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Sixth-Order Radiative Corrections to the Anomalous Magnetic Moment of the Electron

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In order to evaluate the contributions to the anomalous magnetic moment of the electron (a_e), a method entirely based on computer techniques is developed. This method is embedded in the framework of the functional formalism. As a first result, the contributions to the α^3 part of a_3 from diagrams with vacuum polarization insertions are derived.

I. INTRODUCTION

Some of the contributions from the 72 diagrams contributing to the α^3 part of the anomalous magnetic moment of the electron $[a_e = \frac{1}{2}(g_e - 2)]$ have already been calculated.[1-4] Recently the measurement of a_e has been performed by Wesley and Rich[5] with an accuracy of 6 ppm, so that a knowledge of the remaining contributions is urgently needed.

One of the main features of this kind of calculation is the amount of algebra involved. A great part of it can only be done reasonably by computer. In fact, in the above-mentioned calculations, computer techniques were broadly used. Two programs devoted to symbolic manipulations were mainly used: the REDUCE system of Hearn,[6] and SCHOONSCHIP, a machine code program developed by Veltman.[7] In the method we are reporting, a slightly different method is used, and all the manipulations involved in the calculations are performed by a computer. The program written by one of us (J.C.) is intended to be general enough to calculate the contributions of all of the relevant diagrams. To do this, we had to develop a suitable method of renormalization and the appropriate calculational techniques. It appeared that the functional formalism framework[8-10] and the well-known Feynman method were suitable for our purpose.

The formulation of the renormalization theory in the framework of the functional formalism permits us to determine unambiguously the counterterms which have to be subtracted from a diagram to

make its contribution finite. This, and the fact that the tensorial dependence of the skeleton divergence of a diagram contributing to a_e is in γ_μ, have been used to develop a numerical method of cancellation of the ultraviolet divergences. The main features of this formulation are given in Sec. II.

Section III is devoted to a survey of the program. Without entering into technical details, we stress its possibilities and limitations. This program is written in the LISP programming language.[11] It takes as input a set of expressions describing a diagram and gives in the output the contributions to a_e in terms of multidimensional integrals over the Feynman parameters. It is also possible to compute either the anomalous magnetic moment of the muon (a_μ) or the difference $a_e - a_\mu$ of the electron and muon magnetic moments.

The integrals in hand are numerically estimated by means of a subroutine due to Scheppey, Dufner, and Lautrup.[12] Section IV contains a brief description of this subroutine and describes how the infrared divergences are removed, again using a numerical method.

The last section is devoted to the latest results obtained. We have chosen to compute first the diagrams with vacuum polarization insertions. While we were compiling our results, a paper by Brodsky and Kinoshita[3] appeared, reporting on the same calculations. Both results are in agreement.

II. THE FUNCTIONAL FORMALISM FRAMEWORK

The formulation of the renormalization theory in the framework of the functional formalism has been achieved for quantum electrodynamics (QED)[13] and for renormalizable scalar theories.[14-16] We give herein the main feature of this formulation for QED.

Let u_0 be the generating functional of the vacuum expectation values of the chronological products of field operators. It is defined by

$$u_0[n,\bar{n},J;m]=\exp\left(-\tfrac{1}{2}\int \bar{\eta}_\alpha(x)S^{(c)}_{\alpha\beta}(x-x';m)\eta_\beta(x')d^4xd^4x'\right)\exp\left(-\tfrac{1}{4}\int J_\mu(x)\Delta^{(c)}(x-x')J_\mu(x')d^4xd^4x'\right), \tag{2.1}$$

where η, $\bar{\eta}$, and J are the sources corresponding, respectively, to the fermion, antifermion, and photon fields; m is the electron mass. The photon mass is set to be zero. $S^{(c)}$ and $\Delta^{(c)}$ are the usual Green functions. They are defined by

$$\Delta^{(c)}(x)=-\frac{2i}{(2\pi)^4}\lim_{\epsilon=0}\int d^4k\,\frac{e^{ikx}}{k^2-i\epsilon}, \tag{2.2}$$

$$S^{(c)}(x)=-\frac{2i}{(2\pi)^4}\lim_{\epsilon=0}\int d^4k\,\frac{e^{ikx}}{k^2+m^2-i\epsilon}(i\hat{k}-m), \tag{2.3}$$

where $\hat{k}=k_\mu\gamma^\mu$ and $k^2=\vec{k}^2-k_0^2$.

The sources η, $\bar{\eta}$, and J belong to the space $D(R^4)$ and then all the formulas listed in this section are mathematically well defined.[17] The generating functional of the propagators of the interacting fields is defined by

$$u[\eta,\bar{\eta},J;e,m]=\exp\left(ie\int\Gamma(\xi)d\xi\right)u_0[\eta,\bar{\eta},J;m], \tag{2.4}$$

where e is the electron charge and Γ is a differential operator which expresses the interaction. Γ is defined by

$$\Gamma(\xi)=\frac{\delta}{\delta\eta_\alpha(\xi)}\gamma^\nu_{\alpha\beta}\frac{\delta}{\delta\bar{\eta}_\beta(\xi)}\frac{\delta}{\delta J_\nu(\xi)}, \tag{2.5}$$

where $\delta/\delta J_\nu(\xi)$ is the functional derivative with respect to the source J, at the point ξ.[18] The generating functional u is closely related to the scattering S matrix. It is straightforward to show that the functional derivatives of u, once the sources have been set equal to zero, are expressed by the usual Feynman diagrams.[17]

To develop the renormalization procedure, one has to write the u functional as a formal series in the coupling constant e:

$$u[\eta,\bar{\eta},J;e,m]=\sum_n\frac{e^n}{n!}u^{(n)}[\eta,\bar{\eta},J;m]. \tag{2.6}$$

The first step of the renormalization program is to regularize: Each term $u^{(n)}$ of the series (2.6) is regularized, for instance, by means of the Pauli-Villars method.[19] Let us call $u^{(n)}_\lambda$ such a regularized term, with λ denoting the set of auxiliary masses introduced in the regularization. It is always possible to find some λ_0 such that $u^{(n)}_{\lambda_0}=u^{(n)}$. The sum of the formal series of the regularized terms is then called u_λ.

The second step is renormalization. It consists of determining the divergent part $\mathfrak{D}\,u_\lambda$ of u_λ in such a way that

$$\lim_{\lambda=\lambda_0}\{u_\lambda-\mathfrak{D}u_\lambda\}=\mathcal{P}u, \tag{2.7}$$

where $\mathcal{P}u$ means the finite part of u. To realize this step, one has to introduce the following re-

normalization constants: $N(e,m)$, $Z_i(e,m)$, with $i=1,2,3$, $e_0(e,m)$, and $m_0(e,m)$. They are considered to be formal series in e. e_0 and m_0 are bare charge and mass, respectively; Z_1, Z_2, and Z_3 are, respectively, the change in scale of the spinors and photon sources; N is a normalization factor. Z_1 obeys the relation $Z_2 Z_3^{1/2} e_0 = Z_1 e$.

The renormalized generating functional u_{Ren} is then defined by

$$u_{\text{Ren}}[\eta,\bar{\eta},J;e,m] = \frac{1}{N}\, u[Z_2^{-1/2}\eta, Z_2^{-1/2}\bar{\eta}, Z_3^{-1/2}J; e_0, m_0]. \tag{2.8}$$

Paraziuk's theorem shows that there exists both a regularization and a choice of the renormalization constants such that

$$u_{\text{Ren}}[\eta,\bar{\eta},J;e,m] = \mathcal{P}\, u[\eta,\bar{\eta},J;e,m]. \tag{2.9}$$

Using Eq. (2.8) one can show that $\mathcal{P}u$ satisfies the following branching equation[20]:

$$\frac{d}{de}\mathcal{P}u[\eta,\bar{\eta},J;e,m] = D\mathcal{P}u[\eta,\bar{\eta},J;e,m], \tag{2.10}$$

where D is formal series of differential operators $D^{(n)}$. By expressing the quantities contained in (2.10) and (2.8) as formal series in e and expanding them in powers of e, it is straightforward to show that (2.10) leads to the following recurrence formula:

$$\mathcal{P}u^{(n+1)} = i\int d\xi\, :\Gamma(\xi):\mathcal{P}u^{(n)} + \sum_{k=1}^{n}\binom{n}{k} D^{(k)}\mathcal{P}u^{(n-k)}, \tag{2.11}$$

$$\mathcal{P}u^{(0)} = u_0.$$

The symbol : : means that one takes the Wick products of the quantities within it. The differential operators $D^{(k)}$ depend on the renormalization constants introduced in the theory. They are defined in Ref. 14. The meaning of Eq. (2.11) is that when the finite contributions of the diagrams up to the nth order are known, one can derive the finite contributions of the whole set of diagrams of order $(n+1)$. An important property of this equation is that it provides an unambiguous determination of the number and of the types of counterterms which have to be subtracted from a diagram to make its contribution finite. This is of interest at high orders when dealing with overlapping divergences.

III. CALCULATION OF THE CONTRIBUTIONS TO a_e

Equation (2.11) was first used to develop a program which gives the Feynman diagrams at a given order of the perturbative expansion.[21] The next step was the calculation of the Feynman amplitudes. At first it has been specialized to the problem of the anomalous magnetic moment. The corresponding program described in this section is written in the LISP programming language, which is particularly suited to symbolic calculations. This program is slightly different from the one written by Campbell and Hearn[22] which is also devoted to the Feynman-diagram computation.

Among the many methods which can be used to obtain the Feynman amplitudes, the most suitable for computer calculation is the one originally developed by Feynman.[23] This is because it is very systematic and does not need further inspection of the expressions derived during the computation. In fact, once the method is set up for a given low order, the recursive property of the programming language makes it work for any higher order. However, some of the technical difficulties arising from the length of the quantities handled have not yet been solved; temporary solutions were nevertheless available.

We first consider the calculation of contributions to a_e which are free from ultraviolet divergences. Any technical considerations are avoided in the following survey of what the program does. They may be found in Refs. 24 and 25.

A diagram is described by the following quantities: its denominator, numerator, the list of γ matrices appearing in the numerator, the list of the Feynman parameters, lists of internal and external momenta, and, finally, lists of the respective masses included in the denominator and numerator. We shall see later that to eliminate ultraviolet divergences we need to introduce some more information in the input. From the beginning we impose a structure on the numerator. It only involves, apart from the masses, quantities of the form $\hat{p}$ but neither scalar products nor components of momentum. Here, p means either an internal or external momentum. This is obviously true at the beginning when applying Feynman rules to derive the starting form of the amplitude; moreover, this will remain true up to the end of the computation. This choice leads to a particularly simple structure for the numerator, although it becomes somewhat larger.

We now follow through the different steps of the Feynman method. The first is to combine the factors in the denominator according to the identity

$$\frac{1}{a_1\cdots a_n} = (n-1)!\int_0^1 d\alpha_1\cdots d\alpha_n \times \delta\left(1-\sum_1^n \alpha_i\right)\left(\sum_{i=1}^n \alpha_i a_i\right)^{-n}. \tag{3.1}$$

It is also possible to make the double parametrization introduced by Kinoshita.[26] This is allowed by a special form of the α parameters input list. The next step is to find the translations which have to be done to get quadratic forms with respect to the internal momenta. These translations are performed in the order shown in the internal momenta input list. For instance, when this list is $(k_1 k_2)$ and when we have two external momenta p_1 and p_2, the translations are of the form

$$k_1 \to k_1 - c_1 k_2 - c_2 p_1 - c_3 p_2, \qquad k_2 \to k_2 - c_4 p_1 - c_5 p_2. \tag{3.2}$$

These translations are also applied in the numerator. We are now left with both a numerator and a denominator involving coefficients which are complicated combinations of the α_i parameters. A suitable simplification is achieved by replacing these coefficients by symbolic expressions which are printed with their corresponding values.

The summation over the repeated indices of γ matrices does not lead to any difficulty. When a trace calculation has to be performed, a special indicator has to be introduced in the description of the numerator given in the input. When computing the contributions to a_e, a trace calculation always comes with a summation over indices of γ matrices. This fact allows us to simplify the related part of the computation. As a consequence of the general structure imposed on the numerator, scalar products or components of momenta appearing in the trace or summation formulas have to be transformed into products of $i\hat{p}_j$ quantities, with p_j standing either for an internal or external momentum.

The next manipulation on the numerator is to express it as a sum of monomials:

$$N = \sum \beta_k \prod i\hat{p}_j \gamma_\mu \prod i\hat{p}_j . \tag{3.3}$$

Here β_k is a coefficient depending on α_i parameters and masses and in fact is expressed as a product of symbolic variables. γ_μ is the over-all tensorial dependence of the vertex being calculated. Once this decomposition is done, one has to formulate the symmetry properties of the integrals:

$$\int k_\rho k_\nu F(k^2) d^4k = \tfrac{1}{4} \int k^2 \delta_{\rho\nu} F(k^2) d^4k,$$

$$\int k_\rho k_\nu k_\delta k_\sigma F(k^2) d^4k = \tfrac{1}{24} \int [\delta_{\rho\nu}\delta_{\delta\sigma} + \delta_{\rho\delta}\delta_{\nu\sigma} + \delta_{\rho\sigma}\delta_{\delta\nu}] \times (k^2)^2 F(k^2) d^4k. \tag{3.4}$$

To apply these formulas, we have to transform the internal momenta which are of the form k into $k_\rho \gamma^\rho$. Once Eq. (3.4) has been applied, another summation over the twice-repeated indices of γ matrices has to be done in order to keep the desired structure of the numerator.

Then one has only to factorize the integration variables and to apply the following integration formula:

$$\int d^4k \frac{(k^2)^{m-2}}{(k^2 + a^2)^n} = \frac{i\pi^2}{(a^2)^{n-m}} \frac{(m-1)!(n-m-1)!}{(n-1)!}, \tag{3.5}$$

with $n > m \geq 2$. Once again the integrations are performed according to the order shown in the internal momenta input list.

Up to this point the program is general enough to calculate any diagram, and no reference has been made to the fact that we are interested in the evaluation of a_e. In fact, only two limitations appear. The first is that the maximum number of masses in the theory has been restricted to two, plus a possible mass associated with the photon. The second is that in Eq. (3.4) the formula for a product of six components does not appear, because the tensorial dependence of such a product is in γ_μ at the sixth order and therefore does not contribute to the magnetic moment. The removal of both limitations would be straightforward.

We now consider the derivation of the contributions to a_e. The numerator is at the present time a sum of the form (3.3) but in which p_i stands only for an external momentum. Using the anticommutation relations for $\hat{p}_1$ and $\hat{p}_2$ and the fact that the spinors acting on both sides of the numerator are solutions of the Dirac equation, it is always possible to transform the factors in Eq. (3.3) into $\beta' \Pi \hat{p}_j \gamma_\mu \Pi \hat{p}_j$, where $\Pi \hat{p}_j$ is now either $\hat{p}_1 \hat{p}_2$ or $\hat{p}_1$ or $\hat{p}_2$ or 1. As a matter of fact, only ten terms of the form $\Pi \hat{p}_j \gamma_\mu \Pi \hat{p}_j$ contribute to the anomalous magnetic moment. A table of these terms has been introduced into the program. To get the contributions to a_e, the external momenta are placed on their mass shell and this table is then used.

We next consider the computation of diagrams which include ultraviolet divergences. The general way to cancel these divergences is to compute both the diagram and its counterterms and to use the following identity:

$$\frac{1}{a^n} - \frac{1}{b^n} = -n \int_0^1 du \frac{(a-b)}{[(a-b)u + b]^{n+1}} \tag{3.6}$$

to obtain a compact expression of the contribution. This method has been used for renormalizable scalar field theories.[21] Nevertheless, for the present computation, the following facts led us to use a numerical method to cancel these divergences.

First, the skeleton divergence may be dropped, since its tensorial dependence is only γ_μ; furthermore, this means that all integrations over the internal momenta can be performed in those parts of both the diagram and its counterterms which contribute to a_e. Secondly, considering the way these counterterms are obtained, we can always associate the corresponding lines of the diagram and its counterterms with the same α parameter. This remark is important because we shall see in the next section that the points where the integrands are calculated are chosen randomly. The last remark is that one can get the contributions to a_e from the counterterms of a diagram by performing only slight modifications in the above-described program. In fact, the main modification is that at a given step of the calculation some momenta have to be placed on their mass shell; these are the momenta which are external for each subdiagram.

More precisely, let us call $S_{\rho_i}(p_i)$ a diagram which has for external momenta the p_i and which involves summation over the twice-repeated indices ρ_i of γ matrices. If a diagram $G_{\nu\rho\sigma}(p_1, p_2)$ enclosed a subdiagram $S_\nu(k_i)$, one has to sum first over ν to put the k_i on their mass shell and then to sum over ρ and σ to have the contribution of one of the counterterms of G. When S is a self-energy of fermion, the second counterterm of S is obtained from S by applying a derivative and using this same method. These operations are achieved by imposing in the input special names for the momenta which have to be placed on their mass shell and by adding information which tells the program where this replacement should be made.

At the present time this procedure is available for both vertices and fermion self-energy subdiagrams. Nevertheless, in the latter case, it is better to use another feature of the program, which is to introduce finite self-energies or vacuum polarization contributions. This leads to simpler expressions and saves much computing time. This is why the removal of ultraviolet divergences arising from vacuum polarization insertions has still not been implemented.

This program allows one to compute the contributions to the α^3 part of a_e from all of the contributing diagrams. It is also suitable for obtaining the contributions from the diagram contributing to $a_\mu - a_e$, where a_μ is the anomalous part of the muon magnetic moment. Furthermore, it has been developed with the aim of being easily extended to calculations of other physical effects.

IV. NUMERICAL INTEGRATION – REMOVAL OF INFRARED DIVERGENCES

A. Numerical Integration

The LISP program described in the previous section gives the contribution to a_e in terms of multidimensional integrals over the α_i parameters. In fact, at the sixth order, we are left with 8-tuple integrals. At the present time it is not convenient to use non-numerical methods to perform these integrations. We have therefore to use numerical ones.

A program originally developed by Scheppey and improved by Dufner[12] and then by Lautrup allows one to evaluate integrals of the type we are dealing with. Basically, this method of integration is a computation of a Riemann sum. The integration volume is divided into a given maximum number of subvolumes. In each subvolume the points where the integrand is calculated are chosen randomly, and the variance is thus obtained from a crude Monte Carlo calculation. To improve the accuracy of the calculation, the program works iteratively. At each iteration the divisions on the integration axis are dynamically modified according to the variance obtained in the previous iteration. The program stops when the desired accuracy, which is requested in the input, is reached. Otherwise, the number of iterations set in the input is performed. To get the results shown in the next section, we have asked for 10 to 20 iterations and for 10 000 to 800 000 subvolumes. The choice of both these quantities evidently depends on the smoothness of the integral in hand. It has already been emphasized that the cancellation of ultraviolet divergences is performed numerically. More precisely, when both the diagram and its counterterms have been computed by the LISP program, the integrals obtained have to be evaluated in a same run. The elimination of the divergences between the diagram and the counterterms only occurs if the same α parameter is associated with the same line in the diagram and in the counterterms. Because the points at which the integrands are evaluated are random, this method only works if all the integrals which contain an ultraviolet divergence are calculated at the same time.

B. The Infrared Divergences

The removal of infrared divergences (IRD) is in principle straightforward when the integrals are calculated analytically. One needs only to associate a mass with the photon, to drop the factors in either $\ln\lambda$ or $\ln^2\lambda$, where λ is the ratio of the photon and electron masses, and to pass to the limit as $\lambda \neq 0$.

Unfortunately, it is not possible to computerize this procedure at present. The removal of IRD is then much more troublesome, and can only be handled by using least-squares fitting techniques. Furthermore, even using this kind of method, the determination of the IR part of a diagram may not

be obtained with the desired accuracy (less than 5%). This is due to the amount of computing time needed to get an accurate result. A way to partially avoid this difficulty is to compute in the same run a sum of diagrams free from IRD. For this purpose we give the photon a mass and perform the calculations for a set of values of λ (generally $10^{-3}<\lambda<10^{-1}$). The results $F(\lambda)$ obtained in this way are fitted by a curve whose equation is of the form

$$G(\lambda)=A+B\lambda+C\lambda\ln\lambda+D\lambda^2\ln\lambda+E\lambda^2\ln^2\lambda+\cdots . \quad (4.1)$$

The factors in Eq. (4.1) involving λ, which are referred to as background terms, are generally smooth enough to permit a good determination of the constant term A. When a diagram contains an IRD only in $\ln\lambda$, the same method, using

$$G'(\lambda)=G(\lambda)+a\ln\lambda , \quad (4.2)$$

leads to satisfactory results. Although a is obtained within an error, it is possible to determine its best value, which is the one which gives A with the smallest error, by performing first a fit on a and then another fit on A. The accuracy of the result obviously depends on the accuracies of the $F(\lambda)$ calculations which, in their turn, depend chiefly on the amount of machine time spent for their computation.

At the present time this method does not permit us to extract an IRD of the type $(a\ln\lambda+b\ln^2\lambda)$ from the contribution of a diagram. The only available method is to evaluate at the same time several of these diagrams in such a way that their sum is free from IRD.

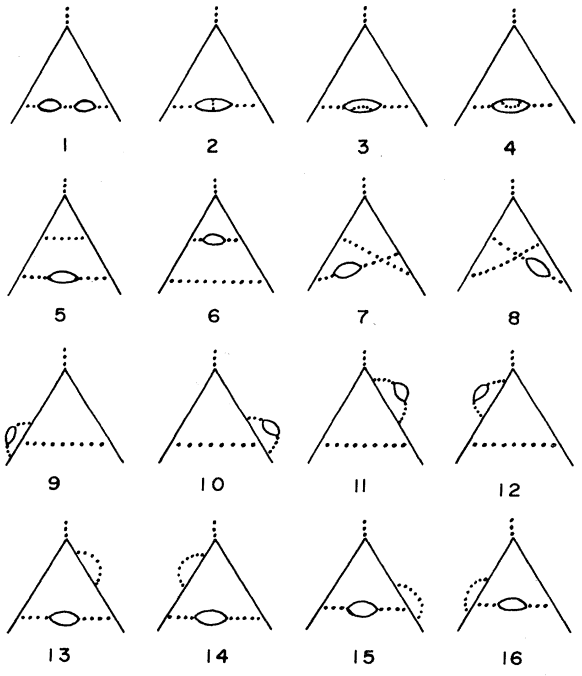

FIG. 1. Diagrams with vacuum polarization insertions contributing to the α^3 part of a_e.

V. RESULTS

The method has been tested first by calculating the contributions to the α^2 part of a_e. The result obtained was $(-0.322\pm0.086)(\alpha/\pi)^2$, which is in good agreement with the analytic value, $-0.328\times(\alpha/\pi)^2$. The details of this calculation are reported in Ref. 24; the error comes from the integration subroutine.

Two diagrams contributing to the α^3 part of $a_\mu - a_e$ have also been evaluated.[24] The results obtained are in agreement with the calculations of Lautrup, Petermann, de Rafael[27] and Brodsky and Kinoshita.[3] We have begun the complete evaluation of the contributions from the 72 diagrams contributing to a_e by first computing those with vacuum polarization insertions (Fig. 1). We have obtained the following results:

$$\begin{aligned}
&\mu^{(1)}=(0.002559\pm0.00015)(\alpha/\pi)^3 ,\\
&\mu^{(2)}+\mu^{(3)}+\mu^{(4)}=(0.05221\pm0.00209)(\alpha/\pi)^3 ,\\
&\mu^{(5)}+\mu^{(6)}=(0.0522\pm0.0010)(\alpha/\pi)^3 ,\\
&\mu^{(7)}+\mu^{(8)}=(-0.0031\pm0.0010)(\alpha/\pi)^3 , \quad (5.1)\\
&\mu^{(9)}+\mu^{(10)}=(0.0274\pm0.0005)(\alpha/\pi)^3 ,\\
&\mu^{(11)}+\mu^{(12)}=(-0.1151\pm0.0009)(\alpha/\pi)^3 ,\\
&\mu^{(13)}+\mu^{(14)}+\mu^{(15)}+\mu^{(16)}=(-0.1121\pm0.0022)(\alpha/\pi)^3 .
\end{aligned}$$

The values $\mu^{(1)}$ through $\mu^{(4)}$ were already known. Mignaco and Remiddi[1] had found by a different method that $\mu^{(1)}=0.00258(\alpha/\pi)^3$ and $\mu^{(2)}+\mu^{(3)}+\mu^{(4)}=0.05289(\alpha/\pi)^3$. While we were compiling our results, a paper by Brodsky and Kinoshita[3] appeared, reporting on the contributions $\mu^{(5)}$ through $\mu^{(16)}$. They have obtained for the sum of these contributions

$$(-0.154\pm0.009)(\alpha/\pi)^3 , \quad (5.2)$$

while we have found

$$(-0.1507\pm0.0056)(\alpha/\pi)^3 . \quad (5.3)$$

Both results are in very good agreement.

Diagrams (1), (9), (10), (11), and (12) of Fig. 1 are free from IRD and are thus easy to compute. The sum of the diagrams (5) and (6) is free from IRD. They have been computed in the same run and the background terms have been removed with the use of (4.1). The same method has been used for the sum (7) plus (8). For diagrams (13) and (14) we have also performed another calculation

in order to check the method of elimination of the IRD. We have used (4.2) to determine the IR part. The result obtained was

$$\mu^{(13)}+\mu^{(14)}=(0.01582\pm0.00026)\ln\lambda^2 -(0.0653\pm0.0003)(\alpha/\pi)^3, \qquad (5.4)$$

while for diagrams (15) and (16) it was straightforward to take off the IRD analytically. The result obtained was

$$\mu^{(15)}+\mu^{(16)}=(-0.0474\pm0.0020)(\alpha/\pi)^3+(\text{IR term}). \qquad (5.5)$$

The IR term was not checked. The sum of (5.4) and (5.5) gives (0.1127 ± 0.0023) for the constant term. This is in agreement with the value listed in (5.1) for $\mu^{(13)}+\mu^{(14)}+\mu^{(15)}+\mu^{(16)}$, which was obtained by calculating these four diagrams in the same run and eliminating the background terms.

Apart from these 16 diagrams, the only known contributions are the six light by light diagram contributions[2] and an estimate of three of the diagrams without any insertion.[4] The evaluation of other diagrams is in progress.

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