PHYSICAL REVIEW D VOLUME 3, NUMBER 12 15 JUNE 1971

Mathematical Quarks from a Bootstrap Principle

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(Received 21 December 1970)

In a previous paper a simple bootstrap hypothesis was applied to the meson-meson and meson-baryon scattering amplitudes of a hypothetical set of arbitrary numbers of even- and odd-parity mesons and baryons. A set of self-consistency relations for the trilinear interaction constants of the hadrons was obtained. In the present paper, this set is reexamined and extended by considering baryon-baryon scattering. The most striking new predictions are that the mesons must interact with the symmetry of quark-antiquark composites, while the baryons behave as quark-quark composites. The meson Regge trajectories must be parity-doubled, while the baryon parity depends on the symmetry with respect to the interchange of the two (mathematical) quarks. Particles of baryon number more than 1 are forbidden.

I. INTRODUCTION

In the last several years many papers have been written concerning the induction of hadron symmetries from a bootstrap hypothesis.[1] In these papers a hypothetical set of hadrons is considered; the interaction constants, and usually the number of hadrons in the set, are left arbitrary. Some simple dynamical hypothesis is applied either to scattering amplitudes or to vertex functions, and used to obtain self-consistency conditions on the interaction constants. The aim is to show that these conditions require that the hadron set possess a simple symmetry.

Recently, the author used a bootstrap hypothesis based on duality to obtain such consistency conditions for meson-meson and meson-baryon scattering.[2,3] The main advantage of the duality formulation is that it allows for mesons and baryons of both parities to be included on the same footing. This is important mathematically as well as physically, for there are no solutions in which mesons of one parity exist without the other, or baryons of one parity exist without the other (except for a trivial solution involving only one meson state).[2,4]

In this paper we reexamine the conditions and extend them to include baryon-baryon scattering. The three most striking new results are: (i) The mesons of each parity must interact as if they were quark-antiquark composites. (ii) No states of baryon number greater than 1 may exist. (iii) The baryons interact as if they were quark-quark composites, the baryon parity corresponding to the symmetry under interchange of the quarks. This quark structure is not assumed but is forced by the bootstrap equations. The quarks are "mathematical" in that no one-quark state can exist in the model.

The consistency equations, derived in previous references, are listed in Sec. II. In this paper, we are concerned particularly with the algebraic implications of the conditions. The implications for mesons and for baryons are given in Sec. III and Sec. IV, respectively. In order to aid comprehension, we have made the logical arguments in Sec. III and Sec. IV complete. Results obtained previously are rederived, if they are necessary to the argument. Old and new results are labeled as such. The general relation of the results to the experimentally observed hadron spectrum is discussed at the conclusion of Sec. III and Sec. V; a detailed comparison with experiment is not made. The consistency conditions used are obtained from considering forward and backward scattering; the problem of extending to other angles is discussed only briefly, in Sec. V.

II. THE CONSISTENCY CONDITIONS

We assume a hypothetical universe of four types of hadrons: mesons of even and odd parities, and baryons of even and odd parities. Each of these four hadron subsets is taken as degenerate, but the numbers of hadrons and their trilinear interaction constants are left arbitrary. A two-hadron → two-hadron interaction may be represented in the s, t, and u Mandelstam channels as follows:

$$\begin{aligned} s&:\ \bar{a}+b\to\bar{c}+d,\\ t&:\ \bar{a}+c\to\bar{b}+d,\\ u&:\ c+b\to a+d, \end{aligned}\tag{2.1}$$

where $\bar{i}$ denotes the conjugate (antiparticle) state of i. The consistency conditions are applied only to forward and backward amplitudes. The particle labels refer both to internal quantum numbers and to spin components along the direction of interaction, taken as the z axis. For collinear amplitudes these spin components behave as internal quantum numbers under crossing.

The bootstrap condition relates the contributions of one-particle poles in any two of the three Mandelstam channels. We use the general coupling constant G_{ijk} to refer to the vertex $k \to i+j$, where i and j are states emitted in the $+z$ and $-z$ directions, respectively, in the k rest system. Reversing the directions of the two emitted particles leads to the symmetry relation

$$G_{ijk} = \eta^{ijk} G_{jik}, \tag{2.2}$$

where η^{ijk} is the orbital parity of the vertex. We define a parity factor $\eta^{ij\cdots m}$ with a variable number of indices, subject to the restrictions that the number of indices referring to states of baryon number 1 is even, and the number referring to states of baryon number -1 is also even. The value of η is the product of the intrinsic parities of the meson states and the relative parities of the pairs of baryon and antibaryon states.

If i is a meson state, the G's are defined so that they have the crossing property

$$G_{ijk} = G^*_{\bar{i}kj} \quad (i = \text{meson state}). \tag{2.3}$$

Baryons are defined as particles with a conserved baryon number; we do not want to specify in advance whether their spins are integral or half-odd-integral. We avoid the consequent ambiguity concerning the intrinsic parity of baryon-antibaryon states by always letting the right-hand subscript of a meson-baryon-baryon coupling constant refer to a baryon or antibaryon, so that one baryon is in the initial state and one is in the final state. Baryons may be crossed two at a time; the corresponding crossing property of G is

$$G_{ijk} = \eta^{ijk} G_{i\bar{k}\bar{j}} \quad (i = \text{meson state}). \tag{2.4}$$

If i, j, and k are all meson states, Eq. (2.4) follows from Eqs. (2.2) and (2.3).

We first consider the cases when a and c are meson states, so that b and d are either both mesons or both baryons. The consistency condition corresponding to the s-u pair of channels is

$$\sum_n G_{d\bar{c}n} G^*_{\bar{a}bn} = \eta^{abcd} \sum_n G_{dan} G^*_{cbn}, \tag{2.5}$$

where the sums are over all possible intermediate states of both parities.[5] Our bootstrap equations are Eq. (2.5), and corresponding equations for the s-t and u-t pairs of channels and for baryon-baryon scattering. It is required that all two-hadron $\to$ two-hadron amplitudes be considered, and that the set of internal particles be identical to the set of external particles.

A model that leads to this condition is discussed in Ref. 3; we give here only the general features of the model. The virtual particles are the lightest states on Regge trajectories. The meson states of one parity lie above those of the other by an interval of $(a')^{-1}$ in the energy squared, where a' is the universal slope of the trajectories; a similar exchange-degeneracy condition applies to the baryon trajectories. The consistency condition results from a simple proportionality assumption concerning the Regge residues, together with the duality condition that the imaginary part of the s-channel resonance contribution to the amplitude in the backward direction be equal to the imaginary part of the u-channel exchange contribution. When the parity factor η^{abcd} is negative, the s- and u-channel amplitudes are odd in the center-of-mass momentum k. The η^{abcd} is included in the condition because k is odd under s-u crossing and must be removed from the amplitude when coupling constants are defined from the residues. For an elastic process ($a=c$ and $b=d$), it is seen from Eq. (2.2) that the contributions of virtual particles of opposite parity to either side of Eq. (2.5) are of opposite sign, as they should be for backward scattering.

In this model, the Regge residues of all meson trajectories are assumed proportional, and those of baryon trajectories are assumed proportional. If μ and m are the masses of the odd-parity mesons and even-parity baryons, the constants G of Eq. (2.5) are essentially the residues when all meson and baryon energy factors are μ and m. Thus the G are equal to physical coupling constants only for interactions involving only odd-parity mesons and even-parity baryons. The physical coupling constants involving even-parity mesons and odd-parity baryons are proportional to the G's. Since the G's have been defined in terms of states of definite spin components, the relation of the states to the total angular momentum operator J^2 is not simple; this is discussed briefly in Sec. V.

We now turn to the s-t-channel consistency conditions. These may be obtained by making the replacements $a \rightleftarrows \bar{b}$ and $\bar{a} \rightleftarrows b$ in Eq. (2.5). However, when baryons are involved, the resulting equations violate our convention of letting the right-hand index refer to a baryon state. Therefore we will write the s-t conditions for meson-meson scattering in a different form, and then use this form as a guide for writing s-t conditions for meson-baryon and baryon-baryon scattering.

We write the s-t condition for meson-meson scattering, and use Eqs. (2.2) and (2.3) to permute indices so that the subscripts corresponding to internal states are first. If this is done for the t-channel term only, the result is

$$\sum_n G_{\bar{c}dn} G^*_{\bar{a}bn} = \kappa \sum_r \eta^{acr} G_{r\bar{c}\bar{a}} G_{rdb}, \tag{2.6}$$

where we have chosen a conjugate representation for the virtual mesons r (i.e., $\bar{r}=r$). The κ factor is 1, but is included in order to make generalization to baryons easier. The expression on the right may be visualized as corresponding to the exchange of t-channel trajectories, for the s-channel amplitude. The η^{acr} corresponds to the Regge signature factor. If a similar permutation of indices is made in the s-channel term also, the result is

$$\sum_r \eta^{cdr} G_{r\,\bar{b}\,\bar{a}} G_{rdc} = \sum_r \eta^{acr} G_{r\,\bar{c}\,\bar{a}} G_{rdb}, \tag{2.7}$$

where again the r are conjugate states.

When considering meson-baryon scattering, we will identify a and c with meson states, and use Eq. (2.6) as the s-t equation, including the arbitrary constant κ to account for the fact that the virtual particles on the left and right sides are baryons and mesons, respectively. In the case of baryon-baryon scattering, we will choose the u channel to correspond to baryon number 2, and use Eq. (2.7) for the s-t condition.

III. THE MESON SYSTEM

A. Form of the Equations

In this section, we consider only meson-meson scattering amplitudes. Throughout the rest of the paper, we work only in representations in which all meson states are self-conjugate. The vertex parity factor η^{ijk} is equal to the product of the intrinsic parities of the three mesons; we denote the G_{ijk} corresponding to positive and negative η factors by d_{ijk} and f_{ijk}, respectively. One may use Eqs. (2.2) and (2.3) to show that the d_{ijk} are real and completely symmetric, while the f_{ijk} are imaginary and completely antisymmetric.

If the product of the $a, b, c,$ and d parities is even, one may use Eqs. (2.2) and (2.3) to write the s-u equation, Eq. (2.5), in the form

$$\sum_r d_{cdr}\, d_{arb} - \sum_r f_{cdr} f_{arb} = \sum_r d_{adr}\, d_{crb} - \sum_r f_{adr} f_{crb}, \tag{3.1}$$

while if the parity factor η^{abcd} is odd, Eq. (2.5) may be written

$$\sum_r d_{cdr} f_{arb} - \sum_r f_{cdr}\, d_{arb} = -\sum_r d_{adr} f_{crb} + \sum_r f_{adr}\, d_{crb}. \tag{3.2}$$

Since all channels are meson-meson channels, the s-t and u-t conditions are superfluous; one may obtain all the conditions by permuting indices in Eqs. (3.1) and (3.2).

It is convenient to define a coupling-constant matrix G_i, with jk matrix element G_{ijk}. Then Eqs. (3.1) and (3.2) are equivalent to the db elements of the following matrix commutator equations:

$$[d_c, d_a] = [f_c, f_a], \tag{3.3}$$

$$[d_c, f_a] = [f_c, d_a]. \tag{3.4}$$

The vector space includes all meson states of both parities.

B. Some Results Obtained Previously

We list here five important implications of Eqs. (3.1) and (3.2) that have been obtained previously.[6]

(1) *The one-state solution.* It is clear that if only one state λ exists, of even parity, with one interaction constant $d_{\lambda\lambda\lambda}$, Eqs. (3.1) and (3.2) are satisfied. We look for another solution that is less trivial.

(2) *Lack of nontrivial solutions involving only d's.* We examine all possible solutions in which all f's vanish. The d_i are real and Hermitian. If all $f_{ijk}=0$, Eq. (3.3) states that all the d_i commute and so may be diagonalized simultaneously by an orthogonal transformation. Such a transformation preserves the self-conjugate property of the basis, so the d's remain completely symmetric. Since all the d's are diagonal and completely symmetric, each d_{ijk} vanishes if any two indices are different. Thus, the change of basis has yielded a set of disjoint, one-state solutions.[7] Nontriviality requires that some f's exist.

(3) *Group properties of the f's.* We sum Eq. (3.1) over all 24 permutations of $a, b, c,$ and d, including a minus sign with the odd permutations. If the antisymmetry property of the f's is used, the result may be written in the form

$$\sum_r (f_{adr} f_{crb} + f_{acr} f_{brd} + f_{abr} f_{drc}) = 0. \tag{3.5}$$

This is the Jacobi-identity condition. Together with the antisymmetry condition, it implies that the f's are proportional to structure constants of a Lie group (with imaginary proportionality constants) and that the meson states correspond to the regular representation of the group.[8,9]

We consider the real Hermitian metric tensor $g_{ab} = -\sum_{ij} f_{ija} f_{jib}$ and diagonalize it with an orthogonal transformation. Each term in the sum contributes a zero or negative amount to g_{aa}. If a diagonal element g_{aa} is zero, then all f_{ija} are zero. Thus, if we consider only the set of states with one or more nonzero f's (called the "antisymmetric set"), the metric g_{ab} is negative definite, implying that the Lie group of this set is compact and semisimple.[9] The f_i matrices are the regular representation of the group generators.

(4) *Existence and group properties of the d's.* Since the f's do not all commute, Eq. (3.3) implies that some d's exist. Thus, mesons of both parities

must exist and Eq. (3.2), as well as Eq. (3.1), must be considered. If Eq. (3.2) is summed over all permutations of the three indices a, b, and c, and the symmetry properties of the f's and d's are used, the result may be written in matrix form,

$$[d_a, f_d]_{bc} = -\sum_r f_{adr}\,(d_r)_{bc}\,. \tag{3.6}$$

This equation implies that each d_a transforms either as the regular or the singlet representation of the group.

(5) *Constant matrix character of the d's for states not in the antisymmetric set.* If a state c is not in the antisymmetric set, so that $f_{cij}=0$, Eq. (3.6) shows that d_c commutes with all the f's, and Eq. (3.3) shows that d_c commutes with all the d's. Thus, by Schur's lemma, d_c is a multiple of the unit matrix in any irreducible subspace of the representation of the f's or d's. We call such a matrix a unit-type matrix. The state c is a singlet state.

C. New Results

If we subtract Eq. (3.4) from Eq. (3.3), the result is the matrix commutator form of the s-u channel condition, i.e.,

$$[p_c, m_a] = 0, \tag{3.7}$$

where $p_i = f_i + d_i$ and $m_i = f_i - d_i$. Two further equations may be obtained by subtracting Eq. (3.2) from Eq. (3.1) and applying the permutation operators $(1 \mp \Pi_{ac})\Pi_{ab}$, where Π_{ij} permutes the indices i and j. The two resulting equations involve the "plus-type" (p_i) matrices, i.e.,

$$[p_c, p_a] = 2\sum_r f_{acr}\,p_r\,, \tag{3.8}$$

$$\{p_c, p_a\}_+ = 2\sum_r d_{acr}\,p_r\,. \tag{3.9}$$

These may be regarded as the difference and sum of the s-t and u-t channel consistency conditions. It is convenient to write two more equations by adding Eqs. (3.1) and (3.2), and then applying the permutation operators $(1 \mp \Pi_{ac})\Pi_{ab}$. The results are

$$[m_c, m_a] = 2\sum_r f_{acr}\,m_r\,, \tag{3.10}$$

$$\{m_c, m_a\}_+ = -2\sum_r d_{acr}\,m_r\,. \tag{3.11}$$

The indices a, c, and r in these equations apply to meson states of both parities, and the vector space in which the matrices are defined includes states of both parities. It is convenient to assign all the even-parity states the lowest row and column indices, so that the matrices consist of four distinct quadrants. Since the f and d are nonzero only when the numbers of interacting odd-parity mesons are odd and even, respectively, each d_i or f_i is nonzero only in one pair of diagonally related quadrants. Thus, if i is an odd-parity state, f_i is nonzero in the diagonal quadrants, and d_i is nonzero in the off-diagonal quadrants. For this reason, the m_i may be determined from the p_i. Thus, the information in Eqs. (3.10) and (3.11) is already contained in Eqs. (3.8) and (3.9); however, it is convenient to consider all these equations.

Because of this redundancy, the requirement that some interactions are nonzero implies that neither the p_i nor the m_i set of matrices may be identically zero. We now limit attention temporarily to states a and c that are members of the antisymmetric set, but consider all meson states in the vector space of the matrix equations. Since the f_{ijk} are proportional to structure constants, Eqs. (3.8) and (3.10) are the conditions that the p_i and m_i matrices are representations of the group generators i.

We turn next to the p_i anticommutation relation, Eq. (3.9). If a state r' contributes to the sum in this equation, and yet does not contribute to the sum in the p_i commutation relation for any a and c, then r' cannot be in the antisymmetric set. For such an r', $f_{r'}=0$ and property (5) of Sec. III B states that $d_{r'}$ is a unit-type matrix. Thus, Eqs. (3.8) and (3.9) show that every product of p_i matrices is a linear combination of the p_i's representing the generators and unit-type matrices. This is sufficient to show that the group is $SU(n)$, and that the p's are a fundamental representation, or a direct sum of a fundamental representation repeated.[10] Thus, the space of the p's is quark space. Since the signs in the p and m commutation relations [Eqs. (3.8) and (3.10)] are the same, and the signs in the anticommutation relations [Eqs. (3.9) and (3.11)] are opposite, the m matrices are the fundamental representation conjugate to that of the p's, i.e., the m's are operators in antiquark space. Furthermore, Eq. (3.7) states that the p's and m's commute, so that each state in the vector space is characterized by a quark index and an antiquark index, and the two indices are independent of each other. This is the quark model for meson-meson-meson interactions.

We now study the question of whether the $SU(n)$-symmetric solution may contain only n^2 meson states, corresponding to one regular and one singlet representation. If the n^2-state solution exists, parity conservation requires that every state of the regular representation with a unique, nonzero set of eigenvalues of the diagonal generators, must correspond to definite parity. If it is possible to connect two such states with just one of the nondiagonal generators of the sets p_i and m_i, a contradiction results, since these generators involve a mixture of parities. This argument shows that if $n>2$, the n^2-state solution is not possible.

The $SU(2)$ case will be discussed later. If $n>2$, the above argument shows that regular representation multiplets must exist for both parities. It can be shown to follow that singlets of both parities must exist also. The simplest possibility is the parity-doubling solution, in which there are $2n^2$ states, a singlet and a regular representation for each parity. The group is $SU(n)\otimes SU(n)$. The two $SU(n)$'s in the reduction of this group correspond to the sum and difference of the states of the two parities. It can be shown that in this solution each d_{ijk} and f_{ijk} is invariant to changing the parities of two of the states i, j, and k. Therefore, we may omit the parity labels from the meson indices, and let these indices and the dimension of the vector space in Eqs. (3.7)–(3.11) range only over the n^2 states of the regular and singlet representations. Each d_{ijk} and f_{ijk} is defined to be the value corresponding to the "right" vertex parity, i.e., d_{ijk} is the value that occurs when all three mesons, or only one meson, is of even parity. The four quadrants of the matrix equations are collapsed into one. The m equations, Eqs. (3.10) and (3.11), are then independent of the p equations, rather than being redundant, but the content of the set of Eqs. (3.7)–(3.11) is the same as before.

In the case that the group is $SU(2)\otimes SU(2)$, if one considers only amplitudes for which the four external mesons are members of the odd-parity triplet and of the even-parity singlet, all internal mesons are also members of this set. Hence this set and its mutual interactions are a special solution, which involves no parity doubling.

For completeness, we give here a simple form for the coupling constants in the parity-doubling, $SU(n)\otimes SU(n)$ solution. The quantum numbers of each of the n^2 meson states of either parity may be represented by the quark-antiquark construction, i.e.,

$$a=\sum_{ij} A_{ij}\,\overline{Q}_i\, Q_j\,, \tag{3.12}$$

where a capital letter denotes the coefficient matrix associated with the meson state denoted by the corresponding small letter. The coefficient matrices are normalized by the criterion $\sum_{ij}|A_{ij}|^2=1$. Those matrices associated with the regular-representation states are a fundamental representation of the generators. The coupling constants are proportional to traces of products of these matrices,

$$d_{abr}=\lambda \mathrm{Tr}[(AB+BA)R]\,, \tag{3.13a}$$

$$f_{abr}=\lambda \mathrm{Tr}[(BA-AB)R]\,, \tag{3.13b}$$

where λ is a real constant. The n^2 Hermitian coefficient matrices of a conjugate representation satisfy the closure property

$$\sum_R R_{ij}R_{kl}=\delta_{il}\,\delta_{jk}\,, \tag{3.14}$$

or, alternately, $\sum_R \mathrm{Tr}(RX)\mathrm{Tr}(RY)=\mathrm{Tr}(XY)$. One may use this closure property to verify that Eqs. (3.13a) and (3.13b) satisfy the basic conditions, Eqs. (3.1) and (3.2).

Experimentally, the lightest meson set is the odd-parity set. The group is $SU(6)$, and the vector- and pseudoscalar-meson nonets fill out the singlet and regular representations. It can be shown that the spin components must be treated by the $SU(2)_W$ prescription.[11] In terms of the model discussed in Refs. 2 and 3, our parity-doubling prediction applies to Regge trajectories of even and odd parities; this prediction appears to be satisfied. The relation to experiment is discussed further in Sec. V.

In the above argument we have used, without citing a proof, the theorem that if all products of matrices representing the algebra of a simple compact Lie group are linear combinations of themselves and of the identity matrix, the group is $SU(n)$ and the representation is fundamental. We can bypass this theorem. We illustrate this by assuming parity doubling, so that the full group is $\mathfrak{M}\otimes\mathfrak{M}$. We assume that $\mathfrak{M}$ is a simple group, and consider Eqs. (3.7)–(3.11) in the collapsed space of the quantum numbers of either parity. Let x be the dimension of the representation of the p_i operators. Then the p_i and m_i generate x^2 states with different sets of quantum numbers, so that no two of the states have associated with them proportional pairs of matrices p_i and m_i. There can be only one state in the set that is excluded from the antisymmetric set, since such a state is represented by $p_i=-m_i=c\underline{1}$, where c is a constant and $\underline{1}$ is the unit matrix. Hence, there are x^2-1 Hermitian matrices p_i of dimension x that represent the generators of $\mathfrak{M}$. This defines simultaneously $SU(n)$ and a fundamental representation.

IV. THE BARYON SYSTEM

A. Meson-Baryon Scattering Conditions

We define a baryon to be a particle with a unit value of a conserved quantum number (the baryon number) not carried by the mesons. Since our bootstrap equations are consistent with mesons alone, the existence of baryons is not required. Nevertheless, we investigate the properties that a set of baryons must have if they satisfy the bootstrap equations.

We use capital letters D and F to denote the interactions of even-parity mesons with baryons of the same and opposite parities, respectively, and script letters $\mathfrak{D}$ and $\mathfrak{F}$ to denote the interactions of odd-parity mesons with baryons of the same and opposite parities, respectively.[12] We define interaction matrices, e.g., $(D_i)_{jk}=D_{ijk}$, only when the first index is the meson index.

In this subsection, we study the meson-baryon scattering conditions, identifying a and c of Eq. (2.1) with self-conjugate meson states. The s-u-channel condition is Eq. (2.5). It can be shown that one can write this condition in terms of the constants $D, F, \mathfrak{D}$, and $\mathfrak{F}$ by using the following substitution rules: If the parity of the meson i is even, one makes the substitutions $d_i \to D_i$ and $f_i \to F_i$ in Eqs. (3.1) and (3.2) [or in Eqs. (3.3) and (3.4)]. If the parity of the meson i is odd, one makes the substitutions $d_i \to \mathfrak{D}_i$ and $f_i \to \mathfrak{F}_i$. After the substitutions, Eq. (3.1) applies when the baryon parities are the same and Eq. (3.2) applies when the baryon parities are opposite. For example, if the parities of the mesons a and c are odd and even, respectively, and the baryon parities are the same, the equation is

$$\sum_n D_{cdn}\mathfrak{D}_{anb} - \sum_n F_{cdn}\mathfrak{F}_{anb} = \sum_n \mathfrak{D}_{adn}D_{cnb} - \sum_n \mathfrak{F}_{adn}F_{cnb}\,. \tag{4.1}$$

The s-t-channel condition is Eq. (2.6). We may write this equation in the form

$$\sum_n G_{cdn}G^*_{abn} = \kappa\Big(\sum_p d_{pca}G_{pdb} - \sum_q f_{qca}G_{qdb}\Big), \tag{4.2}$$

where the sum over n includes baryons of both parities, while the sums over p and q include mesons with even and odd meson-meson-meson vertex parities, respectively. We assume that the group is not $SU(2)$, so that the meson states are parity-doubled, as discussed in Sec. III. The d and f are the meson-meson-meson interaction constants of Eqs. (3.13a) and (3.13b).

We define plus and minus combinations of the meson-baryon-baryon interaction matrices by the equations

$$P_i = F_i + D_i\,, \quad M_i = F_i - D_i\,,$$
$$\mathcal{P}_i = \mathfrak{F}_i + \mathfrak{D}_i\,, \quad \mathfrak{M}_i = \mathfrak{F}_i - \mathfrak{D}_i\,.$$

We consider the case in which the meson states a and c are of even parity. We use the s-t-channel condition [Eq. (4.2)] to write equations first for the case when the baryons are of the same parity and then for the case when the baryons are of opposite parity. If these two equations are added, the result may be written in terms of the P_i and $\mathcal{P}_i$ matrices, i.e.,

$$P_c P_a = \kappa\Big(\sum_r f_{acr}\mathcal{P}_r + \sum_r d_{acr}P_r\Big). \tag{4.3}$$

The vector space of such matrix equations includes all baryon states of both parities. We assign the lowest row and column indices to the even-parity baryon states, so that the equations have a quadrant structure similar to that of Sec. III. The D and $\mathfrak{D}$ are nonzero only in the diagonal quadrants, and the F and $\mathfrak{F}$ are nonzero only in the off-diagonal quadrants.

The lone script operator in Eq. (4.3) results from odd-parity virtual mesons in the t channel. Because of this operator, Eq. (4.3) is inconvenient, so we adopt the following procedure: Equations similar to Eq. (4.3) are written for all four possible assignments of parities to the meson states a and c. The indices a and c are then permuted in each equation, and the difference and sum taken. This results in equations for all possible commutators and anticommutators of the operators $P_a, P_c, \mathcal{P}_a$, and $\mathcal{P}_c$. These equations are then used to write the commutators and anticommutators of the sums and differences of the P and $\mathcal{P}$ operators of corresponding even- and odd-parity meson states. The result is

$$[(P\pm\mathcal{P})_c, (P\pm\mathcal{P})_a] = \pm 4\kappa \sum_r f_{acr}(P\pm\mathcal{P})_r\,, \tag{4.4}$$

$$\{(P\pm\mathcal{P})_c, (P\pm\mathcal{P})_a\}_+ = 4\kappa \sum_r d_{acr}(P\pm\mathcal{P})_r\,, \tag{4.5}$$

$$[(P\pm\mathcal{P})_c, (P\mp\mathcal{P})_a] = 0, \tag{4.6}$$

$$\{(P\pm\mathcal{P})_c, (P\mp\mathcal{P})_a\}_+ = 0, \tag{4.7}$$

where the upper signs go together.

The algebraic argument concerning these equations is a modification of that used after Eqs. (3.8)–(3.11). Now, however, the d and f constants are known from Sec. III, and correspond to the group $SU(n)$. Thus, the two pairs of equations, Eqs. (4.4) and (4.5), state that each of the operator sets $(P+\mathcal{P})$ and $(P-\mathcal{P})$ is either identically zero, or corresponds to a fundamental representation of $SU(n)$. The last two equations show that all products of a $(P+\mathcal{P})_i$ and a $(P-\mathcal{P})_j$ are zero. Thus, if we require that all pairs of baryons are connectable by some series of interactions, one of the sets $(P+\mathcal{P})$ and $(P-\mathcal{P})$ must be identically zero. Since changing the sign of all odd-parity meson states converts $\mathcal{P}_i$ into $-\mathcal{P}_i$, we lose no generality by setting $(P-\mathcal{P})_i = 0$. Since the D and $\mathfrak{D}$ operate in the diagonal quadrants, and the F and $\mathfrak{F}$ operate in the off-diagonal quadrants, this implies

$$\mathfrak{D}_i = D_i\,, \quad \mathfrak{F}_i = F_i\,, \tag{4.8}$$

i.e., the baryonic interactions of corresponding even- and odd-parity mesons are the same.[12]

We use Eq. (4.8) to eliminate the $\mathcal{P}$ from Eqs. (4.4) and (4.5), yielding the equations[13]

$$[P_c, P_a] = 2\kappa \sum_r f_{acr}P_r\,, \tag{4.9}$$

$$\{P_c, P_a\}_+ = 2\kappa \sum_r d_{acr} P_r . \qquad (4.10)$$

Since the D and F operate in different pairs of quadrants, these equations imply the following equations for the "minus-type" operators:

$$[M_c, M_a] = -2\kappa \sum_r f_{acr} M_r , \qquad (4.11)$$

$$\{M_c, M_a\}_+ = -2\kappa \sum_r d_{acr} M_r . \qquad (4.12)$$

If Eq. (4.8) is used, the s-u-channel equations for different meson-parity assignments are equivalent, as explained earlier in this section. One obtains the matrix form of the s-u equation by replacing m_i and p_i by capital letters in Eq. (3.7). The result is

$$[P_c, M_a] = 0. \qquad (4.13)$$

Two classes of solutions to equations equivalent to Eqs. (4.9)–(4.13) are found in Ref. 2. We list these here and show that they are the only classes of solutions.

(i) *The parity-doubled solution.* We look for a solution in which either the $P(F+D)$ or $M(F-D)$ operator sets is identically zero. Since the D and F operate in different pairs of quadrants, this is only possible if parity doubling exists for baryons as well as mesons, so that we may collapse the vector space as was done in Sec. III. It is required that each meson-baryon-baryon interaction constant is unchanged if the parities of the two baryons are changed. The baryon vector space is collapsed into that of the states for either parity alone, and D and F are replaced by the values in their respective nonzero quadrants of the uncollapsed space. The M equations [Eqs. (4.11) and (4.12)] are then independent of the P equations [Eqs. (4.9) and (4.10)]. We may then set $F-D$ equal to zero, in which case the P equations require that the multiplet of the baryons of either parity correspond to the fundamental (one-quark) representation. The values of the D and F constants are

$$D_{rab} = F_{rab} = kR_{ab} , \qquad (4.14)$$

where k is a real proportionality constant, and the R_{ij} are the matrices of Eq. (3.12). (We again use capital R, A, and C to denote the matrices corresponding to the meson states r, a, and c.) This solution is generalized later in this subsection.

(ii) *The parity-undoubled solution.* The other possibility is that neither the P_i nor the M_i set is identically zero. The situation differs from that of Sec. III in that the relative sign of the P and M commutator equations, Eqs. (4.9) and (4.11), is the same as the relative sign of the P and M anticommutator equations. Therefore, the P and M must correspond to the *same* fundamental representation. The s-u equation, Eq. (4.13), shows that the P and M quantum numbers are independent. Thus, the baryons interact as quark-quark composites.

We call the quarks α and β quarks. Since the signs in the P and M commutator equations, Eqs. (4.9) and (4.11), are opposite, the P and M are related oppositely to the α- and β-quark generator matrices, i.e.,

$$P_a = 2kA^{\alpha}, \quad M_a = -2kA^{\beta}, \qquad (4.15)$$

where $2k$ is a proportionality constant. In a representation in which the baryons are direct products of α and β quarks, i.e., $b = |\alpha\beta\rangle$ and $b' = |\alpha'\beta'\rangle$, the quark-generator matrices are defined by the equation

$$\langle b' | A^{\alpha} | b\rangle = A_{\alpha'\alpha} \delta_{\beta'\beta}, \qquad (4.16)$$

where the $A_{\alpha'\alpha}$ are the matrices defined in Eq. (3.12). The constant k is the product of the proportionality constants of Eqs. (3.13) and (4.2), i.e., $k = \lambda\kappa$. The D and F operators, obtained from Eq. (4.15), are

$$D_a = k(A^{\alpha} + A^{\beta}), \quad F_a = k(A^{\alpha} - A^{\beta}). \qquad (4.17)$$

Since parity doubling leads to extra multiplets, we attempt to avoid it. It is seen from Eq. (4.17) that this may be accomplished if and only if baryon states of opposite parity correspond to opposite symmetry under interchange of the α and β quarks. Then the D connect only states of the same parity, and the F connect only states of opposite parity, as is required.

Because of the fact that the meson-baryon scattering diagram contains only one baryon line which goes "straight through" the diagram, each of the two above solutions may be generalized by adding a set of passive quantum numbers, not all of which are the same. Thus, in the parity-undoubled solution, the baryon may be represented by $|\alpha\beta\gamma\rangle$, where α and β are the interacting quarks and γ denotes a set of passive quantum numbers. If γ represents a third quark and the group is $SU(6)$, the predicted multiplets of one parity (those symmetric in two of the quarks) are $\underline{56}$ and $\underline{70}$, and those of the opposite parity are $\underline{70}$ and $\underline{20}$. This is the solution of the meson-baryon scattering equations that is closest to experimental observation. It is discussed in Ref. 2.

B. Baryon-Baryon Scattering Equations

In this section we test the solutions to the meson-baryon scattering equations by considering baryon-baryon scattering. Only the "physical" three-quark solution has been subjected to this test previously. The three-quark solution fails.[14]

We take all external particles in the u channel to be of baryon number 1, so the barred states of Eq. (2.1) are states of negative baryon number. We examine first the u-t-channel consistency condition, which may be obtained by making the substitutions $a \rightleftarrows \bar{c}$ and $c \rightleftarrows \bar{a}$ in Eq. (2.6). The result is

$$\sum_n G_{adn} G^*_{cbn} \sim \sum_r \eta^{acr} G_{rac} G_{rdb} . \tag{4.18}$$

The sum on the left-hand side is over states of baryon number 2, and the sum on the right-hand side (t channel) is over conjugate meson states. Because of the η^{acr} factor, the equalities $\mathfrak{D}=D$ and $\mathfrak{F}=F$ [Eq. (4.8)] imply that the contributions of corresponding mesons of opposite parities are opposite. Thus, the right-hand side of Eq. (4.18) vanishes for all baryon-baryon amplitudes. Similarly, the meson-exchange contribution to the s-u-channel equation vanishes, so the u-channel sums must vanish. Therefore, no states of baryon number 2 exist in the model.[12] Consequently, no states of baryon number greater than 1 exist, in agreement with the observed hadron spectrum.

The only further condition that must be considered is the s-t-channel condition, Eq. (2.7). If Eq. (2.4) is used to express the antibaryon coupling constants of Eq. (2.7) in terms of baryon coupling constants, the result may be written

$$\eta^{abcd} \sum_r G_{rab} G_{rdc} = \sum_r G_{rac} G_{rdb} , \tag{4.19}$$

where both sums are over conjugate meson states. In this case it follows from Eq. (4.8) that the contributions of corresponding mesons of even and odd parities to either the s- or t-channel term are equal.

We use Eq. (4.19) to test first the parity-doubled, one-quark solution of the meson-baryon equations. The coupling constants are given by Eqs. (4.8) and (4.14). One may use the closure property of Eq. (3.14) to show that the consistency condition implies the relation

$$\eta^{abcd} \delta_{ac} \delta_{bd} = \delta_{ab} \delta_{cd} . \tag{4.20}$$

This equation is not satisfied for arbitrary baryon states. It is clear that the addition of passive quantum numbers (as discussed in Sec.IV A) would not cause it to be satisfied. The parity-doubled solution to the meson-baryon equations fails the baryon-baryon test.

We consider next the two-quark parity-undoubled solution to the meson-baryon equations. The D, F, $\mathfrak{D}$, and $\mathfrak{F}$ interaction constants are given by Eqs. (4.8) and (4.17); an expression that gives these constants simultaneously is

$$G_{rab} = k(R^{\alpha}_{ab} + \eta^{ab} R^{\beta}_{ab}). \tag{4.21}$$

Because of the fact that baryons of opposite parity have opposite symmetry with respect to exchange of the α and β quarks in this solution, the contributions of the R^α and R^β terms in Eq. (4.21) are equal for each vertex, provided each baryon state is of definite parity. Thus, the sum on the left-hand side of Eq. (4.19) may be written in terms of R^α matrices alone, i.e.,

$$\sum_r G_{rab} G_{rdc} = 8k \sum_R R^{\alpha}_{ab} R^{\alpha}_{dc} , \tag{4.22}$$

where the sum over R is over the n^2 states of the regular and singlet representations, and the factor of 8 arises because of the equal contributions of even- and odd-parity mesons, and the equal contributions of the R^α and R^β terms at each vertex. If the right-hand side of Eq. (4.19) is written in terms of the β-quark generators, the result is similar, but the η factor of Eq. (4.21) appears, i.e.,

$$\sum_r G_{rac} G_{rdb} = 8k\eta^{ac} \eta^{db} \sum_R R^{\beta}_{ac} R^{\beta}_{db} . \tag{4.23}$$

A simple equation results if Eqs. (4.22) and (4.23) are substituted into Eq. (4.19), i.e.,

$$\sum_R R^{\alpha}_{ab} R^{\alpha}_{dc} = \sum_R R^{\beta}_{ac} R^{\beta}_{db} . \tag{4.24}$$

We will test Eq. (4.24) in the direct-product basis, in which each baryon state is the product of an α-quark state and a β-quark state. These states are not eigenfunctions of parity, but they may be used because there are no factors in Eq. (4.24) that depend on baryon parity. We use the matrix elements of Eq. (4.16) and the closure property of Eq. (3.14) to write Eq. (4.24) in the form

$$\delta^{\alpha}_{ac} \delta^{\alpha}_{bd} \delta^{\beta}_{ab} \delta^{\beta}_{dc} = \delta^{\beta}_{ab} \delta^{\beta}_{cd} \delta^{\alpha}_{ac} \delta^{\alpha}_{db} , \tag{4.25}$$

where δ^{α}_{ij} refers to the α-quark states of the baryon states i and j. This equation is an identity; the two-quark solution passes the test.[15]

The test would not be passed if a set of passive quantum numbers were present that were not the same for all baryon states. For example, if a passive γ quark were present (as discussed in Sec. IV A), the left-hand and right-hand sides of Eq. (4.25) would contain extra factors of $\delta^{\gamma}_{ab} \delta^{\gamma}_{dc}$ and $\delta^{\gamma}_{ac} \delta^{\gamma}_{db}$, respectively. These are not equal for all amplitudes. Thus, the baryon-baryon conditions are satisfied by only one of the solutions to the meson-baryon conditions, the parity-undoubled two-quark solution.

V. CONCLUDING REMARKS

A bootstrap theory is like other physical theories in the sense that the ideal is to find mathematical equations with the following two properties:
(i) The actual universe satisfies the equations.
(ii) Other conceivable universes do not. The pres-

ent paper differs from most hadron theoretical papers in that we are concerned particularly with the second property. For this reason, we have made very few initial assumptions concerning the nature of the existing hadrons. On the other hand, we have not written a complete theory in the sense that we have not made use of every well-established invariance principle. For example, we have considered only collinear scattering amplitudes. The philosophy is that if a limited set of consistency equations forces the hadrons to behave like composites of quarks and antiquarks, this hadron property will not be changed if other consistency conditions are added.

Because of the restriction to collinearity, the only angular momentum conservation law applied has been conservation of the z component. In order for our solution to be physical, it should be possible to meet one of the most important requirements of J^2 conservation, the requirement that the physical particles have definite spins. It has been shown that one may meet this requirement for the odd-parity mesons and lightest-baryon set (the even-parity baryons) by using the W-spin prescription for the $SU(2)$ subgroup that applies to the spins.[11] On the other hand, if the vertices are to be dominated by low orbital angular momenta, the even-parity meson states and odd-parity baryon states must be superpositions of states of different spins.[16] Thus, the parity-doubling partner of a vector-meson trajectory must be a trajectory whose lowest state is a spin-2 meson, but this meson should be degenerate with even-parity mesons of lower spin. We will not discuss the correspondence to physical particles further in this paper.

It has been pointed out by Belinfante and Renninger that in the static limit, meson-baryon s-u-channel consistency conditions based on $SU(n)_W$ symmetry are valid at all angles.[17] However, if such noncollinear mechanisms as two-particle intermediate states were included in the dynamics, the $SU(n)_W$ vertex symmetry would be broken. It is possible that the experimental degree of accuracy of $SU(6)_W$ vertex symmetry depends on the degree of dominance by collinear processes of the dynamical processes that determine the interaction constants, whatever those processes may be.

Our conditions require that the baryons behave as two-quark composites, in contrast to the three-quark behavior observed experimentally. (Thus, the particles of unit baryon number in the model are of integral spin.) However, the prediction that the two baryon parities correspond to different symmetries under quark interchange does agree with experiment. [Experimentally, the $SU(6)$ representations of the baryon trajectories are the completely symmetric $\underline{56}$ for even parity, and the mixed-symmetry $\underline{70}$ for odd parity.]

The predictions are stronger than was generally believed to follow from bootstrap conditions of the type used. Group symmetry must be present, the group must be $SU(n)$ (although the value of n is not predicted), and the mesons and baryons must correspond to specific multiplets. The meson predictions agree with experiment, and the baryon predictions disagree by one quark. It may be that self-consistency conditions, rather than the existence of physical quarks, are the reason that the quark model is successful.

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*Supported in part by the U. S. Atomic Energy Commission and in part by the U. S. Bureau of Standards.

†On leave from Purdue University, Lafayette, Ind.

[1]Two early papers on this subject are R. H. Capps, Phys. Rev. Letters $\underline{10}$, 312 (1963); R. E. Cutkosky, Phys. Rev. $\underline{131}$, 1888 (1963).

[2]R. H. Capps, Phys. Rev. D $\underline{2}$, 780 (1970).

[3]The consistency equations are derived in Sec. II C of Ref. 2, but a more thorough derivation is given by R. H. Capps, Phys. Rev. D $\underline{2}$, 2640 (1970).

[4]R. H. Capps, Phys. Rev. $\underline{171}$, 1591 (1968).

[5]This equation is equivalent to Eqs. (2) and (4) of Ref. 2. The derivation of Ref. 3 does not eliminate the possibility of an extra minus sign in front of the u-channel term. It can be shown that if such a minus sign is present, there are no nontrivial solutions for mesons.

[6]In the case of meson-meson scattering, the set of consistency equations used here is algebraically the same as a set obtained in Ref. 4 from superconvergence relations and a simple dynamical assumption. The results of Sec. III B of the present paper are listed in Ref. 4.

[7]This type of proof for the absence of nontrivial symmetric solutions was first used by R. E. Cutkosky in connection with a scalar-meson model (private communication).

[8]The first model in which a bootstrap hypothesis led to the Jacobi identity condition was the vector-meson model of R. E. Cutkosky, Ref. 1.

[9]A lucid discussion of Lie groups, and a list of the theorems that we use concerning the structure constants, are given by M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1962). See, particularly, Chap. 8, Sec. 10.

[10]We have found many people who believe this theorem, but have not found a complete and rigorous proof. It is possible for us to bypass the theorem, as is shown at the end of Sec. III C.

[11]The fact that the interactions of Eqs. (3.13a) and (3.13b) satisfy the meson-meson scattering conditions, and the fact that the spin must be treated by the $SU(2)_W$ prescription, are shown by R. H. Capps, Phys. Rev.

168, 1731 (1968).

[12]In the treatment of self-consistency equations in Ref. 2, the absence of states of baryon number 2 is assumed *a priori* and used to deduce the equality of the baryonic interactions of corresponding even- and odd-parity meson states.

[13]One of the first models in which baryonic interactions are forced to satisfy a commutation relation similar to Eq. (4.9) is that of J. C. Polkinghorne, Ann. Phys. (N.Y.) 34, 153 (1965). In the Polkinghorne model, which involves vector mesons and baryons of one parity, any group representation is satisfactory for the baryons.

[14]This point has been emphasized by J. L. Rosner, Phys. Rev. Letters 21, 950 (1968); 21, 1468(E) (1968).

[15]The nature of this proof shows that the two-quark solution works for the same reason that it satisfies the "duality diagram" test of Harari and Rosner. See H. Harari, Phys. Rev. Letters 22, 562 (1969); J. L. Rosner, *ibid.* 22, 689 (1969).

[16]Examples of such superpositions are the virtual (resonance) states calculated in the $SU(6)_W$-potential model of R. H. Capps, Phys. Rev. 158, 1433 (1967); 165, 1899 (1968).

[17]J. G. Belinfante and G. H. Renninger, Phys. Rev. 148, 1573 (1966).

PHYSICAL REVIEW D VOLUME 3, NUMBER 12 15 JUNE 1971

Persistence of the "Photon" in Conformal-Dual Models*

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(Received 15 March 1971)

The spin-1 zero-mass "photon" of the (conformal) Veneziano model persists in the spectra of all "new" (conformal) dual models of the Bardakci-Halpern type. In these generalized models, however, the "photon" does not, in general, decouple through statistics.

The conformal Veneziano[1] model (all Ward identities[2] working) involves unit leading-trajectory intercept, and hence a spin-1 zero-mass "photon" Γ. As far as we know, this particle is not related to the real photon (of electromagnetism). In any case, Γ decouples from the model via Bose statistics, so it is no problem. Recently, Bardakci and the author introduced a new class of dual-conformal models[3] that includes spin. In each of the simple examples we discussed, Γ appeared again in the spectrum. What we want to show here is that this feature is completely general: For any model of our type, Γ persists in the spectrum. Moreover, it does not in general decouple through statistics; for example, in the additive models, it couples to baryon-antibaryon pairs. For models with spin-orbit forces, we cannot determine the coupling until after the gauge states are removed.

We begin by describing our models in general terms.[4] We assume we have found a set of conformal generators J_m, constructed as Fourier components of a density $J(\theta)$:

$$J_m = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-im\theta} J(\theta), \tag{1}$$

where the density is itself constructed of sums of bilinears in local fields and currents, which we denote by $\{\pi(\theta)\}$ and $\{V(\theta)\}$, respectively. The $\{V(\theta)\}$ carry Lorentz and internal-symmetry labels, while $\{\pi(\theta)\}$ in general carries Lorentz and fifth (etc.) operator labels. For reference, we record the projective subalgebra of the J_m,

$$[J_0, J_{\pm 1}] = \mp J_{\pm 1}, \tag{2a}$$

$$[J_{+1}, J_{-1}] = 2J_0, \tag{2b}$$

together with the equation for mass-shell states $|\Psi\rangle$,

$$J_0|\Psi\rangle = |\Psi\rangle\ . \tag{3}$$

We can exhibit the dependence of J_m on the external 4-momentum p^μ ($\mu = 0, 1, 2, 3$) in the form

$$J_m = -ap^2\delta_{m,0} + p\cdot A_m + B_m, \tag{4}$$

where the (assumed known) number a and the operators A_m^μ, B_m are p^μ-independent (though the operators may depend on fifth operators). In the sector where fifth quantum numbers are zero, we have a vacuum state $|0\rangle$ such that[5]

$$A_m^\mu|0\rangle = 0, \quad m \geq 0 \tag{5a}$$

$$B_m|0\rangle = 0, \quad m \geq -1\ . \tag{5b}$$

Equations (1)–(5) complete our statement of the generalized models; it includes, of course, the ordinary Veneziano model. Now we want to show that any such system contains a "photon" Γ. The particle will occur in the sector with zero fifth (etc.) quantum numbers, so we will confine ourselves to this case.

Consider Eq. (2a). By differentiating with re-