$SU(2)$ the set would be $A^{\mu}_{-1}$, $V^{\mu 9}_{-1}$, $V^{\mu 8}_{-1}$, $V^{\mu 3}_{-1}$, $T^{\mu\nu 8}_{-1}$, $T^{\mu\nu 3}_{-1}$, where $A^{\mu}_{-1}$ is a linear combination of all the other vectors including $\pi^{\mu}_{-1}$, as in Eq. (21).

[7]Here we are speaking only of the models with nonzero spin-orbit forces.

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# Asymptotic Behavior and the Possibility of Duality without Ghosts in Feynman-Diagram Models*

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Elastic scattering of two spinless particles with equal or unequal masses is considered in Feynman-diagram models of the Van Hove type with the usual couplings and with arbitrary spectra of masses, spins, and coupling constants, including infinitely large masses and spins. It is shown that if the amplitude defined by analytic continuation of the sum of all lowest-order diagrams with $s$-channel poles has no unphysical or $u$-channel singularities, then for fixed $t$ this amplitude is not bounded by any power of $s$ as $|s|\to\infty$ in any infinitely multiply connected domain which excludes only neighborhoods of the poles. Since a dual Born term with no $u$-channel poles can be represented entirely by the sum of $s$-channel pole diagrams (or by the sum of $t$-channel pole diagrams), this bad asymptotic behavior cannot be canceled in the Born approximation in a dual multiparticle theory. Regge asymptotic behavior is also supposed to be associated with duality, and consequently one sees that Feynman-diagram models of the Van Hove type cannot exhibit duality. The result is independent of the order of summation of diagrams. It depends crucially on the requirement that coupling constants be real.

## I. INTRODUCTION

It is well known that it is possible to construct Feynman-diagram models[1-4] with simple couplings which exhibit Regge asymptotic behavior in one channel. At first, one might hope that by introducing infinitely many particles and adjusting all the masses and coupling constants, a model of the Van Hove type[1-4] possessing Regge behavior in two channels could be found. The existence of dual-resonance models[5,6] would tend to support this hope if implicit poles in $t$ were allowed in the sum of $s$-channel pole diagrams. One might even hope that some model of the Van Hove type could fully incorporate duality, in which case the infinite sum of $s$-channel pole diagrams would be equal to a similar sum of $t$-channel pole diagrams. However, we will prove that the asymptotic behavior of any such generalized Van Hove model (with the usual couplings[7]) is unacceptable from the point of view of duality. The factors which are primarily responsible for this negative result are (1) *the absence of ghosts* and (2) *the simple couplings of the arbitrarily high-spin particles which are present in the model*. We prove that if the sum of all lowest-order diagrams with $s$-channel poles has no implicit unphysical or $u$-channel singularities, then for fixed $t$, the amplitude defined by this sum is not bounded by any power of $s$ as $|s|\to\infty$ in any infinitely multiply connected domain which excludes only neighborhoods of the poles. No assumption is made about the domain of convergence of the sum of diagrams other than its existence.

From the point of view of applications of the Van Hove model, the result indicates that outside the usually accepted region of applicability the model is quite badly behaved.

From the point of view of duality the result is mildly unfortunate because of its generality. It implies that no dual multiparticle theory with trajectories and daughters of any shape or spacing can correspond to a ghost-free theory with the simple and conventional couplings[7] of the model. Of course, the results of this paper do not imply that all dual multiparticle theories must have ghosts. Thus, there is no contradiction between the present conclusions and the general belief that the Veneziano model with unit intercept is free of ghosts.

Although duality and positivity do come into conflict within the fairly general framework treated here, the results which are obtained do not rule out the possibility of a ghost-free dual Feynman-diagram model associated with a somewhat more complicated theory such as a theory with nonminimum derivative couplings. Abarbanel[8] has considered the possibility that duality can be exhibited by a rather general Born term (which includes an infinite number of arbitrary functions of $s$) in a strictly localizable field theory. He shows by example that at least one dual-resonance model, the Veneziano model, is contained in his Born term as a special case. In this example, the coupling constants are not explicitly required to be real. If all the coupling constants are indeed real, then according to the results of this paper, the acceptability of this example must be due to the fact that the field theory of the Abarbanel model differs from that of the Van Hove model.[2-4] Since the Born term considered by Abarbanel is different from the one treated in this paper, there is no contradiction between Abarbanel's example and the conclusions reached here. Thus, although the Veneziano model cannot be obtained from a Feynman-diagram model[2-4] of the Van Hove type, Abarbanel's construction, as well as much of the work on the generalized Veneziano model,[5] indicates that there may be an acceptable ghost-free theory in which a Veneziano amplitude does appear as the first Born term.

The general idea of the proof presented here is that power boundedness implies the existence of a convergent series (partial-fraction expansion) and that the infinite sum of Feynman diagrams can be conveniently and rigorously studied by comparison with this convergent series. If all the coupling constants are real, then the comparison leads to an inconsistency which involves the couplings of the arbitrarily high-spin particles. Roughly speaking, the difficulty, which is quite severe, comes from the numerator $s$ dependence of the Feynman diagrams together with the fact that positivity prevents cancellation of this $s$ dependence when diagrams are summed. This description of the origin of the difficulty is considerably oversimplified in that it makes no reference to the crucial questions of convergence which are analyzed in detail in Sec. III. To put the situation in perspective, we include two further observations. The first concerns pole models which are obtained by replacing numerator factors of $s$ with the appropriate masses squared. The methods used in this paper will reveal no problems in connection with such models. Our second observation is that the difficulty might also disappear if we inject more $s$ dependence through the introduction of form factors.

## II. DUALITY CONDITIONS AND THE MODEL

We consider the elastic scattering of two spinless particles with masses $m_1$ and $m_2$ which couple[7] to an infinite set of particles. As in the Van Hove model, we examine the sum of lowest-order diagrams with poles in one channel. The explicit expression for this sum is[2-4]

$$M(s,t)=\sum_{i=1}^{\infty}\frac{(2J_i+1)g_i^2}{s_i-s}(q_i^2)^{J_i}P_{J_i}(z_i)\,, \tag{1}$$

where

$$s=(p_1+p_2)^2,\quad s_i=M_i^2,\quad t=(p_1-p_3)^2\,,$$

$$4q_i^2=s-a_i\,, \tag{2}$$

$$a_i=2m_1^2+2m_2^2-\frac{(m_1^2-m_2^2)^2}{s_i}\,, \tag{3}$$

$$z_i=1+\frac{2t}{s-a_i}\,, \tag{4}$$

and the $g_i$ are coupling constants. In Eq. (1), the $P_l$ are Legendre polynomials which can be expressed in terms of Gauss's hypergeometric series[9] as follows[10]:

$$P_l(z)=F(-l,\,l+1;\,1;\,\tfrac{1}{2}(1-z))\,. \tag{5}$$

Substituting Eqs. (2)–(5) in Eq. (1), we find that

$$M(s,t)=\sum_{i=1}^{\infty}(s_i-s)^{-1}\sum_{n=0}^{J_i}C(J_i,n)t^n(s-a_i)^{J_i-n}\,, \tag{6}$$

where

$$C(J_i,n)=\frac{(2J_i+1)g_i^2}{4^{J_i}(n!)^2}\frac{(J_i+n)!}{(J_i-n)!}\,. \tag{7}$$

We observe that reality of the coupling constants $g_i$ and the positivity of the factors which come from the expansion of $P_l$ as a power series (5) in $t$ are responsible for the fact that

$$C(J_i,n)>0\,. \tag{8}$$

We wish to examine the consequences of imposing five conditions which are a subset of the necessary conditions one would associate with a dual multiparticle theory. The first four are conditions which one might associate with the Van Hove model alone.

(i) It is required that the infinite sum in Eq. (6) be uniformly convergent in some unspecified domain of $s$ and $t$, and that the Born term $M(s,t)$ can be defined elsewhere by analytic continuation.

(Note that the order in which diagrams are summed has been left arbitrary by not having specified the dependence of $s_i$ and $J_i$ on $i$. Here we continue to allow for the possibility that the order of summation is important by not demanding absolute convergence.)

(ii) It is required that $J_i$ be unbounded. [If $J_i$ were bounded then $M(s,t)$ would be a polynomial in $t$. This would correspond to behavior as $|t|\to\infty$ for fixed $s$ which is uninteresting from the point of view of either the Van Hove model or duality.]

(iii) Particles with finite mass are required to have finite spins and finite degeneracy. (This condition is satisfied[11,12] by dual multiparticle theories. Except for this requirement, the spin content and degeneracy are completely arbitrary.)

(iv) For any fixed $t$ with the possible exception of a set of isolated points, $M(s,t)$ is required to be a meromorphic function of $s$ whose poles and residues are just poles and residues of terms in the infinite sum in Eq. (6).

[There is no restriction on possible implicit singularities in $t$ at isolated points because it is an essential requirement of duality that we allow for the possibility of implicit poles in $t$ or $u$. Having allowed for the possibility of singularities in $t$, it is convenient to exclude $u$-channel singularities by considering only processes in which there are no $u$-channel contributions to the Born term. This eliminates one source of implicit singularities in the variable $s$ when $t$ is held fixed. We also exclude as "unphysical" any non-$u$ singularities whose position in $s$ is $t$-dependent. Such singularities have no interpretation as particle contributions to our Born term and should not be present in any dual model. Finally, we note that no $s$-channel singularities are supposed to be "dual" to other $s$-channel singularities. Therefore, in condition (iv) we forbid implicit singularities in $s$ while imposing no restrictions on the explicit "physical" $s$-channel poles of Eq. (6).]

(v) It is tentatively required that for any fixed $t$, with the possible exception of isolated points, $M(s,t)$ is bounded by some power of $s$ as $|s|\to\infty$ in a multiply connected domain, excluding only neighborhoods of the poles but including infinitely many segments of the positive real $s$ axis.

(Note that neighborhoods of all poles must be excluded. On the other hand, segments of the positive real $s$ axis out to infinity are included because dual theories are supposed to have some relevance in the asymptotic part of the physical region. The resulting infinitely multiply connected domain need not include segments of the $s$ axis between every pair of adjacent poles.)

## III. INCOMPATIBILITY OF THE DUALITY CONDITIONS AND THE MODEL

For subsequent purposes we define a function $\overline{M}(s,t)$ which differs from the amplitude $M(s,t)$ only through the omission of those diagrams whose poles are below threshold. Thus, we write

$$\overline{M}(s,t)=\sum_{i=1}^{\infty}\theta(s_i-s_0)(s_i-s)^{-1}\sum_{n=0}^{J_i}C(J_i,n)t^n(s-a_i)^{J_i-n}\,, \tag{9}$$

where

$$s_0=(m_1+m_2)^2\,,$$

$$\theta(x)=1 \quad \text{for } x\geq 0\,,$$

$$\theta(x)=0 \quad \text{for } x<0\,.$$

The requirements of meromorphy and finite degeneracy imply that only a finite number of diagrams have poles below threshold. Thus, the domain of convergence of the infinite series in Eq. (9) completely overlaps the domain of convergence of the infinite series in Eq. (6). If we denote the maximum value of spin for the particles below threshold by $\overline{J}$, then we see that the function $(M-\overline{M})\sim s^{\overline{J}-1}$ as $|s|\to\infty$ with $t$ held fixed. Thus, $\overline{M}(s,t)$ is bounded along with $M(s,t)$ by some power of $s$ in the domain described in condition (v).

The ordered sequence of points at which the $s$-channel poles of $\overline{M}(s,t)$ are located will be denoted by $\sigma_p$, $p=1,2,3,\ldots$, where $\sigma_p<\sigma_{p+1}$. Since essential singularities are not allowed, we must have $\sigma_p\to\infty$ as $p\to\infty$. In general, any given pole will appear in more than one diagram. The set of values of $i$ for which $s_i=\sigma_p$ will be denoted by $I(p)$ and for these values of $i$ we write $a_i=b_p$.

We next consider the contour integral

$$I_m(s,t)=\frac{1}{2\pi i}\int_{C_m}ds'\frac{\overline{M}(s',t)}{(s'-s)^{k+1}} \tag{10}$$

taken along a circle $C_m$ which has its center at the origin and which encloses the first $m$ pole positions $\sigma_1,\ldots,\sigma_m$. Here $k$ is an integer, $k\geq K=K(t)$ and $K(t)$ is the least integer for which $M(s,t)/s^K\to 0$ as $|s|\to\infty$ in the domain described in condition (v). Condition (v) guarantees that we can find an infinite sequence of circles[13] $C_m$ on which $|I_m(s,t)|\to 0$ as $m\to\infty$ for any finite $s$ away from the poles. This in turn implies that we can obtain a uniformly convergent series[13] by evaluating the residues of poles of the integrand in Eq. (10) and considering the limit $m\to\infty$. In this way we obtain

$$\overline{M}^{(k)}(s,t)=k!\sum_{p=1}^{\infty}(\sigma_p-s)^{-k-1}\times\sum_{i\in I(p)}\sum_{n=0}^{J_i}C(J_i,n)t^n(\sigma_p-b_p)^{J_i-n}\,, \tag{11}$$

where the $k$th derivative of $\overline{M}$ with respect to $s$ is denoted by $\overline{M}^{(k)}$. In arriving at Eq. (11), we have invoked condition (iv).

We next make use of the convergence of the $p$ sum in Eq. (11) at $s=\sigma$ and $t=\tau$, where $\sigma$ and $\tau$ are positive real numbers, $k\geq K(\tau)$ and $\sigma\neq\sigma_p$, $p=1,2,\ldots$. In this case all terms in the infinite series with $p>p_0$ are positive, $p_0$ being the least integer for which $\sigma_p\geq\sigma$. Note that because poles below threshold do not appear in $\overline{M}(s,t)$, we have $\sigma_p\geq(m_1+m_2)^2$, and consequently

$$\sigma_p \geq 2m_1{}^2 + 2m_2{}^2 - \frac{(m_1{}^2 - m_2{}^2)^2}{\sigma_p}, \tag{12}$$

or, in terms of the quantities which appear in Eq. (11),

$$(\sigma_p - b_p) \geq 0\,. \tag{13}$$

Thus, the infinite series in Eq. (11) with $s=\sigma$ and $t=\tau$ converges absolutely. Absolute convergence for other values of $s$ and $t$ can be inferred from

$$\begin{aligned}|(\sigma_p - s)^{-k-1}\sum_{i\in I(p)}\sum_{n=0}^{J_i} C(J_i,n)t^n(\sigma_p-b_p)^{J_i-n}| &\leq |\sigma_p - s|^{-k-1}\sum_{i\in I(p)}\sum_{n=0}^{J_i} C(J_i,n)|t|^n(\sigma_p-b_p)^{J_i-n}\\ &\leq (\sigma_p-\sigma)^{-k-1}\sum_{i\in I(p)}\sum_{n=0}^{J_i} C(J_i,n)\tau^n(\sigma_p-b_p)^{J_i-n}\end{aligned} \tag{14}$$

for $p \geq p_0$, $\mathrm{Re}\, s \leq \sigma$, and $|t| \leq \tau$. Through the Weierstrass $M$ test,[14] this inequality shows that the series in Eq. (11) converges uniformly and absolutely for $|t| \leq \tau$, $k \geq K(\tau)$, $\mathrm{Re}\, s \leq \sigma$, and $s \neq \sigma_p$, $p = 1, 2, \ldots$. Note that for $|t| < \tau$ we might have had $K(t) \leq K(\tau)$. By taking $\sigma$ to be arbitrarily large, we see that we have established uniform and absolute convergence in the finite part of the $s$ plane excluding neighborhoods of the poles. In contrast, the domain of convergence in $t$ is correlated with the index $k$ and thus may be different for the series which represent the various functions $\overline{M}^{(k)}(s,t)$. However, we can take $\tau$ to be as large as we wish and still find some series [those with $k \geq K(\tau)$] which converge uniformly and absolutely inside a circle of radius $\tau$.

We can now do some rearrangement of the series in Eq. (11). Absolute convergence of the repeated series $\sum_p \sum_{i\in I(p)}$ taken in that order implies the absolute convergence of the corresponding double series[15] whose terms can therefore be summed in any order[16] as a single series without changing the value of the sum. We choose the new order of summation to be the same as the order of summation of diagrams in Eq. (9). Thus, the series representation

$$\overline{M}^{(k)}(s,t) = k!\sum_{i=1}^{\infty}\theta(s_i - s_0)(s_i - s)^{-k-1} \times \sum_{n=0}^{J_i} C(J_i,n)t^n(s_i - a_i)^{J_i-n} \tag{15}$$

converges absolutely and uniformly for $|t| \leq \tau$, $k \geq K(\tau)$, $|s| < \infty$, and $s \neq s_i$, $i = 1, 2, \ldots$.

By differentiating Eq. (9) $k$ times with respect to $s$, we obtain the series representation

$$\overline{M}^{(k)}(s,t) = \sum_{i=1}^{\infty}\theta(s_i - s_0)\sum_{n=0}^{J_i} C(J_i,n) \times t^n\left(\frac{\partial}{\partial s}\right)^k \frac{(s-a_i)^{J_i-n}}{s_i - s}, \tag{16}$$

which for arbitrary $k$ converges[17] in the same unspecified domain of convergence as Eqs. (6) and (9).

Now let us focus on an arbitrary point $s_0$, $t_0$ in the unspecified domain of convergence of Eq. (16). If we simply choose $\tau$ sufficiently large so that $\tau > |t_0|$ then $s_0$, $t_0$ will also be a point in the domain of convergence of a series of the form (15) with $k \geq K(\tau)$. Thus, we can find a common domain of convergence of two series (15) and (16) which represent the same function $\overline{M}^{(k)}(s,t)$ provided that we have $k \geq K(\tau)$. If in this domain we subtract Eq. (16) from Eq. (15), we obtain a nontrivial representation of zero,[18]

$$0 = \sum_{i=1}^{\infty}\theta(s_i - s_0)\sum_{n=0}^{J_i} C(J_i,n)t^n H(J_i - n, i, k, s)\,, \tag{17}$$

where

$$H(l,i,k,s) = \left(\frac{\partial}{\partial s}\right)^k \frac{(s_i - a_i)^l - (s-a_i)^l}{s_i - s} \tag{18}$$

$$= (s_i - a_i)^{l-k-1}\sum_{j=0}^{l-k-1}\frac{(j+k)!}{j!}\left(\frac{s-a_i}{s_i - a_i}\right)^j \tag{19}$$

for $l \geq k+1$. For $l \leq k$, $H=0$.

We next show that the domain of convergence of Eq. (17) actually extends over the full domain of convergence of the series in Eq. (15). This is accomplished by comparing moduli of terms in Eqs. (15) and (17).

For sufficiently large $s_i$, i.e., $s_i > R$, we have

$$\left|\frac{s-a_i}{s_i - a_i}\right| < \epsilon < \frac{1}{k+1}\,. \tag{20}$$

This condition also holds for sufficiently large $i$ since we can choose $i > \bar{i}$, where $\bar{i}$ is the largest index of any $s_i$ with $s_i \leq R$. From Eq. (19) we see that

$$|H(l,i,k,s)| < (s_i - a_i)^{l-k-1}\sum_{j=0}^{l-k-1}\frac{(j+k)!}{j!}\epsilon^j \tag{21}$$

for $i > \bar{i}$.

If we call the $j$th term in this last sum $C_j$, then we observe that $C_{j+1} \leq (k+1)\epsilon C_j$ and consequently

$$|H(l,i,k,s)| < (s_i - a_i)^{l-k-1}\frac{k!}{1-(k+1)\epsilon}\,. \tag{22}$$

Furthermore, it is easy to show that

$$|H(l,i,k,s)| < \frac{(s_i - a_i)^l}{|s_i - s|^{k+1}} (1+\epsilon)^{k+1} \frac{k!}{1-(k+1)\epsilon} \tag{23}$$

or

$$|H(l,i,k,s)| < \text{const } k! \frac{(s_i - a_i)^l}{|s_i - s|^{k+1}} \tag{24}$$

for sufficiently large $i$. The important feature of this last inequality is that the constant does not depend on $i$, $l$, or $s$. The inequality (24) involves the $s$-dependent factors which appear in Eqs. (15) and (17). We now reidentify $l$ with $J_i - n$ and use our last result (24) to establish that the modulus of the $i$th term in Eq. (17) with $|t| \leq \tau$ is less than some constant times the modulus of the $i$th term in Eq. (15) with $t = \tau$ for all $i > \bar{i}$. Thus, the series in Eq. (17) is found to converge absolutely and uniformly in the same domain as Eq. (15).

We note that within our "enlarged" domain of convergence of Eq. (17), $s$ and $t$ can simultaneously take on positive real values and the only restriction on $s$ is that $s \neq s_i$, $i = 1, 2, \ldots$. For real $s > 2m_1^2 + 2m_2^2$, we have $s - a_i > 0$ and we observe from Eq. (19) that $H(l,i,k,s) \geq 0$ for all $l$ and $i$. It should be recalled that the constants $C(J_i, n)$ are also positive. The net result is that for real $t > 0$ and real $s > 2m_1^2 + 2m_2^2$, each nonvanishing contribution to Eq. (17) is positive. At the same time the series in Eq. (17) has been shown to be convergent and its sum must be zero. Thus, we have arrived at a contradiction.

## IV. DISCUSSION

Since most of the proof involved comparison with a series whose convergence relies on the polynomial boundedness condition (v), we have shown that any model which satisfies the other conditions cannot also be polynomially bounded in the infinitely multiply connected domain described in condition (v).

A physically acceptable way of avoiding the contradiction found in Sec. III would be to replace the coupling constants with form factors. It would then be interesting to see what properties the form factors must have in order to generate acceptable behavior as $|s| \to \infty$ for fixed $t$.

It should be noted that the contradiction might disappear in a theory with ghosts, in which case the $C(J_i, n)$ would not all be positive. Since there can be no problem with polynomial boundedness in $s$ if we restrict ourselves to finite spins, it is worthwhile to point out what happens to Eq. (17) when only finite spins are present. In this case, we can easily find a finite value of $k$, e.g., any value greater than the maximum spin, such that Eq. (17) is satisfied simply because all the $H(J_i - n, i, k, s) = 0$ for $k \geq J_i$ [see Eq. (18)]. This last observation also shows that the lack of polynomial boundedness is not due to the finite-spin particles and could not be remedied by introducing ghosts with only finite spin.

Since the conditions which were imposed in Sec. II are well accepted features of dual multiparticle theories, we see that Feynman-diagram models of the Van Hove type cannot exhibit duality.

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# Superconvergence and Algebraic Realizations of Chiral Symmetry*

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The consequences of assuming that there is no isospin $T=2$ part in the commutator $[\dot{Q}_5^a, \dot{Q}_5^b]$ of time derivatives of the axial charges $Q_5^a$ and $Q_5^b$ have been investigated. The superconvergence condition derived from this assumption provides constraints on Weinberg's mass matrix. The mixing angles introduced by Weinberg for the $\pi$-$A_1$-$\rho$-$\sigma$ system of mesons and for the nucleon resonances belonging to the $(1,\frac{1}{2})$ and $(\frac{1}{2},0)$ representations of the chiral group are now determined. When only $p$-wave pion interactions are considered, it is shown that the new constraints on the mass matrix lead to mass degeneracy for the symmetric representations of $SU(2)\otimes SU(4)$. The constraints on the spin spectrum derivable from the superconvergence conditions are briefly mentioned.

Weinberg[1] evaluated the pion scattering amplitudes using a chiral Lagrangian, and by demanding that they have good high-energy behavior, he derived the following restrictions on the axial-vector coupling matrix $\vec{X}(\lambda)$, the isospin matrix $\vec{T}$, and the mass-squared matrix $m^2$:

$$[X_a(\lambda), X_b(\lambda)] = i\epsilon_{abc} T_c\,, \tag{1}$$

$$[X_a(\lambda), [X_b(\lambda), m^2]] \propto \delta_{ab}\,, \tag{2}$$

where $\lambda$ is the helicity.

Fubini and Furlan[2] have shown that Eqs. (1) and (2) can be derived by saturating the following commutation relations between the axial charges $Q_5^a(t)$:

$$[Q_5^a, Q_5^b] = i\epsilon_{abc} Q^c\,, \tag{3}$$

$$[Q_5^a, \dot{Q}_5^b]_{T=2} = 0, \tag{4}$$

with single-particle states in the infinite-momentum frame. Here $Q^c$ is the vector charge and $Q^c \equiv T_c$. The consequences of saturating a further commutator

$$[\dot{Q}_5^a, \dot{Q}_5^b]_{T=2} = 0, \tag{5}$$

are examined in this paper.

Weinberg[2] investigated possible saturation schemes for Eqs. (1) and (2) by assigning particles to reducible representations of the chiral $SU(2) \times SU(2)$ group and showed that one obtains good results for coupling constants and particle masses for both mesons and baryons provided the angles specifying mixing between states of various representations of the chiral group were chosen to be 45°. Cronström and Noga[3] studied the algebraic realizations of chiral symmetry for $p$-wave coupling matrices $D_{\alpha a}$ and showed that they generate the algebra of $SU(2)\otimes SU(4)$. The mass-squared matrix which they obtained for the symmetric representations of $SU(4)$ is

$$m^2(T,J) = m_0{}^2 + c[J(J+1) - T(T+1)]\,. \tag{6}$$

The mass spectrum represented by Eq. (6) does not correspond to the experimentally observed spectrum.

If the commutation relation (5) is saturated in the infinite-momentum frame with *single-particle* states, it leads to the predictions that Weinberg's mixing angle should indeed be 45° and that a totally degenerate mass matrix is the only solution for the $p$-wave algebraic realizations of chiral symmetry as viewed by Cronström and Noga.[3]

The commutator (5) can be rewritten as follows:

$$[[Q_5^a, H], [[Q_5^b, H], H]]_{T=2} = 0. \tag{7}$$

Using the techniques explained by Fubini and Furlan,[2] the matrix elements of the above commutator (7) are evaluated by sandwiching it between the single-particle states $|a, \vec{p}_a, \lambda\rangle$ and $|b, \vec{p}_b, \lambda'\rangle$ and saturating it in the infinite-momentum frame with *single-particle* states. This gives the following extra condition on the mass matrix and the axial-vector coupling matrix:

$$[[X^a(\lambda), m^2], [[X^b(\lambda), m^2], m^2]]_{T=2} = 0. \tag{8}$$