146, 980 (1966).

[11]A. Bettini *et al.*, Nuovo Cimento 47A, 643 (1967).

[12]The reasoning leading us to the conclusion that an isovector ϵ, if existent at all, is low lying or weakly coupled is based on phenomenological observations. This is not surprising from the theoretical point of view. The inelastic process $\bar{p}p \to \bar{n}n$ is in a pure $I_t = 1$ state. Annihilation contributions to its absorptive part does not add coherently, since it lacks the positivity condition which applies only to the imaginary part of the *elastic* amplitude. Indeed, $\mathrm{Im}[A(\bar{p}p \to \bar{n}n)]_{\mathrm{ann}} \propto \frac{1}{2}[\sigma_{\mathrm{ann}}(\bar{p}p) - \sigma_{\mathrm{ann}}(\bar{p}n)]$, which is small as has already been pointed out.

[13]Here we assume that, if α_ϵ were to choose nonsense, β would have a simple zero at $\alpha_\epsilon = 1$. This is so in the simple Veneziano model, for example.

[14]C. B. Chiu, S. Y. Chu, and L. L. Wang, Phys. Rev. 161, 1563 (1967).

[15]C. Schmid, Phys. Letters 20, 689 (1968); C. B. Chiu and A. Kotanski, Nucl. Phys. B7, 615 (1968).

[16]M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters 8, 261 (1962).

[17]We note here that resonance production cross section in the multiparticle final states for $\bar{p}p$ annihilation at rest has been explained in terms of the statistical model plus final-state interactions. [See L. Montanet, in Proceedings of the Fifth International Conference on Elementary Particles, Lund, Sweden, 1969, p. 218 (unpublished).] This suggests that the number of events involving ϵ_1 formation in the final states having n pions (typically for $n \geq 5$) is directly correlated to the coupling of $\epsilon_1 \to 3\pi$. Since $\Gamma(\epsilon_1 \to 3\pi) \ll \Gamma(\omega \to 3\pi)$, we expect that the number of ϵ_1 events is suppressed as compared to that of ω. The ϵ_1 events should be much less contaminated if one looks only at the peripheral processes in $\bar{p}p \to \pi^0\epsilon_1$, $\pi^+n \to \epsilon_1 p$, $K^-p \to \Lambda^0\epsilon_1$, etc.

[18]We thank Dr. R. Eisner for providing us with the 3π invariant-mass plot from the reaction $K^-p \to \Lambda\pi^+\pi^-\pi^0$.

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Infinite Set of Quasipotential Equations from the Kadyshevsky Equation*

Robert J. Yaes
Center for Particle Theory, Department of Physics, University of Texas at Austin, Austin, Texas 78712
and Department of Physics, Illinois Institute of Technology, Chicago, Illinois 60616†
(Received 22 February 1971)

We show that by modifying the propagator in the Kadyshevsky equation, we can obtain an infinite set of quasipotential equations which satisfy both Lorentz covariance and elastic unitarity and of which the Logunov-Tavkhelidze-Blankenbecler-Sugar-Alessandrini-Omnes equation and the Gross equation are special cases. We also show that the perturbation scheme of Chen and Raman, for using the quasipotential equation to obtain approximations to the Bethe-Salpeter equation, can be greatly simplified by the use of resolvent-identity-type arguments.

In potential theory, the off-shell T matrix satisfies the Lippmann-Schwinger equation, the integral form of the Schrödinger equation. Since the free-particle Green's function has the appropriate discontinuity, elastic unitarity is guaranteed by the equation if the potential is real and symmetric.[1]

In the relativistic case, we do not have a simple equation like the Schrödinger equation, so we must resort to the techniques of field theory. However, equations of the same form as the Lippmann-Schwinger equation can prove useful. The most common example is the Bethe-Salpeter equation in the ladder approximation. The terms obtained from iterating this equation correspond to individual Feynman diagrams, the so-called ladder graphs. There is thus a simple immediate connection with field theory. In addition, it can be shown that elastic unitarity is exactly satisfied between the elastic threshold and the threshold for production.[2]

There are, however, serious difficulties associated with the Bethe-Salpeter equation. Since it is a four-dimensional integral equation, it only reduces to a two-dimensional integral equation upon taking a partial-wave projection. In addition, there are the difficulties associated with the indefiniteness of the Lorentz metric, making the equation difficult to deal with except in simple models such as the Wick-Cutkosky model.

In order to circumvent these difficulties, a simpler equation has been proposed by Logunov and Tavkhelidze,[3] by Blankenbecler and Sugar,[4] and by Alessandrini and Omnes.[5] This equation is manifestly covariant, and the Green's function is chosen to have the discontinuity that will insure that the solution satisfies elastic unitarity for a real symmetric "potential." The equation, however, is three-dimensional, and it reduces to a form which is very similar to the Lippmann-Schwinger equation in the center-of-mass system. The equation for a given partial wave is a one-dimensional integral equation on which ordinary Fredholm techniques can be used.

This equation is, however, not unique. Alternative equations, with the same desirable properties, have been proposed by Kadyshevsky[6] and by Gross.[7] This nonuniqueness stems from the fact that the unitarity does not uniquely determine the Green's function, only its discontinuity. (In the nonrelativistic case the Green's function is unique, since it is the Green's function for the Schrödinger equation with incoming plane-wave boundary conditions.)

Following Kadyshevsky, we shall refer to any relativistic equation with the above-mentioned properties as a "quasipotential equation." We wish to point out in this note that by modification of the Kadyshevsky Green's function we can obtain an infinite set of such quasipotential equations of which the Logunov-Tavkhelidze, Blankenbecler-Sugar, Alessandrini-Omnes (LTBSAO) equation and the Gross equation are special cases.

In the Kadyshevsky equation,[6] the intermediate state is on the mass shell but off the energy-momentum shell. In order to accomplish this, a quasiparticle, with no quantum numbers, is introduced to carry off the extra four-momentum so that energy-momentum conservation can be maintained. The equation is[8] (Fig. 1)

$$\langle p_1, p_2, \lambda\kappa|T|q_1, q_2, \lambda\kappa'\rangle = \langle p_1, p_2, \lambda\kappa|V|q_1, q_2, \lambda\kappa'\rangle + \int (2\pi)^{-3} d^4k_1 d^4k_2 d\kappa'' \langle p_1, p_2, \lambda\kappa|V|k_1, k_2, \lambda\kappa''\rangle \delta^+(k_1{}^2 - m^2)\delta^+(k_2{}^2 - m^2) \times (\kappa'' - i\epsilon)^{-1}\delta^4(k_1 + k_2 - \lambda\kappa'' - q_1 - q_2 + \lambda\kappa')\langle k_1, k_2, \lambda\kappa''|T|q_1, q_2, \lambda\kappa'\rangle , \quad (1)$$

where λ is an arbitrary four-vector satisfying $\lambda^2 = 1$, which determines the direction in which we go off the energy-momentum shell. κ, κ', and κ'' are parameters with the dimensions of energy, T is the T matrix, V is the Born term, and $\delta^+(k_1{}^2 - m^2) = \theta(k_1{}^0)\delta(k_1{}^2 - m^2)$. The Kadyshevsky Green's function,

$$G(k_1, k_2, \lambda\kappa'') = (2\pi)^{-3}(\kappa'' - i\epsilon)^{-1}\delta^+(k_1{}^2 - m^2)\delta^+(k_2{}^2 - m^2) , \quad (2)$$

satisfies

$$G - G^\dagger = (2\pi)^{-2} i\delta(\kappa'')\delta^+(k_1{}^2 - m^2)\delta^+(k_2{}^2 - m^2) . \quad (3)$$

It can thus be easily seen[5] that if V is real and symmetric, $V = V^\dagger$, and if we put the external states on shell by setting $\kappa = \kappa' = 0$, we obtain the on-shell elastic unitarity relation

$$\mathrm{Im}T(p_1, p_2, q_1, q_2) = \frac{1}{8\pi^2}\int d^4k_1 d^4k_2 T(p_1, p_2, k_1, k_2)\delta^+(k_1{}^2 - m^2)\delta^+(k_2{}^2 - m^2)\delta^4(k_1 + k_2 - q_1 - q_1)T^\dagger(k_1, k_2, q_1, q_2) . \quad (4)$$

In order to simplify Eq. (1), we choose λ collinear with $p_1 + p_2$ and hence with $q_1 + q_2$ and $k_1 + k_2$, i.e.,

$$\lambda = \frac{p_1 + p_2}{[(p_1 + p_2)^2]^{1/2}} = \frac{q_1 + q_2}{[(q_1 + q_2)^2]^{1/2}} = \frac{k_1 + k_2}{[(k_1 + k_2)^2]^{1/2}} . \quad (5)$$

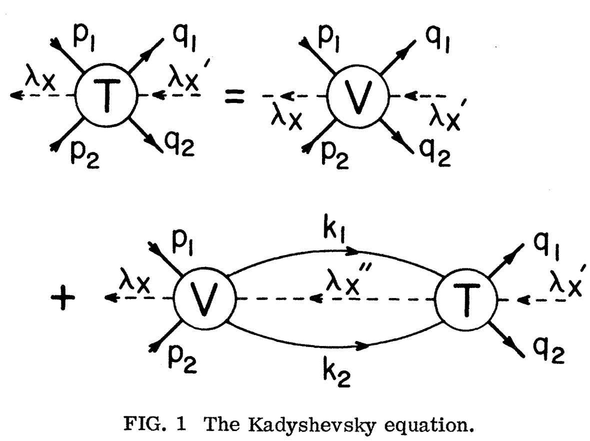

FIG. 1 The Kadyshevsky equation.

We then go to the center-of-mass system,

$$\vec{p}_1 = -\vec{p}_2 = \vec{p}, \quad \vec{q}_1 = -\vec{q}_2 = \vec{q} .$$

In this frame, λ will have only a time component, $\lambda = (1, 0, 0, 0)$. Since the external lines are on the mass shell, we also have

$$p_1^0 = p_2^0 = (\vec{p}^{\,2} + m^2)^{1/2} = E_p , \quad q_1^0 = q_2^0 = (\vec{q}^{\,2} + m^2)^{1/2} = E_q . \quad (6)$$

The mass-shell δ functions for the intermediate state have the form

$$\delta^+(k_1^2 - m^2) = \frac{1}{2(\vec{k}_1^2 + m^2)^{1/2}}\delta(k_1^0 - (\vec{k}_1^2 + m^2)^{1/2}) . \quad (7)$$

To obtain the half-off-shell equation, we further set $\kappa' = 0$.

We can then use the six δ functions in Eq. (1) to integrate over six of the variables by inspection, and we have

$$\vec{k}_1 = -\vec{k}_2 = \vec{k}\,, \quad k_1^0 = k_2^0 = (\vec{k}^2 + m^2)^{1/2} = E_k\,,$$
$$\kappa'' = 2(E_q - E_k)\,. \tag{8}$$

The initial, final, and intermediate states are thus each specified by a single three-vector, $\vec{p}$, $\vec{q}$, and $\vec{k}$, respectively, and Eq. (1) becomes

$$T(\vec{p},\vec{q}) = V(\vec{p},\vec{q}) + (4\pi)^{-3}\int d^3k\; V(\vec{p},\vec{k}) \times \frac{1}{E_k^{\,2}(E_k - E_q - i\epsilon)}\, T(\vec{k},\vec{q})\,, \tag{9}$$

which is the Kadyshevsky quasipotential equation. By our previous argument, we can easily see that

$$\mathrm{Im}\, T(\vec{p},\vec{q}) = (8\pi)^{-2}\int d\Omega_k \left(\frac{E_k^{\,2} - m^2}{E_k^{\,2}}\right)^{1/2} T(\vec{p},\vec{k})T^{+}(\vec{k},\vec{q}) \tag{10}$$

with $E_k = E_p = E_q$, which is just Eq. (4) after we have made use of the δ function to integrate over 6 of the variables.

The LTBSAO quasipotential equation and the Gross quasipotential equation are usually obtained by replacing the product of Feynman propagators in the Bethe-Salpeter equation with a single propagator which is chosen in such a way as to guarantee that the elastic unitarity condition, Eq. (4), is satisfied. In the first case, we replace

$$-i\,(k_1^{\,2} - m^2 - i\epsilon)^{-1}(k_2^{\,2} - m^2 - i\epsilon)^{-1}$$

by[5]

$$4\pi\delta(Q\cdot k)\left(\frac{s}{m^2 - k^2}\right)^{1/2}[4(m^2 - k^2) - s + i\epsilon]^{-1}\,, \tag{11}$$

where[9]

$$Q = p_1 + p_2 = q_1 + q_2 = k_1 + k_2$$

and (12)

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2 = (k_1 + k_2)^2\,.$$

In the center-of-mass system, this gives us the equation

$$T(\vec{p},\vec{q}) = V(\vec{p},\vec{q}) + \frac{1}{4(2\pi)^3} \times \int V(\vec{p},\vec{k})\,\frac{1}{E_k[E_k^{\,2} - (E_q + i\epsilon)^2]}\,T(\vec{k},\vec{q})\,d^3k\,. \tag{13}$$

Gross puts one particle on the mass shell, substituting for the Bethe-Salpeter propagator,

$$2\pi\delta^{+}(k_1^{\,2} - m^2)(k_2^{\,2} - m^2 - i\epsilon)^{-1}\,,$$

which gives us the equation

$$T(\vec{p},\vec{q}) = V(\vec{p},\vec{q}) + \int \frac{d^3k}{(4\pi)^3}\, V(\vec{p},\vec{k}) \times \frac{1}{E_q E_k(E_k - E_q - i\epsilon)}\, T(\vec{k},\vec{q})\,. \tag{14}$$

Going back to Eq. (1), we note that the invariants associated with the intermediate state are $s_k = (k_1 + k_2)^2$ and κ''. The discontinuity condition for the Green's function, Eq. (3), which guarantees elastic unitarity, involves the value of the Green's function at $\kappa''=0$ because of the $\delta(\kappa'')$ factor. Hence, we can multiply the Kadyshevsky Green's function, Eq. (2), by any dimensionless function of s_k and κ'', $f(s_k, \kappa'')$, which satisfies for all s_k

$$f(s_k, 0) = 1 \tag{15}$$

and can still maintain both elastic unitarity and Lorentz invariance.

As an example, consider the function

$$f(s_k, \kappa'') = \left(\frac{\sqrt{s_k} + \gamma\kappa''}{\sqrt{s_k} + \beta\kappa''}\right)^{\alpha}\,, \tag{16}$$

which obviously satisfies the conditions. From Eq. (8), we see that in the center-of-mass system with $\kappa' = 0$, $\sqrt{s_k} = 2E_k$ and $\kappa'' = 2(E_q - E_k)$. Hence, choosing $\alpha = \beta = 1$, $\gamma = 0$, we would multiply the Green's function in Eq. (9) by E_k/E_q giving us Eq. (14), and choosing $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = 0$, we would multiply the Green's function by $2E_k/(E_k + E_q)$ giving Eq. (13).

In the nonrelativistic limit,

$$\vec{p}^{\,2}, \vec{k}^2, \vec{q}^{\,2} \ll m^2, \quad E_p \to m + \vec{p}^{\,2}/2m\,,$$
$$E_q \to m + \vec{q}^{\,2}/2m,\; E_k \to m + \vec{k}^2/2m\,.$$

Hence, in this limit, keeping only terms to lowest order in $|\vec{p}|/m$, Eq. (9) becomes

$$T(\vec{p},\vec{q}) = V(\vec{p},\vec{q}) + \frac{2}{(4\pi)^3 m}\int d^3k\, V(\vec{p},\vec{k}) \times \frac{1}{\vec{k}^2 - \vec{q}^{\,2} - i\epsilon}\, T(\vec{k},\vec{q})\,, \tag{17}$$

which is the Lippmann-Schwinger equation.

We should further note that in the center-of-mass system with $\kappa' = 0$, Eq. (16) becomes

$$f(s_k, \kappa'') = \left[\frac{\gamma E_q + (1-\gamma)E_k}{\beta E_q + (1-\beta)E_k}\right]^{\alpha}\,. \tag{18}$$

However, in the nonrelativistic limit, $E_q, E_k \to m$, so

$$f(s_k, \kappa'') \to 1 \tag{19}$$

for the choice of f in Eq. (16). Hence, all the quasipotential equations obtained in this way go over into the Lippmann-Schwinger equation in the nonrelativistic limit. Since any value of α, β, and γ will do, or, for that manner, any f satisfying Eqs. (15) and (19), we see that there are an infinite number of equations all having the same desirable properties and there seems to be no particular reason to prefer any one of them over the others.

These equations could be considered either as dynamical equations in their own right[3,6] or as approximations to the Bethe-Salpeter equation.[4,5,7] In particular, Chen and Raman in a recent paper,[10] have proposed a method of using the LTBSAO equation to obtain successive approximations to the Bethe-Salpeter equation. We wish to point out that by use of the resolvent-identity-type arguments which we have previously introduced[11] in the context of the Faddeev equations, the perturbation scheme of Chen and Raman can be greatly simplified.

We begin with two equations of the same form and the same Born approximation but with different Green's functions,

$$\hat{T} = V + V\hat{G}\hat{T}\,, \tag{20}$$

$$T = V + VGT\,. \tag{21}$$

For example, Eq. (20) could be the Bethe-Salpeter equation and Eq. (21) the LTBSAO equation or the Gross equation. We can also write Eq. (21) as

$$T = V + TGV = [1 + TG]V \tag{22}$$

and Eq. (20) as

$$\hat{T} = V[1 + \hat{G}\hat{T}]\,. \tag{23}$$

Multiplying Eq. (23) by $[1+TG]$, we have

$$[1+TG]\hat{T} = [1+TG]V[1+\hat{G}\hat{T}]$$

or

$$\hat{T} + TG\hat{T} = T[1+\hat{G}\hat{T}] = T + T\hat{G}\hat{T}\,,$$

and bringing the second term on the left over to the right, we have

$$\hat{T} = T + TG'\hat{T}\,, \tag{24}$$

with

$$G' = \hat{G} - G\,.$$

If Eq. (20) is the Bethe-Salpeter equation and Eq. (21) is a quasipotential equation, then Eq. (24) is a four-dimensional integral equation which is no easier to deal with than the Bethe-Salpeter equation itself. However, if we want a series of approximations, we need merely iterate,

$$\hat{T} = T + TG'T + TG'TG'T\,, \text{ etc.}$$

It can easily be seen that this perturbation series is equivalent to that of Chen and Raman (except for the fact that they have chosen to use the R matrix rather than the T matrix. However, since the T matrix and R matrix equations have the same form, the argument would be the same for the R-matrix formalism as for the T-matrix formalism). However, since we have already effectively used the resolvent identity to "solve" the equations which Chen and Raman have written down, at each stage, we have only to evaluate an integral rather than solve an integral equation.

If we are interested in the problem of the difference between two different quasipotential equations, i.e., if Eqs. (20) and (21) are both quasipotential equations, then Eq. (24) is actually a three-dimensional integral equation which can be solved by the same techniques that we would use on the quasipotential equation itself. Thus, while this problem is both interesting and not very difficult, to our knowledge there has been no numerical work done on it as yet.

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†Present address.

[1]Inelasticity can be taken into account by giving the potential an imaginary part.

[2]Elastic unitarity is more difficult to prove for the Bethe-Salpeter equation than for the Lippmann-Schwinger equation or the quasipotential equation since the propagator consists of a product of two energy denominators instead of a single one. Both the principal-value parts and the δ-function parts contribute to the discontinuity [A. Pagnamenta, in *Lectures in Theoretical Physics*, edited by W. E. Britten *et al.* (Gordon and Breach, New York, 1968), Vol. XI-D].

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[7]F. Gross, Phys. Rev. 186, 1448 (1969).

[8]In this equation and the others which follow, an overall energy-momentum conservation δ function between the initial and final states, present in each term, is implicitly understood.

[9]In the Bethe-Salpeter equation, we are off the mass shell but on the energy-momentum shell.

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Numerical Studies of the Bethe-Salpeter Equation and the Multiperipheral Integral Equation of Amati, Bertocchi, Fubini, Stanghellini, and Tonin*

H. W. Wyld, Jr.
Physics Department, University of Illinois, Urbana, Illinois 61801
(Received 25 January 1971)

We present some numerical results for Regge poles determined from the Bethe-Salpeter equation with scalar couplings. Both the trajectories and residue functions are determined. We find that it is a good approximation to ignore the coupling between different $O(4)$ states. The effect of a second-order correction to the potential (the crossed-box graph) is studied and evaluated numerically. The relation of the Bethe-Salpeter equation with the multiperipheral integral equation is reviewed, and we show how to solve the latter equation by numerical iteration. Some results are given which do not exhibit any oscillations in the total cross section.

I. INTRODUCTION

For twenty years the Bethe-Salpeter equation[1,2] has been of great interest in particle physics because it provides a relativistically covariant, yet tractable, equation for a two-body bound state or scattering state. In its simplest form (the ladder approximation) the equation sums the series of Feynman graphs illustrated in Fig. 1 and is thus formally similar to the nonrelativistic Schrödinger equation with a potential corresponding to a rung of the ladder. In 1962 a big advance was made by Lee and Sawyer,[3] who showed that the Bethe-Salpeter scattering amplitude in the ladder approximation with scalar couplings is meromorphic in the complex angular momentum half-plane $\mathrm{Re}\, l \geq -\frac{3}{2}$ with at least one Regge pole in this region. In simple terms the Regge poles are just bound states for arbitrary (nonintegral) values of l. More recently extensive use was made of the Bethe-Salpeter equation by Domokos and Suranyi[4] and by Freedman and Wang[5] in their study of daughter trajectories.

The important point, with respect to the daughter trajectories, is the four-dimensional rotational invariance of the equation as applied to a bound state with total energy zero [$P=0$ in Eq. (1) below]. For nonzero values of the total energy the equation has the usual three-dimensional rotational invariance. At zero energy the additional symmetry implies that the Regge poles appear in families, a leading trajectory at $l=\alpha$ with daughters at $l=\alpha-1$, $\alpha-2, \ldots$. This $O(4)$ symmetry is also extremely important for the practical purpose of solving the equation numerically, and this is the point we are most interested in for this paper. Because of its covariant structure the Bethe-Salpeter equation is a four-dimensional integral equation. If one makes the usual angular momentum decomposition, one obtains a two-dimensional integral equation. While it may be feasible to solve such an equation numerically on a computer in simple cases, it is certainly difficult and expensive. On the other hand, for total energy zero the additional symmetry allows us to expand in four-dimensional spherical harmonics. The equations decouple and we are left with a one-dimensional integral equation. This can be easily solved numerically by approximating it by a matrix equation. At nonzero total energy the four-dimensional symmetry of the equation is broken. However, we can still expand in four-dimensional spherical harmonics to obtain coupled one-dimensional integral equations. If we are sufficiently close to zero total energy, the coupling between amplitudes will be small and we need keep only a few coupled amplitudes to obtain an accurate result. Some numerical calculations of the Regge poles $\alpha(t)$, using this method, have been made by zur Linden.[6] Earlier, less complete results were obtained by Chung and Snider[7] using the two-dimensional integral equation. We present in this paper some additional calculations of $\alpha(t)$, and also some