numerical results depend on this model, our treatment of threshold conditions applies equally well to the relativistic case.

A wide variety of methods have been used to treat the threshold problem. The most common is the use of a threshold or phase-space factor, [1-4] which is factored out of the original amplitude before applying the N/D method. If this factor is chosen to vanish as $(s-s_0)^l$ at threshold, the resulting amplitude also has this property. An alternative technique is to add background pole terms of the form $A/(s-a)$ to the left-hand cut terms.[5] The positions and residues of these poles can be chosen to give a solution with the correct threshold behavior. These additional forces may be interpreted as representing short-range effects neglected in using the hypothesis that nearby singularities dominate[1,6] – that is, approximating the left-hand cut by the first few Born terms.

The first method suffers from the fact that the simple choice $\rho=(s-s_0)^l$ produces an amplitude which violates unitarity for $l \geq 1$.[6] In fact, it is easily seen that for large s, ρ must be bounded by a constant. Thus, the simplest choice of ρ is $\rho(s) =(s-s_0)^l/\mathcal{P}_l(s)$, where $\mathcal{P}_l(s)$ is an lth-degree polynomial in s with zeros on the left. To avoid arbitrariness, many authors simply use $\rho(s)=(s-s_0)^l$ and cut off their integrals at some large value to avoid divergences.[7,8] As we shall see, this corresponds to choosing $\mathcal{P}_l(s)$ such that all its zeros have been pushed to infinity. Since a suitable threshold factor has l-poles, it is obviously quite similar to the use of background poles. In fact, the two methods can be shown to be identical; the poles in $\rho(s)$ correspond exactly to the positions of the background poles.[9,10] Hence, the cut-off N/D equations are effectively including additional forces of zero range.

We first investigate in detail the interrelation of the methods of insuring threshold zeros. We show that the use of threshold factors is completely equivalent to using background poles and that the latter method can be employed either by introducing poles into the equation for $N(s)$ or zeros into the equation for $D(s)$. This last formulation resembles the equations in the presence of a Castillejo-Dalitz-Dyson ambiguity.[10] Finally, we note that choosing these additional background poles at infinity is equivalent to solutions obtained by using a cutoff in the N/D equations.

In order to study the problems of threshold factors and the left-hand discontinuity in a physical problem, we have compared various approximate solutions to the N/D equations for p-wave scattering with the exact solution of the Schrödinger equation. We work with an attractive Yukawa force with $g^2/\mu=5$ which produces no resonances but has a phase shift significantly larger than the first Born approximation so that unitarity corrections are important. This potential strength is comparable to that found in strong interaction physics where the π-π force due to ρ-meson exchange has been estimated to be $g^2/\mu \sim 3$.[11] We have not included long-range repulsive forces which are thought to be important for resonance formation[12] but restrict our considerations to finding an accurate approximation to the solution with attraction only.

Luming[7] and Collins and Johnson[8] have studied the effects of using higher-order Born approximations for the left-hand cut within the framework of the cut-off N/D equations. We also study the effect of using higher Born approximations for the left-hand cut but within the context of a more general treatment of the threshold problem. We find that with our coupling strength, no combination of threshold effects with the first Born approximation for the left-hand cut produces accurate solutions. However, if we include the second Born approximation we obtain excellent results, whereas Collins and Johnson were forced to include the third Born terms in their approach to achieve similar accuracy.

In Sec. II, we review the physical problem of scalar-scalar Yukawa scattering and formulate the usual N/D equations with a threshold factor, $\rho(s)$. In Sec. III, we show that the threshold factor is equivalent to the use of background poles and show that these equations can be written either with poles in the $N(s)$ equation or poles in the $D(s)$ equation. Section IV is devoted to a comparison of exact solutions to the scattering problem with various approximations to the left-hand discontinuities.

II. BACKGROUND AND NOTATION

We consider the scattering of equal-mass scalar particles due to an attractive Yukawa force in the nonrelativistic Schrödinger equation. Our units are such that $\hbar = c = 1$ and the external mass $\mu = 1$. We also choose the exchange mass $m = 1$. In the c.m. system, the radial Schrödinger equation has the form

$$u''(r) + \left[k^2 - V(r) - \frac{l(l+1)}{r^2}\right]u(r) = 0, \tag{1}$$

where k is the c.m. momentum. The potential is chosen to be

$$V(r) = -g^2 e^{-r}/r$$

and our numerical results are presented for $g^2 = 5$.

We normalize our scattering amplitudes such that on the right-hand cut,

$$\mathrm{Im} f_l = k|f_l|^2,$$

and we expect that $f \sim k^{2l}$ at threshold. We disperse in the variable $s = 4(1+k^2)$ so that $f \sim (s-4)^l$ at threshold. We shall work with the functions $N(s)$ and $D(s)$ normalized such that

$$\lim_{s\to\infty} N(s) \to 0,$$

$$\lim_{s\to\infty} D(s) \to 1,$$

and assume that $D(s)$ and $N(s)$ have only the right- and left-hand cuts of $f_l(s)$, respectively.

Within this normalization we introduce a threshold factor

$$\rho(s) = (s-4)^l/\mathcal{P}_l(s),$$

where $\mathcal{P}_l(s)$ is an lth-degree polynomial in s with zeros only to the left of $s=4$. Without loss we choose $\lim_{s\to\infty}\rho(s) \to 1$. Our N/D equations are now written for the new amplitude

$$\bar{f}(s) \equiv \frac{1}{\rho(s)} f_l(s),$$

where now on the right

$$\mathrm{Im}\,\bar{f}(s) = k\rho(s)|\bar{f}\,|^2,$$

and unless $\bar{f}(s)$ accidentally has threshold zeros, $f_l(s)$ will automatically behave correctly at threshold. We now define

$$\bar{f}(s) = \bar{N}(s)/\bar{D}(s),$$

which yields the following equations:

$$\bar{D}(s) = 1 - \frac{1}{\pi}\int_R \frac{k'\rho(s')\bar{N}(s')d(s')}{s'-s}, \tag{2}$$

$$\bar{N}(s) = \frac{1}{\pi}\int_L \frac{\bar{D}(s')\mathrm{Im}\,\bar{f}(s')ds'}{s'-s}, \tag{3}$$

where R and L imply integrations over the right- and left-hand cuts, respectively. We now write

$$f = \rho(s)\bar{N}(s)/\bar{D}(s) \equiv N(s)/D(s)$$

and

$$\mathrm{Im}\,\bar{f} = [1/\rho(s)]\,\mathrm{Im}\,f(s).$$

(Note that $1/\rho$ is an analytic function on the left-hand cut.) Since we do not generally know $\mathrm{Im}\,f(s)$ exactly on the left-hand cut, our principal physical assumption will be to replace $\mathrm{Im}\,f$ by some approximation, $\mathrm{Im}B$, in this region. Then our equations become

$$D(s) = 1 - \frac{1}{\pi}\int_R \frac{k'N(s')ds'}{s'-s}, \tag{4}$$

$$N(s) = \frac{\rho(s)}{\pi}\int_L \frac{D(s')\mathrm{Im}B(s')\,ds'}{\rho(s')(s'-s)}. \tag{5}$$

Introducing $D(s)$ into the expression for $N(s)$ and interchanging the order of integration, we find

$$N(s) = B_\rho(s) + \frac{1}{\pi}\int_R ds'\,\frac{k'N(s')}{s'-s}\left[\frac{\rho(s)}{\rho(s')}B_\rho(s') - B_\rho(s)\right], \tag{6}$$

where

$$B_\rho(s) \equiv \frac{\rho(s)}{\pi}\int_L \frac{\mathrm{Im}B(s')ds'}{\rho(s')(s'-s)}$$

and $\mathrm{Im}B(s)$ is chosen to be some approximation (to be specified later) of $\mathrm{Im}\,f(s)$ on the left-hand cut. In Sec. III, we shall compare these equations with other methods of enforcing threshold behavior.

III. RELATIONS AMONG THRESHOLD TECHNIQUES

(A) In this section, we shall sketch the equivalence of the N/D equations of Sec. II with threshold factor $\rho(s)$ to the method of using background poles. Since $\rho(s)$ is the ratio of two polynomials, we can rewrite $\rho(s)$ in partial fractions as

$$\rho(s)=1+\sum_{i=1}^{l}\frac{\gamma_i}{s-\alpha_i}\,, \tag{7}$$

where the $\{\alpha_i\}$ are the zeros of the polynomial, $\mathcal{P}_l(s)$. Likewise we can write

$$\frac{\rho(s)}{s'-s}=\frac{\rho(s')}{s'-s}+\sum_{i=1}^{l}\frac{\gamma_i}{(s-\alpha_i)(s'-\alpha_i)}\,. \tag{8}$$

Using this expression in Eq. (5), we obtain

$$N(s)=\frac{1}{\pi}\int_L\frac{D(s')\mathrm{Im}B(s')ds'}{s'-s}+\frac{1}{\pi}\sum_{i=1}^{l}\frac{\gamma_i}{s-\alpha_i}\int_L\frac{D(s')\mathrm{Im}B(s')ds'}{\rho(s')(s'-\alpha_i)}\,. \tag{9}$$

On the other hand, if we set $\rho(s)=1$ in Eq. (5) and replace $\mathrm{Im}B(s')$ by

$$\mathrm{Im}B(s')\to\mathrm{Im}B(s')-\sum_{i=1}^{l}\sigma_i\delta(s'-\alpha_i)\,,$$

then we obtain

$$N(s)=\frac{1}{\pi}\int_L\frac{D(s')\mathrm{Im}B(s')ds'}{s'-s}+\frac{1}{\pi}\sum_{i=1}^{l}\frac{\sigma_i D(\alpha_i)}{s-\alpha_i}\,. \tag{10}$$

In both cases, the equation for $D(s)$, Eq. (4), remains the same but the equation for $N(s)$ is of the form

$$N(s)=\frac{1}{\pi}\int_L\frac{D(s')\mathrm{Im}B(s')ds'}{s'-s}+\sum_{i=1}^{l}\frac{R_i}{s-\alpha_i}\,. \tag{11}$$

To show that Eqs. (9) and (10) are completely equivalent, we must demonstrate that if the $\{\alpha_i\}$ in Eq. (10) are chosen to produce threshold behavior like $(s-s_0)^l$, then the residues of the poles $\{R_i\}$ are the same in both cases. This is straightforward to show and has been carried out elsewhere.[9,10] We shall not repeat the proof here. Note also that if some α_i's are equal, our expressions change because the simple partial-fraction expansion, Eq. (8), is invalid. However, it can be shown that if this limit is taken carefully, in Eq. (9), we obtain the correct expression.

In practice one works with Eq. (11) for $N(s)$ and chooses the $\{R_i\}$ to produce solutions with correct threshold behavior or works with Eq. (5) for $N(s)$ with a threshold factor. If the poles $\{\alpha_i\}$ are the same, identical results are obtained. Although this is obvious, it is rarely apparent in other works because of rather different choices for the $\{\alpha_i\}$. Authors using Eq. (5) have often chosen forms, such as $\rho(s)=q^{2l}/s^l$ [6] or (in case of meson-baryon scattering) $\rho(s)=q^{2l}/(q+m)^{2l}$,[13] which correspond to small $\{\alpha_i\}$, that is, long-range forces. However, in using Eq. (11), the tendency is to interpret these poles in $B(s)$ as short-range effects and use large values of $\{\alpha_i\}$. In Sec. III B, we note the extreme case of $\lim\{\alpha_i\}\to\infty$.

(B) In solving Eqs. (4) and (5), it is convenient to introduce a large cutoff into the integrations. Jones and Tiktopoulos[14] have shown that such equations can be solved by matrix means as in the Fredholm case. With this cutoff to maintain our integrals finite, we can formally pass to the limit where $\{\alpha_i\}\to\infty$ and make the replacements

$$\frac{\rho(s)}{\rho(s')}\to\left(\frac{s-4}{s'-4}\right)^l,$$

$$B_\rho(s)\to\frac{(s-4)^l}{\pi}\int_L\frac{\mathrm{Im}B(s')ds'}{(s'-4)^l(s'-s)}\,.$$

Our Eqs. (4) and (6) become, respectively (R_c indicates a cutoff at large s),

$$D(s)=1-\frac{1}{\pi}\int_{R_c}\frac{k'N(s')ds'}{s'-s}$$

and

$$N(s)=(s-4)^l B_c(s)+\frac{(s-4)^l}{\pi}\times\int_{R_c}\frac{k'N(s')}{(s'-s)}[B_c(s')-B_c(s)]\,ds'\,,$$

where

$$B_c(s)=\frac{1}{\pi}\int_L\frac{\mathrm{Im}B(s')ds'}{(s'-4)^l(s'-s)}\,.$$

For completeness, we note that $B_c(s)$ can be written in the form

$$B_c(s)=\frac{B(s)}{(s-4)^l}-\frac{1}{\pi}\int_R\frac{\mathrm{Im}[B(s')/(s'-4)^l]\,ds'}{s'-s}\,.$$

Making the identification $N(s)=\tilde{N}(s)\times(s-4)^l$, we secure

$$D(s)=1-\frac{1}{\pi}\int_{R_c}\frac{k'(s'-4)^l\tilde{N}(s')ds'}{s'-s}$$

$$\tilde{N}(s)=B_c(s)+\frac{1}{\pi}\int_{R_c}\frac{k'(s'-4)^l\tilde{N}(s')}{s'-s}\times[B_c(s')-B_c(s)]\,ds'\,,$$

which are the equations used by Luming[7] and by Collins and Johnson.[8] Although we have not shown that the limit $\{\alpha_i\}\to\infty$ exists uniformly, it is clear that the most plausible interpretation of these cutoff equations is that they correspond to subtracting forces of zero range to produce correct threshold behavior. It should be noted that no forces [infinities in $B(s)$] have been neglected by using the above cutoff procedure. Rather, it has been assumed that the nonunitary behavior of the unitary integral is precisely the same, in an opposite sense, as that of $B(s)$, so that together they approach zero for high energies.

Although it is a rather strong assumption to consider background forces of zero range, it does serve to bypass two obvious problems with the more general form $\rho(s)=(s-4)^l/\mathcal{P}_l(s)$. First, as we go to larger l, the degree of arbitrariness in $\mathcal{P}_l(s)$ increases, which is contrary to our knowledge that the simple Born approximation is valid in this limit. Second, if we wish to look for Regge poles in our amplitude, we must analytically continue in l. Therefore $\mathcal{P}_l(s)$ must be chosen to have suitable analytic structure in l as well as s. The choice $\mathcal{P}_l(s)=$ constant serves to eliminate both of these difficulties; however, an alternative choice, $\mathcal{P}_l(s)=(s-a)^l$, also works and does not encounter the difficulty of zero range forces. As we shall see in Sec. IV, for the case, $l=1$, we find better agreement with the exact solution for $\mathcal{P}(s)=(s-a)$ than for $\mathcal{P}(s)=$ constant using the same approximation for the left-hand cut.

(C) In passing from Eqs. (2) and (3) to Eqs. (4) and (5), we implicitly assumed that

$$\rho\overline{N}=N,\quad \overline{D}=D\,.$$

Expanding $\rho(s)/(s'-s)$ into partial fractions then leads to an equation of the form, Eq. (11), with poles in the equation for $N(s)$. This was shown to be equivalent to adding background, polelike terms to $\mathrm{Im}B(s)$. At times it may be more convenient to work with equations such that these poles appear in the equation for $D(s)$ instead of $N(s)$.

To find such a form, we again begin with Eqs. (2) and (3) but now define

$$\overline{N}=\hat{N},$$

$$\overline{D}=\rho\hat{D},$$

so that again,

$$f(s)=\rho(s)\overline{N}(s)/\overline{D}(s)=\hat{N}(s)/\hat{D}(s)\,.$$

The equations for $\hat{N}(s)$ and $\hat{D}(s)$ are

$$\hat{D}(s)=\frac{1}{\rho(s)}-\frac{1}{\pi\rho(s)}\int_R\frac{k'\rho(s')\hat{N}(s')ds'}{s'-s},$$

$$\hat{N}(s)=\frac{1}{\pi}\int_L\frac{\hat{D}(s')\mathrm{Im}B(s')ds'}{s'-s}\,.$$

Now $1/\rho(s)=\mathcal{P}_l(s)/(s-4)^l$ has a multiple pole at $s=4$ and the expansion into partial fractions takes the form

$$\frac{1}{\rho(s)}=1+\sum_{i=1}^{l}\frac{\tau_i}{(s-4)^i}$$

and

$$\frac{1}{\rho(s)(s'-s)}=\frac{1}{\rho(s')(s'-s)}+\sum_{i=1}^{l}\sum_{j=1}^{i}\frac{\tau_i}{(s-4)^i(s'-4)^{i-j+1}}\,.$$

Using these we can write for $\hat{D}(s)$

$$\hat{D}(s)=1-\frac{1}{\pi}\int_R\frac{k'\hat{N}(s')ds'}{s'-s}+\sum_{i=1}^{l}\frac{\tau_i}{(s-4)^i}-\frac{1}{\pi}\sum_{i=1}^{l}\sum_{j=1}^{i}\frac{\tau_i}{(s-4)^j}\int_R\frac{k'\rho(s')\hat{N}(s')ds'}{(s'-4)^{i-j+1}}\,.$$

Thus, we can write for $\hat{N}(s)$ and $\hat{D}(s)$

$$\hat{N}(s)=\frac{1}{\pi}\int_L\frac{\hat{D}(s')\mathrm{Im}B(s')ds'}{s'-s},\tag{12}$$

$$\hat{D}(s)=1-\frac{1}{\pi}\int_R\frac{k'\hat{N}(s')ds'}{s'-s}+\sum_{i=1}^{l}\frac{z_i}{(s-4)^i}\,.\tag{13}$$

In the formulation of Eq. (11) we had to choose the $\{R_i\}$ such that $N(s)$ vanished like $(s-4)^l$ at threshold, but the poles $\{\alpha_i\}$ were automatically on the left (by choice). Here we automatically give $D(s)$ a pole of order l at $s=4$ but we choose the $\{z_n\}$ to make sure that its zeros lie to the left to avoid physical poles. In this scheme, Eq. (13) is quite similar to adding an lth-order CDD pole at threshold except that we want to choose the $\{z_n\}$ so that no resonances are produced. This will again produce solutions identical to the threshold factor method if these zeros of $D(s)$ are chosen at the points $\{\alpha_i\}$.

IV. RESULTS AND CONCLUSIONS

Having discussed threshold behavior in detail in Sec. III, we now examine the phase shifts generated by using various approximations for the discontinuity across the left-hand cut. We restrict the calculations to the case of p-wave scattering.

(A) Our first approximation is that which has enjoyed the widest application in actual numerical calculations; namely, $B(s)=$ first Born approximations, $B^{(1)}(s)$, plus a single background pole term, $A/(s-a)$.

In applying this method we arbitrarily choose the position a and adjust the residue A until a threshold zero appears. This procedure is tantamount to using the first Born approximation and a threshold factor of the form $\rho(s)=(s-4)/(s-a)$. Calcula-

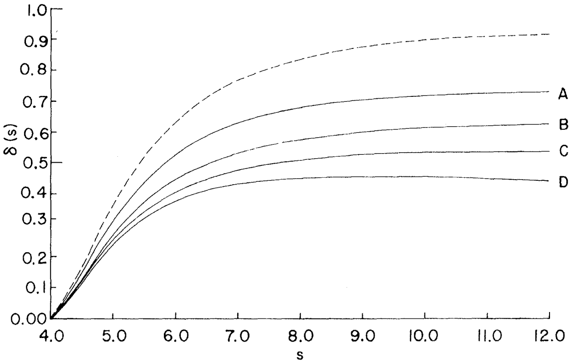

FIG. 1. The phase shift resulting from the input $B(s) = B^{(1)}(s) + A/(s-a)$ for various values of a. (A) $a=-2$, (B) $a=-6$, (C) $a=-20$, (D) $a=-200$. The dashed line results from integration of the Schrödinger equation.

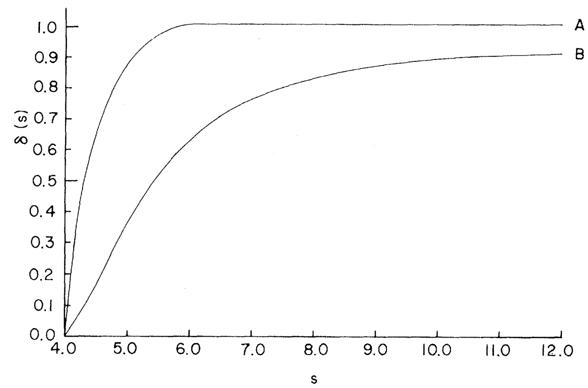

FIG. 2. The phase shifts resulting from the input (A) $B(s) = B^{(1)}(s) + B^{(2)}(s)$ and (B) the actual solution.

tions were made using both of the above procedures and, as expected, identical results, to within the accuracy of our calculations, were found. In accordance with the idea that the background pole represents additional forces, the parameter a was restricted to values less than zero. Figure 1 shows the phase shifts produced for various values of a. The strong dependence of the solutions on a is clearly shown.[15] However, for no value of a in our range of values does the approximate phase shift rise to more than 80% of the actual phase shifts. This discrepancy was found to increase as the couplings increased.

(B) We next included additional forces through the second Born approximation,[8] $B^{(2)}(s)$; that is, we tried to represent other forces as being approximately proportional to $B^{(2)}(s)$. This idea was implemented by using a left-hand cut contribution of the form $B(s) = B^{(1)}(s) + A\overline{B}^{(2)}(s)$, where

$$\overline{B}^{(2)}(s) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\operatorname{Im}B^{(2)}(s')ds'}{s'-s}.$$

The purpose was to determine the parameter A by requiring the presence of a threshold zero. In this way we had hoped to find a parameter-free solution. However, even for a wide range of A $(0.1 \leq A \leq 10.0)$, it was not possible to satisfy the threshold requirment. A typical solution $(A=1.0)$ is shown in Fig. 2 along with the exact solution determined from integration of the Schrödinger equation. The lack of proper threshold behavior of this approximation is evident.

(C) Finally, we tried a combination of the above methods. We included the left-hand-cut contribution through

$$B(s) = B^{(1)}(s) + B^{(2)}(s) + \frac{A}{s-a}, \tag{14}$$

again determining a and A as in Sec. IV, paragraph (A). The phase shifts resulting for various values of a are shown in Fig. 3. For $a=-6$ to -8 our solution is within 2% of the exact solution for $s \leq 12$. The phase shift shown in Fig. 3, curve D is for the large value $a=-200$, which is essentially equivalent to choosing the pole at infinity. We include this value in order to make a comparison of our method with the cutoff method previously discussed.

In the case under investigation we have found an approximation scheme that reproduces the exact solutions determined from the Schrödinger equation very well. In addition, the scheme has the very desirable property that it can be continued in angular momentum without violating unitarity. To accomplish this, the equivalence of the threshold factor and background pole methods are taken advantage of by including a threshold factor of the form $\rho(s) = (s-4)/(s-a)$ with the first and second Born approximations for the left-hand cut. Thus, we can extend our scheme to Reggeized calculations by using the above threshold factor and using, for instance, the new form of the strip approximation to determine our input.[4]

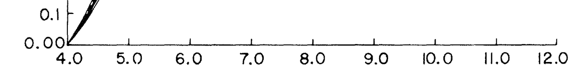

FIG. 3. The phase shifts resulting from the input (A) $B(s) = B^{(1)}(s) + B^{(2)}(s) + A/(s-a)$ for various values of a. (A) $a=-2$, (B) $a=-6$, (C) $a=-20$, (D) $a=-200$. The dashed line results from integration of the Schrödinger equation.

It is, of course, essential in an actual calculation to be able to estimate the correct value of a. Although this is an unresolved problem, we note that a behaves as expected when higher-order Born terms are included in our approximation to $B(s)$. That is, when $B(s) \simeq B^{(1)}(s)$, a very small value of $|a|$ is required to approximate the actual solution (see Fig. 1). When $B(s) \simeq B^{(1)}(s) + B^{(2)}(s)$, the proper value of a increases to about $a \simeq -6$. For the case $B(s) \simeq B^{(1)}(s) + B^{(2)}(s) + B^{(3)}(s)$, Collins and Johnson found that $a = -\infty$ produced reasonably accurate solutions. Thus, the background forces approximated by $A/(s-a)$ represent shorter-range effects in a potential model, one might hope to make a reasonable approximation for a in a physical calculation.

As a concluding remark, we note that in some recent work Collins and Johnson[16] made a ρ-meson bootstrap calculation in which they obtained very good numbers for the position and the width of the ρ meson. Previous calculations have generally found a width which is too large by a factor of about three. This is usually the fault of the phase shift turning over too fast, as do our phase shifts in Fig. 1. The interesting question then presents itself: Are higher-order elastic forces, as included in our Eq. (14), enough to produce the required narrowing or are inelastic effects, which Collins and Johnson also included, necessary? We hope to answer this question as well as a similar one with respect to the N-N^*_{33} system in the near future.

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