[7]The discussion given in Ref. 2 on this point is not at all clear.

[8]Note that the index of the distinguished pion can be assigned to only one of the operators $(n\cdot \vec{\nabla})\cdot(\vec{\pi}\times\partial_t\vec{\pi})$.

[9]The expansion of a symmetrized multiple commutator like

$$\sum_{\mathrm{perm}[i(1),\ldots,i(n)]}[X^{i(1)},\ldots[X^{i(n)},[X^{i(n+1)},m^2]]\ldots]$$

gives rise to the occurrence of binomial coefficients $\binom{n}{m}$ in the following way: A typical term in the expansion consisting in m factors $X^{i(l)}$ $(l\neq n+1)$ to the left of $[X^{i(n+1)}, m^2]$ and $(n-m)$ factors to its right may be assembled in $n!$ ways. However, symmetrization allows us to reorder each of the two groups of m and $(n-m)$ factors $X^{i(l)}$ and only $n!/[m!(n-m)!]$ of these arrangements are distinguishable. The alternation of sign follows, of course, from the antisymmetry of the commutator.

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Algebraic Structure of Pion-Hadron Transition Amplitudes and Hadron Mass Spectra*

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Making use of generally accepted assumptions about the equal-time commutator algebra of axial charges and axial divergences, the spin and helicity dependence of Weinberg's algebraic relations is entirely determined for pions in arbitrary partial waves. It is shown that the algebraic structure of axial-vector coupling matrices may be given by the Lie algebra of the group $SO(4,3)$, and so hadron states must be assigned to unitary representations of this group. Furthermore it is proved that the mass-spectrum operator is given as a sum of a scalar and a component of a 35-dimensional totally antisymmetric irreducible tensor of the group $SO(4,3)$. The general form of the mass spectrum is exhibited as a linear combination of the Clebsch-Gordan coefficients of the group $SO(4,3)$. Application to hadron states of fixed intrinsic quantum numbers leads to the conclusion that mass-squared values of hadrons must be a linear function of spin. This result is a unique and exact consequence of the structure of certain algebraic relations.

I. INTRODUCTION

Recently Weinberg[1] has derived extremely powerful and elegant algebraic relations involving the pion-hadron decay amplitudes and the hadron mass-squared matrix. These relations have the following form:

$$[X^\alpha, X^\beta]=i\epsilon^{\alpha\beta\gamma}I^\gamma \tag{1.1}$$

and

$$[X^\alpha,[m^2,X^\beta]]=\tfrac{1}{3}\delta^{\alpha\beta}[X^\gamma,[m^2,X^\gamma]], \tag{1.2}$$

where $\alpha,\beta=1,2,3$ are isospin indices of the pion. The meaning of the various symbols in the previous two equations is as follows. $(X^\alpha)_{ba}$ is a matrix element in the space of the internal quantum numbers b and a such as isospin, spin, hypercharge, parity, etc. It is related to the invariant Feynman amplitude $M^\alpha_{ba}(p'q;p)$ for any collinear (helicity-conserving) transition process

$$a(p,\lambda_a)\to b(p',\lambda_b)+\pi^\alpha(q) \tag{1.3}$$

of the massless pion π^α by

$$M^\alpha_{ba}(p'q;p)=2F_\pi^{-1}(m_a^2-m_b^2)(X^\alpha)_{ba}, \tag{1.4}$$

where $a(p,\lambda_a)$ and $b(p',\lambda_b)$ denote hadrons with momenta p and p', helicities λ_a and λ_b, and masses m_a and m_b, respectively. I^α is the isospin generator matrix, m^2 is the diagonal mass-squared operator, and $F_\pi\approx 190$ MeV is the pion decay amplitude. The matrices X^α are diagonal in helicity, i.e.,

$$(X^\alpha)_{b\lambda_b,a\lambda_a}=\delta_{\lambda_b\lambda_a}(X^\alpha)_{ba}. \tag{1.5}$$

The essential assumptions used by Weinberg in his derivation of the aforementioned relations were:

(a) Tree-graph contributions to the forward scattering amplitude with massless pions, calculated from a chirally invariant Lagrangian, should not violate the asymptotic behavior predicted by Regge-pole theory. This requirement is equivalent to the saturation of dispersion-theoretic sum rules by single-particle states, as was demonstrated by Weinberg.[1]

(b) There should be no so-called exotic states having isospin $I=2$.

The algebraic relations (1.1), along with the relations involving the isospin generator matrices I^α of the isospin group $SU(2)_I$, define the Lie algebra of the chiral group $SU(2)\otimes SU(2)$, and this implies that the hadron states must, for each helicity and various isospins and spins, be assigned to unitary (irreducible or reducible) representations of the chiral group. The commutator (1.1) then determines the transition amplitudes among the hadrons accommodated in the single unitary representation of the group in question. Once the matrices X^α are known, they can be inserted in the second algebraic relation (1.2) which then gives the form of the mass spectrum of the hadrons under consideration.

As can be seen, the method demonstrated by Weinberg has a great amount of appeal since it provides a scheme for calculating the pion-hadron transition processes and hadron mass spectra – the goal of strong-interaction physics.

This treatment has been extended to multipion production processes by McDonald,[2] and also to the higher chiral group $SU(3)\otimes SU(3)$ by Ram Mohan.[3]

The Weinberg algebraic relations can be obtained in many ways. For example, they are equivalent to the relations derived by Gilman and Harari.[4] A very important method for deriving these relationships is based on the infinite-momentum saturation of commutators of $SU(2)\otimes SU(2)$ [or $SU(3)\otimes SU(3)$] axial charges and axial divergences,[5,6,7] and in particular (1.1) is derived by saturating the equal-time commutator between axial charges,[8]

$$\left[\int d^3x A_0^\alpha(\vec{x},t), \int d^3z A_0^\beta(\vec{z},t)\right] = i\epsilon^{\alpha\beta\gamma}I^\gamma, \qquad (1.6)$$

while (1.2) is obtained by saturating the double commutator,[9]

$$\left[\int d^3x A_0^\alpha(\vec{x},t), \left[m^2, \int d^3z A_0^\beta(\vec{z},t)\right]\right] = \delta^{\alpha\beta}\Lambda, \quad (1.7)$$

where $A_0^\alpha(\vec{x},t)$ is the time component of the axial-vector current and Λ is an isoscalar–Lorentz-scalar operator. It should be mentioned that the double commutator (1.7) may also have an isospin-2 component, in general. This is assumed to be absent as exotic states have not been seen in nature. This means that the algebraic structure of the relations (1.1) and (1.2) has a profound origin in the equal-time commutator algebra which charges and axial divergences satisfy, as was pointed out by Fubini and Furlan.[6]

Unfortunately (1.1) and (1.2) do not provide any information on how the representations with different helicities are related to each other. As was pointed out by Weinberg,[1] the helicity and spin dependence of the matrices X^α can be determined if one assumes that only a few partial waves predominate in the pion-hadron transition processes (1.3). Then (1.1) and (1.2) become finite matrix equations in the internal variable space, with the range of the labels restricted to a finite discrete set. The solutions to these relations have been found in a series of papers.[1,3,10,11] It was found that predictions following from (1.1) and its generalization to p waves were in good agreement with experiment.[3,10] However, results following from (1.2) and its generalization were disappointing in that they predicted that the masses of the hadrons either decrease with increasing quantum numbers[10] or are degenerate.[3]

The purpose of the present paper is to generalize Weinberg's algebraic relations (1.1) and (1.2) to pions in arbitarary partial waves, i. e., to find the complete and general dependence of the matrices X^α and m^2 on spins and helicities. One possible way to get information about the spin and helicity dependence of the matrices X^α is to perform a partial-wave decomposition of the Feynman amplitude (1.4) as indicated by Weinberg.[1] We know that this approach works very well if only one[1,3,10] or two[1,11] partial waves are assumed to predominate in pion-hadron interactions. However, in a completely general case, where all partial waves should be included, one is forced to deal with an infinite number of pion-hadron coupling matrices, i. e., with one matrix for each partial wave. Thus this treatment would lead to cumbersome infinitely many algebraic relations, and that is why we shall use a different approach to this problem.

Our approach consists of employing a boost transformation of the homogeneous Lorentz group, along with an information following from one-particle-state saturation of the equal-time commutators (1.6) and (1.7). We recall once again that the commutators (1.6) and (1.7) can be considered as a starting point for the derivation of the relations (1.1) and (1.2). We shall also show that these commutators are powerful enough to determine the spin and helicity dependence of Weinberg's algebraic relations (1.1) and (1.2) when completed by Lorentz transformations.

In Sec. II it is shown that the spin and helicity dependence of Weinberg's matrices X^α is fully governed by the Lie algebra of the noncompact group $SO(4,3)$. This group is generated by the matrices of isospin, relativistic spin, and by axial-vector current operators. In Sec. III it is proved that the mass-squared diagonal matrix behaves as a sum of two terms which transform as a scalar and a component of the 35-dimensional totally antisymmetric tensor of the group $SO(4,3)$. The general form of the mass spectrum is given in terms of the Clebsch-Gordan coefficients of the group $SO(4,3)$. In Sec. IV the derived results are applied to hadron states with the same intrinsic quantum numbers, such as the third component of isospin, hypercharge, and parity, and it is shown that the mass squared of these hadrons is a linear function of their spins, a result always assumed in Regge phenomenology. Concluding remarks are devoted to the discussion of the relationship between different algebraic approaches proposed for the description of pion-hadron dynamics.

II. ALGEBRAIC STRUCTURE OF AXIAL-VECTOR-CURRENT MATRIX ELEMENTS

Consider a completely general pion-hadron transition process,

$$a(p) \to b(p') + \pi^\alpha(q), \tag{2.1}$$

where $a(p)$ and $b(p')$ denote arbitrary hadron states with momenta p and p', respectively, and $\pi^\alpha(q)$ is a pion of momentum q and isospin index α. The interaction of a single pion with a target hadron can be described by means of the following Lagrangian[1]:

$$\mathcal{L}_I = -F_\pi^{-1} A_\mu^\alpha(x)\,\partial^\mu \varphi^\alpha(x), \tag{2.2}$$

where $A_\mu^\alpha(x)$ is a phenomenological axial-vector current and $\varphi^\alpha(x)$ represents a pion field.

Using the Lehmann-Symanzik-Zimmermann reduction technique,[12] the invariant Feynman amplitude $M_{ba}^\alpha(p'q;p)$ for the process (2.1) can be written as

$$M_{ba}^\alpha(p'q;p) = F_\pi^{-1}(2\pi)^3(4p_0'p_0)^{1/2}(p-p')^\mu\langle b\vec{p}'\,|A_\mu^\alpha(0)|\,a\vec{p}\,\rangle, \tag{2.3}$$

where the hadron states $|a\vec{p}\,\rangle$ and $|b\vec{p}'\rangle$ have been normalized to

$$\langle a\vec{p}\,|\,b\vec{p}'\rangle = \delta_{ab}\delta^3(\vec{p}-\vec{p}'). \tag{2.4}$$

Since the Feynman amplitude (2.3) is Lorentz-invariant, one may assume without loss of generality that the initial state is at rest, i.e.,

$$|a\vec{p}\,\rangle = |a\vec{0}\rangle \equiv |a\rangle. \tag{2.5}$$

To proceed futher, assume the existence of a Hilbert space H spanned by all particle states at rest, $|a\rangle$. The Hilbert space H of rest states must be invariant under rotations; hence one can decompose it with respect to its spin contents. Therefore one can assume that a contains the spin labels s and s_z among other possible quantum numbers. If one wants to describe any relativistic interaction between particles, he must specify how the physical states transform under Lorentz transformations. To put it in a better way, one must define a representation of the Lorentz group on the Hilbert space of the physical states. There are two distinctive ways to handle this problem.

(i) The most usual way to do this is to increase the Hilbert space H to the Hilbert space H', spanned by states $|ap\rangle$, by adding the momentum p as an additional quantum number, and representing the Lorentz-group elements Λ separately for every spin s by means of the Wigner rotation $W^s(\Lambda, p)$,[13]

$$U(\Lambda)\,|a,p\rangle = |a',\Lambda^{-1}p\rangle W^s_{a'a}(\Lambda,p). \tag{2.6}$$

(ii) The second possibility is to leave the Hilbert space H the same and to represent the Lorentz group directly on the rest states $|a\rangle$. In this case each moving particle state $|a\vec{p}\,\rangle$ of three-momentum $\vec{p}$ is characterized by its rapidity vector $\vec{\xi}$, which has the direction of $\vec{p}$ and the magnitude given by

$$\tanh|\vec{\xi}\,| = |\vec{p}\,|/p_0. \tag{2.7}$$

This moving state is obtained by boosting the state $|a\rangle$ at rest by means of the homogeneous Lorentz transformation, i.e.,

$$|a\vec{p}\,\rangle \equiv |a\vec{\xi}\,\rangle = (m_a/p_0)^{1/2}e^{i\vec{\xi}\cdot\vec{N}}\,|a\rangle \equiv B(\vec{\xi})\,|a\rangle, \tag{2.8}$$

where $N_k = J_{k0}$ is the Lorentz boost, m_a is the mass of the particle a, the factor $(m_a/p_0)^{1/2}$ is due to the chosen normalization (2.4), and $B(\xi)$ is the known matrix of the finite Lorentz transformation.[14] This definition of the representation of the Lorentz group on the Hilbert space of the physical states is always used in theories of dynamical groups and infinite-component wave equations, and it will also be used in the remaining part of this section.

In view of what we have said, the Feynman amplitude (2.3) can now be rewritten in the following form:

$$M_{ba}^\alpha(p',q;p) = F_\pi^{-1}(2\pi)^3(4m_bm_a)^{1/2}(p-p')^\mu \times\langle b|e^{-i\vec{\xi}\cdot\vec{N}}A_\mu^\alpha(0)\,|\,a\rangle \tag{2.9}$$

where $|a\rangle$ and $|b\rangle$ are hadron states at rest.

In order to find the algebraic structure of the invariant amplitude (2.9), one may define matrix elements of the relativistic momentum operators $J_{\mu\nu}$ as well as elements of the isospin matrices I^α on the hadron states at rest. In addition to these, twelve matrices X_μ^α are constructed as follows:

$$(X_\mu^\alpha)_{ba} = (2\pi)^3\langle b\,|A_\mu^\alpha(0)\,|a\rangle. \tag{2.10}$$

It was shown in the work of Ref. 15, using several quite general assumptions listed below, that matrices X_μ^α, $J_{\mu\nu}$, and I^α may form a closed Lie algebra. The most general form of this algebra is isomorphic to the Lie algebra of the noncompact group $G = SO(3,1)\otimes SO(4,3)$. Here the factor algebra $SO(3,1)$ is generated by matrices $M_{\mu\nu}$ defined as

$$M_{\mu\nu} = J_{\mu\nu} - S_{\mu\nu}, \tag{2.11}$$

where the $S_{\mu\nu}$ may be interpreted as matrices of the relativistic spin. The $M_{\mu\nu}$ then play the role of external momentum matrices. Since our considerations are restricted only to a rest frame, the matrices $M_{\mu\nu}$ can be omitted. Denoting the generator matrices of the group $SO(4,3)$ by

$$S_{ab}=-S_{ba},$$

its Lie-algebraic properties can be written as follows:

$$[S_{ab},S_{cd}]=i(g_{bc}S_{ad}-g_{bd}S_{ac}-g_{ac}S_{bd}+g_{ad}S_{bc}), \qquad (2.12)$$

where $a,b,c,d=0,1,2,3,5,6,7$ and the metric tensor g_{ab} is defined as

$$g_{00}=g_{55}=g_{66}=g_{77}=+1,$$

$$g_{11}=g_{22}=g_{33}=-1,$$

and

$$g_{ab}=0 \quad \text{if} \quad a\neq b.$$

The range of numbers $a,b,c,\ldots=1,2,3,0$ is associated with the space-time indices of the Minkowski space, while the remaining set of numbers $a,b,c,\ldots=5,6,7$ is connected with the isospin labels α,β,γ. The matrix elements of the matrices S_{ab} represent measurable physical quantities and are associated with the matrix elements of the physical observables as follows:

$$S_{\alpha\beta}=-\epsilon^{\alpha\beta\gamma}I^{\gamma}, \quad \alpha,\beta,\gamma=5,6,7; \quad \epsilon^{567}=+1 \qquad (2.13\text{a})$$

$$S_{\alpha\mu}=X_{\mu}^{\alpha}, \qquad (2.13\text{b})$$

and $S_{\mu\nu}$ with $\mu,\nu=1,2,3,0$ are matrices of the internal angular momenta.

The above-mentioned algebraic structure of the axial-vector-current matrix elements was derived in the work of Ref. 15, using the following set of assumptions:

(1) The axial-vector current $A_{\mu}^{\alpha}(x)$ transforms like an isovector and Lorentz four-vector. This requirement guarantees the isospin and Lorentz invariance of strong interactions.

(2) The equal-time commutator algebra[8] (1.6) is valid. This condition is in some sense equivalent to Weinberg's first algebraic relation (1.1).

(3) Exotic states, i.e., those having isospin $I=2$, are excluded. This is the same requirement as that which leads to the commutators (1.2) and (1.7).

(4) The matrices X_{μ}^{α} transform like algebraic isovectors and Lorentz four-vectors.

It should be stressed that since the group G is noncompact, Eqs. (2.12) are matrix relations among 21 infinite-dimensional matrices. The rows and columns of these matrices are labeled by all quantum numbers of hadrons. From this follows that the operators $e^{-i\vec{\xi}\cdot\vec{N}}$ and $A_{\mu}^{\alpha}(0)$, which appear explicitly in the final expression for the scattering amplitude (2.9), can be represented by infinite-dimensional matrices acting on the vector space which forms a representation space (generally reducible) of the group $SO(4,3)$. Since the successive operation upon the vector $|a\rangle$ with the two operators $A_{\mu}^{\alpha}(0)$ and $e^{-i\vec{\xi}\cdot\vec{N}}$ is well defined, the scattering amplitude (2.9) is then determined by the representation of the group in question. Once the right representation is chosen to describe hadrons, the mathematics of the group takes over and the calculation of this scattering amplitude is straightforward.

The matrix relations (2.12) can be considered as the generalization of Eq. (1.1) to the form determining the spin and helicity dependence of the matrices X^{α}. This follows from the fact that both (1.1) and the algebraic structure (2.12) can be uniquely derived from the equal-time commutator of axial charges (1.6) if one demands the absence of exotic states. This generalized algebraic relation implies that hadron states mutually correlated through pion transition processes must be assigned, in general, to unitary reducible representations of the noncompact group $SO(4,3)$.

If the representation to which the physical hadron states are assigned is irreducible, the matrix elements of S_{ab} will be uniquely determined and consequently the invariant Feynman amplitude will be completely known. So this method for determining three-body scattering amplitudes almost coincides with the approach used in the relativistic framework of dynamical groups proposed by Barut.[16] Note that all dynamical groups $SO(3,1)$, $SO(3,2)$, and $SO(4,2)$, which have been successfully applied to strong decays of mesons[17] and baryons[18] as well as to the study of mass spectra and form factors of hadrons,[19,20] are the subgroups of the group $SO(4,3)$.

However, if the hadrons belong to a reducible representation of $SO(4,3)$, the invariant Feynman amplitude (2.9) will not be uniquely determined, but will contain a number of free parameters, usually known as mixing angles.[1]

The remaining task is to find the restriction on the mass spectrum of hadrons accommodated in the single unitary (irreducible or reducible) representation of the group $SO(4,3)$. This problem is solved in Sec. III.

III. ALGEBRAIC STRUCTURE OF THE MASS MATRIX

A. Mass-Spectrum Condition

In order to find the dependence of the mass-squared operator m^2 on the internal quantum numbers, the double commutator (1.7) is saturated with a complete set of hadron states. Recall that this double commutator, under the assumption of single-particle-states saturation in the infinite-momentum limit,[6,7,9] leads to the second Weinberg algebraic relation (1.2).

Before proceeding further, Eq. (2.12) is rewritten in terms of the matrices of the axial-vector current X_μ^α, the isospin generators I^α, and the relativistic spin operators $S_{\mu\nu}$ as follows, using the definitions (2.13):

$$[I^\alpha, I^\beta] = i\epsilon^{\alpha\beta\gamma} I^\gamma, \tag{3.1}$$

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(g_{\nu\rho}S_{\mu\sigma} - g_{\nu\sigma}S_{\mu\rho} - g_{\mu\rho}S_{\nu\sigma} + g_{\mu\sigma}S_{\nu\rho}), \tag{3.2}$$

$$[I^\alpha, S_{\mu\nu}] = 0, \tag{3.3}$$

$$[I^\alpha, X_\mu^\beta] = i\epsilon^{\alpha\beta\gamma} X_\mu^\gamma, \tag{3.4}$$

$$[S_{\mu\nu}, X_\rho^\beta] = i(g_{\nu\rho} X_\mu^\beta - g_{\mu\rho} X_\nu^\beta), \tag{3.5}$$

and

$$[X_\mu^\alpha, X_\nu^\beta] = ig_{\mu\nu}\epsilon^{\alpha\beta\gamma} I^\gamma - i\delta^{\alpha\beta} S_{\mu\nu}, \tag{3.6}$$

where the convention has been used that the superscripts α, β, γ label isospin indices, while the subscripts μ, ρ, σ, ν are associated with the Lorentz indices in Minkowski space. The relation (3.4) defines the transformation properties of isovectors, while Eq. (3.5) specifies the transformation law for Lorentz four-vectors.

Now consider the double commutator (1.7) for $t=0$, sandwiched between two hadron states $|b\vec{p}'\rangle$ and $|a\vec{p}\rangle$, i.e.,

$$\left\langle b\vec{p}'\left|\left[\int d^3x\, A_0^\alpha(\vec{x},0), \left[m^2, \int d^3z\, A_0^\beta(\vec{z},0)\right]\right]\right|a\vec{p}\right\rangle = \delta^{\alpha\beta}(\Lambda)_{ba}\delta^3(\vec{p}-\vec{p}'). \tag{3.7}$$

To evaluate the left-hand side of the last equation a complete set of the intermediate particle states $|n\vec{p}_n\rangle$ is inserted, and the integrations over the spatial variables $\vec{x}, \vec{z}$, and momentum variables $\vec{p}_n$ are performed by using the translation invariance of matrix elements. This yields the result

$$\begin{aligned}(2\pi)^6 \sum_n [&\langle b\vec{p}|A_0^\alpha(0)|n\vec{p}\rangle m_n^2 \langle n\vec{p}|A_0^\beta(0)|a\vec{p}\rangle \\ &- \langle b\vec{p}|A_0^\alpha(0)|n\vec{p}\rangle\langle n\vec{p}|A_0^\beta(0)|a\vec{p}\rangle m_a^2 \\ &- m_b^2\langle b\vec{p}|A_0^\beta(0)|n\vec{p}\rangle\langle n\vec{p}|A_0^\alpha(0)|a\vec{p}\rangle \\ &+ \langle b\vec{p}|A_0^\beta(0)|n\vec{p}\rangle m_n^2\langle n\vec{p}|A_0^\alpha(0)a\vec{p}\rangle] \\ &= \delta^{\alpha\beta}(\Lambda)_{ba},\end{aligned} \tag{3.8}$$

where the fact that the mass-squared operator m^2 is diagonal has been used, and in addition the common factor $\delta^3(\vec{p}-\vec{p}')$ has been cancelled on both sides. It should be stressed that the above relation can only be derived for states $|b\vec{p}'\rangle$ and $|a\vec{p}\rangle$ with the same three-momentum $\vec{p}$. In view of this, and without loss of generality, only hadron states at rest are going to be considered. Using the definition (2.10) for the matrix X_0^α, i.e.,

$$(X_0^\alpha)_{bn} = (2\pi)^3\langle b|A_0^\alpha(0)|n\rangle, \tag{3.9}$$

and defining the matrix M^2 by

$$(M^2)_{bn} = m_b^2\delta_{bn}, \tag{3.10}$$

Eq. (3.8) is rewritten in the following matrix form:

$$[X_0^\alpha, [M^2, X_0^\beta]] = \delta^{\alpha\beta}\Lambda. \tag{3.11}$$

This property of the double commutator, combined with the fact that M^2 commutes with both the isospin I^α and spin $S_{\mu\nu}$ generators, is now used to derive the mass-spectrum condition for hadrons classified according to unitary representations of the group $SO(4,3)$. In order to do this the 144 matrices $D_{\mu\nu}^{\alpha\beta}$ are defined by

$$D_{\mu\nu}^{\alpha\beta} \equiv [X_\mu^\alpha, [M^2, X_\nu^\beta]]. \tag{3.12}$$

The Jacobi identity applied to the double commutator (3.12) along with Eq. (3.6) gives

$$D_{\mu\nu}^{\alpha\beta} = D_{\nu\mu}^{\beta\alpha}. \tag{3.13}$$

This implies that the matrices $D_{\mu\nu}^{\alpha\beta}$ are symmetric with respect to the interchange of pairs of indices $(\alpha\mu)$ and $(\beta\nu)$ and therefore their number is reduced to 78 independent matrices. Hence the matrices $D_{\mu\nu}^{\alpha\beta}$ behave as the sum of completely symmetric and completely antisymmetric isotensors and Lorentz tensors, and the most general decomposition of the double commutator (3.12) takes the form

$$[X_\mu^\alpha, [M^2, X_\nu^\beta]] = \epsilon^{\alpha\beta\gamma} A_{[\mu\nu]}^\gamma + U_{\{\mu\nu\}}^{\{\alpha\beta\}}, \tag{3.14}$$

where $A_{[\mu\nu]}^\gamma$ and $U_{\{\mu\nu\}}^{\{\alpha\beta\}}$ are matrices, and the symbols $[\rho\sigma]$ and $\{\rho\sigma\}$ are abbreviations for antisymmetry and symmetry, respectively, in the corresponding pair of indices. It can be shown, by using the method outlined in Appendix A of Ref. 15, that $A_{[\mu\nu]}^\gamma \equiv A_{\mu\nu}^\gamma = -A_{\nu\mu}^\gamma$ is an 18-dimensional isovector and antisymmetric Lorentz tensor, and that $U_{\{\mu\nu\}}^{\{\alpha\beta\}}$ transforms as a 60-dimensional totally symmetric isotensor and Lorentz tensor. The restriction of Eq. (3.14) to the time components, i.e.,

$$[X_0^\alpha, [M^2, X_0^\beta]] = U_{\{00\}}^{\{\alpha\beta\}}, \tag{3.15}$$

must give Eq. (3.11). This implies that

$$U_{\{\mu\nu\}}^{\{\alpha\beta\}} = \delta^{\alpha\beta}\Lambda. \tag{3.16}$$

Since the matrix Λ is an isoscalar and Lorentz scalar, it follows (formally, by taking the repeated commutators $[U_{\{00\}}^{\{\alpha\beta\}}, S_{\mu\nu}]$ and $[U_{\{00\}}^{\{\alpha\beta\}}, I^\gamma]$) that $U_{\{\mu\nu\}}^{\{\alpha\beta\}}$ is an invariant totally symmetric isotensor and Lorentz tensor. This implies that $U_{\{\mu\nu\}}^{\{\alpha\beta\}}$ is restricted to the form

$$U^{\{\alpha\beta\}}_{\{\mu\nu\}} = \delta^{\alpha\beta} g_{\mu\nu}\Lambda. \tag{3.17}$$

Using this result, Eq. (3.14) is rewritten as follows:

$$[X^\alpha_\mu, [M^2, X^\beta_\nu]] = \delta^{\alpha\beta} g_{\mu\nu}\Lambda + \epsilon^{\alpha\beta\gamma} A^\gamma_{\mu\nu}. \tag{3.18}$$

Λ and $A^\alpha_{\mu\nu}$ can now be written in terms of X^α_μ and M^2 as

$$\Lambda = \tfrac{1}{12} g^{\mu\nu}[X^\alpha_\mu, [M^2, X^\alpha_\nu]] \tag{3.19}$$

and

$$A^\gamma_{\mu\nu} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}[X^\alpha_\mu, [M^2, X^\beta_\nu]]. \tag{3.20}$$

Inserting the expressions (3.19) and (3.20) into Eq. (3.18), a dynamical equation is obtained, the solution to which determines the mass spectrum of hadrons accommodated in a single unitary (irreducible or reducible) representation of the group $SO(4,3)$.

In the practical applications of Eqs. (3.18)–(3.20), each physical particle state of definite spin s and isospin I is written, in general, as a sum of unitary irreducible representations of the group $SO(4,3)$. The matrices X^α_μ are then entirely determined by the unitary irreducible representations of the group in question, and by the mixing angles which define the coefficients of the irreducible representations in this sum. The known matrices X^α_μ are then inserted into the mass condition (3.18) and a nontrivial relation for the unknown matrix M^2 is obtained. A solution to this equation will represent the mass spectrum of hadrons assigned to the considered representation. Needless to say, the physical masses will be dependent, in general, on the internal quantum variables such as isospin and spin, and also on the mixing angles. It is very easy to imagine that this approach to the mass spectrum gives rise to tedious calculations and does not have the elegance normally associated with group theory. However, an elegant statement can be made about M^2, namely, that it behaves as a component of an irreducible or reducible tensor of the group generated by the Lie algebra of the axial-vector coupling matrices X^α_μ, as will now be shown.

B. Group Properties of the Mass-Squared Matrix

It is possible to show, quite generally, that the mass-squared matrix M^2 behaves as the sum of a scalar and a component of the 35-dimensional irreducible tensor of the group $SO(4,3)$. The known tensorial character of the matrix in question provides us the straightforward method for writing down the mass spectrum of hadrons as a sum of the Clebsch-Gordan coefficients of the group $SO(4,3)$. To do this, use is made of a method developed by Weinberg[1] for the chiral group $SU(2)\otimes SU(2)$.

In order to proceed, an isovector–Lorentz-four-vector matrix Z^α_μ is defined by

$$[M^2, X^\alpha_\mu] \equiv -iZ^\alpha_\mu, \tag{3.21}$$

and Eq. (3.18) is rewritten as

$$[X^\alpha_\mu, Z^\beta_\nu] = i\delta^{\alpha\beta} g_{\mu\nu}\Lambda + i\epsilon^{\alpha\beta\gamma} A^\gamma_{\mu\nu}. \tag{3.22}$$

The matrices Λ and $A^\gamma_{\mu\nu}$ can now be expressed in terms of X^α_μ and Z^β_ν, and making use of the Jacobi identity the commutators $[X^\alpha_\mu, \Lambda]$ and $[X^\alpha_\mu, A^\gamma_{\rho\sigma}]$ can be written as follows (see the Appendix):

$$[X^\beta_\rho, \Lambda] = -iZ^\beta_\rho \tag{3.23}$$

and

$$[X^\alpha_\rho, A^\beta_{\mu\nu}] = i\epsilon^{\alpha\beta\gamma}(g_{\rho\mu}Z^\gamma_\nu - g_{\rho\nu}Z^\gamma_\mu) + i\delta^{\alpha\beta}R_{\rho\mu\nu}. \tag{3.24}$$

Here $R_{\rho\mu\nu}$ is defined as

$$R_{\rho\mu\nu} \equiv -\tfrac{1}{3}i[X^\alpha_\rho, A^\alpha_{\mu\nu}], \tag{3.25}$$

and it is a matrix which transforms like an isoscalar and totally antisymmetric Lorentz tensor of the third rank, obeying the following commutation relation:

$$[X^\alpha_\sigma, R_{\rho\mu\nu}] = -i(g_{\rho\sigma}A^\alpha_{\mu\nu} - g_{\sigma\mu}A^\alpha_{\rho\nu} + g_{\sigma\nu}A^\alpha_{\rho\mu}). \tag{3.26}$$

In the derivation of Eqs. (3.23)–(3.26) the tensorial properties of the matrices Λ, Z^α_μ, $A^\gamma_{\mu\nu}$, and $R_{\rho\mu\nu}$ have been exploited. All of them transform as proper irreducible tensors of the groups $SU(2)$ and $SL(2,C)$, and therefore obey the standard commutation relations

$$[I^\alpha, \Lambda] = [S_{\mu\nu}, \Lambda] = [I^\alpha, R_{\rho\mu\nu}] = 0, \tag{3.27}$$

$$[I^\alpha, A^\beta_{\mu\nu}] = i\epsilon^{\alpha\beta\gamma}A^\gamma_{\mu\nu}, \tag{3.28a}$$

$$[I^\alpha, Z^\beta_\mu] = i\epsilon^{\alpha\beta\gamma}Z^\gamma_\mu, \tag{3.28b}$$

$$[S_{\mu\nu}, Z^\alpha_\rho] = i(g_{\nu\rho}Z^\alpha_\mu - g_{\mu\rho}Z^\alpha_\nu), \tag{3.29a}$$

$$[S_{\mu\nu}, A^\alpha_{\rho\sigma}] = i(g_{\nu\rho}A^\alpha_{\mu\sigma} - g_{\nu\sigma}A^\alpha_{\mu\rho} - g_{\mu\rho}A^\alpha_{\nu\sigma} + g_{\mu\sigma}A^\alpha_{\nu\rho}), \tag{3.29b}$$

and

$$[S_{\mu\nu}, R_{\rho\sigma\kappa}] = i(g_{\nu\rho}R_{\mu\sigma\kappa} - g_{\nu\sigma}R_{\mu\rho\kappa} + g_{\nu\kappa}R_{\mu\rho\sigma} - g_{\mu\rho}R_{\nu\sigma\kappa} + g_{\mu\sigma}R_{\nu\rho\kappa} - g_{\mu\kappa}R_{\nu\rho\sigma}). \tag{3.29c}$$

Equations (3.22)–(3.29) show that the matrices Λ, Z^α_μ, $A^\alpha_{\mu\nu}$, and $R_{\rho\mu\nu}$ may form a $1+12+18+4=35$-dimensional tensor of the group $SO(4,3)$. The only 35-dimensional tensor of the group $SO(4,3)$ which contains an isoscalar–Lorentz-scalar Λ and an isoscalar–Lorentz-four-vector $V^\mu \equiv \epsilon^{\mu\nu\rho\sigma}R_{\nu\rho\sigma}$, is the totally antisymmetric third-rank tensor T_{abc}

$(a, b, c = 1, 2, 3, 0, 5, 6, 7)$, with

$$T_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}\Lambda, \quad T_{\mu\nu\rho} = R_{\mu\nu\rho}, \quad T_{\mu\alpha\nu} = A^{\alpha}_{\mu\nu}, \quad T_{\alpha\beta\mu} = \epsilon_{\alpha\beta\gamma}Z^{\gamma}_{\mu}. \tag{3.30}$$

Using the $SO(4,3)$ definitions of I^{α}, $S_{\mu\nu}$, and X^{α}_{μ} given in Eqs. (2.12) and (2.13), Eqs. (3.12)–(3.29) may be compactly written as follows:

$$[S_{ab}, T_{cde}] = i(g_{bc}T_{ade} - g_{bd}T_{ace} + g_{ae}T_{bcd} - g_{ac}T_{bde} + g_{ad}T_{bce} - g_{be}T_{acd}). \tag{3.31}$$

This is nothing but the statement that the 35 matrices T_{abc} defined by Eqs. (3.30) form a 35-dimensional totally antisymmetric irreducible tensor of the third rank of the group $SO(4,3)$.

It can now be shown, using Eqs. (3.21), (3.23), and (3.27), that the sum of matrices $M^2 + \Lambda$ commutes with all generators S_{ab}, i.e.,

$$[S_{ab}, M^2 + \Lambda] = 0. \tag{3.32}$$

This relation and Schur's lemma require that the sum

$$M^2 + \Lambda = m_0^{\,2} \tag{3.33}$$

must be equal to a multiple of a unit matrix for each irreducible representation of the group $SO(4,3)$. Thus equation (3.33) implies that the mass-squared matrix M^2 of the physical hadron states behaves as a sum of two terms, $m_0^{\,2} - \Lambda$, where $m_0^{\,2}$ is an invariant and Λ is a component of the 35-dimensional totally antisymmetric irreducible tensor T_{abc} of the group $SO(4,3)$.

It must be emphasized that Eq. (3.33) is not an approximation based on some assumption of weak chiral-symmetry breaking (as it is in the Gell-Mann–Oakes–Renner[21] model); rather it is an exact consequence of our assumptions about the commutators between axial charges and axial divergences as expressed by Eqs. (1.6) and (1.7). Therefore there is no reason to expect that the term Λ which breaks the chiral symmetry will be smaller than $m_0^{\,2}$.

Equation (3.33) is now used to determine the mass spectrum of hadrons. Each physical particle state of definite spin s and isospin I and third components s_z and I_z, respectively, can be written as a sum of unitary irreducible representations of the group $SO(4,3)$.

The unitary irreducible representations of the group $SO(4,3)$ are specified by three eigenvalues c_1, c_2, and c_3 of the corresponding three Casimir operators C_1, C_2, and C_3, respectively, defined as

$$C_1 = S_{ab}S^{ba}, \tag{3.34a}$$

$$C_2 = S_{ab}S^{bc}S_{cd}S^{da}, \tag{3.34b}$$

and

$$C_3 = W_a W^a, \tag{3.34c}$$

where

$$W^a = \epsilon^{abcdefg}S_{bc}S_{de}S_{fg}. \tag{3.34d}$$

The reduction-chain decomposition of $SO(4,3)$ which is of physical interest in describing the hadrons is

$$SO(4,3) \quad SU(2)_I \otimes SL(2,C). \tag{3.35}$$

This differs from the standard reduction chain, and therefore the unitary irreducible representations of the groups $SU(2)_I$ and $SL(2,C)$ may occur with multiplicity greater than 1 in the aforementioned decomposition. However, the following discussion is limited to the class of the so-called degenerate unitary irreducible representations, which are of multiplicity 1, of the group $SO(4,3)$. Then a state within a single unitary irreducible representation of $SO(4,3)$, specified by c_1, c_2, and c_3, will be labeled by six numbers I, I_z, s, s_z, j_0, and λ, where j_0 and λ are connected with eigenvalues of the two Casimir operators of the group $SL(2,C)$. This state will be denoted by

$$|c_1, c_2, c_3;\, I I_z, j_0 \lambda\, s\, s_z\rangle \equiv |f; f_z\rangle, \tag{3.36}$$

where f signifies the set of the values c_1, c_2, c_3, and f_z is an abbreviation for the six numbers I, I_z, j_0, λ, s, and s_z. A physical hadron state of definite spin s and isospin I can now be written as

$$|I I_z, s\, s_z\rangle = \sum_f \varphi(f)\,|f; f_z\rangle, \tag{3.37}$$

where $\varphi(f)$ are the so-called mixing angles. In the basis defined by Eq. (3.37) the matrix equation (3.33) becomes

$$M^2(I,S) = m_0^{\,2} - \sum_{ff'} \varphi(f)\varphi^*(f')\rho(f,f')\begin{pmatrix} f & t & f' \\ f_z & t_z & f'_z \end{pmatrix}, \tag{3.38}$$

where the symbol $(\vdots\ \vdots\ \vdots)$ stands for the Clebsch-Gordan coefficient of the group $SO(4,3)$; t signifies the tensorial character of the matrix Λ expressed in terms of c_1, c_2, and c_3; t_z denotes the set of values $I = I_z = s = s_z = j_0 = 0$, $\lambda = 1$; and $\rho(f,f')$ is a reduced matrix element. Thus the mass spectrum of hadrons (3.38) as a function of their spin s and isospin I is entirely determined by the Clebsch-Gordan coefficients of the group $SO(4,3)$.

The application of the results given by Eqs. (3.38) and (3.18) to physical particle states will be given in Sec. IV.

IV. APPLICATIONS

The purpose of this section is to apply the derived results to the mass spectrum of physical hadron states. A knowledge of the matrix elements of the generators and of the Clebsch-Gordan coefficients of the group $SO(4,3)$ [in the reduction chain 3.35)] would enable one to determine all pion-hadron transition processes and all hadron mass spectra. Unfortunately, neither matrix elements nor Clebsch-Gordan coefficients of the group in question have yet been studied in the literature. Explicit predictions can only be made about physical hadron states with the same third component of isospin and with the same parity. It can be shown that hadron states of this type transform as unitary representations of a group which is a subgroup of the $SO(4, 3)$ group, and therefore it is possible to write the explicit form of the mass spectrum for this restricted class.

The restriction of hadron states to those of fixed third component of isospin rules out all generators of the group $SO(4, 3)$ which can make a transformation between two states of different third component of isospin, i.e., matrices such as I^1, I^2, X^1_μ, and X^2_μ. If these are ruled out, the algebra (3.1)–(3.6) reduces to the form

$$[S_{\mu\nu}, S_{\rho\sigma}] = i(g_{\nu\rho}S_{\mu\sigma} - g_{\nu\sigma}S_{\mu\rho} - g_{\mu\rho}S_{\nu\sigma} + g_{\mu\sigma}S_{\nu\rho}), \tag{4.1}$$

$$[S_{\mu\nu}, X_\rho] = i(g_{\nu\rho}X_\mu - g_{\mu\rho}X_\nu), \tag{4.2}$$

and

$$[X_\mu, X_\nu] = -iS_{\mu\nu}, \tag{4.3}$$

where $X_\mu \equiv X^3_\mu$. The algebra given by Eqs. (4.1)–(4.3) is exactly the Lie algebra of the group $SO(3, 2)$, and so hadron states of fixed third component of isospin with both positive and negative parities must be assigned to unitary representations of the group $SO(3, 2)$. Note that this group was proposed as a dynamical group by Barut *et al.*[20] to calculate electromagnetic form factors and hadron mass spectra on a phenomenological basis. It is interesting to note that the same group occurs in the framework of Majorana's[22] and Bhabha's[23] infinite-component wave equations.

Next the group $SO(3, 2)$ is extended by the parity-operator matrix P which represents the reflection in the space 1, 2, 3. The commutation relations between the matrices $S_{\mu\nu}$, X_μ, and P are the following:

$$PS_{ik}P^{-1} = S_{ik}, \quad i, k = 1, 2, 3 \tag{4.4a}$$

$$PX_kP^{-1} = X_k, \tag{4.4b}$$

$$PS_{0k}P^{-1} = -S_{0k}, \tag{4.5a}$$

and

$$PX_0P^{-1} = -X_0. \tag{4.5b}$$

From the above equations it is evident that the matrices S_{0k} and X_0 change the parity of a state. If only hadron states of fixed parity (positive or negative) are considered these matrices must be ruled out. Thus, the algebraic relations (4.1)–(4.3) become

$$[S_i, S_j] = i\epsilon_{ijk}S_k, \tag{4.6a}$$

$$[S_i, X_j] = i\epsilon_{ijk}X_k, \tag{4.6b}$$

and

$$[X_i, X_j] = -i\epsilon_{ijk}S_k, \tag{4.6c}$$

where

$$S_i = \epsilon_{ijk}S_{jk}. \tag{4.6d}$$

The algebraic relations (4.6) are isomorphic to the Lie algebra of the homogeneous Lorentz group $SO(3, 1) \approx SL(2, C)$. The physical implication of these relations is that hadron states of fixed third component of isospin, hypercharge, and parity must be assigned to unitary representations of the "Lorentz group" $SO(3, 1)$, or to its covering group $SL(2, C)$. We have put quotation marks about "Lorentz group" generated by the Lie algebra (4.6), since that is a group in the 1, 2, 3, and 7 space and has no direct relation with the physical Lorentz group in the 0, 1, 2, 3 of the Minkowski space. To avoid any confusion, the group generated by the Lie algebra (4.6) is referred to as the iso-Lorentz group.

Now consider the constraints on the mass spectra of hadrons assigned to unitary representations of this iso-Lorentz group. The general equation (3.18) becomes, when restricted to the case under consideration,

$$[X_j, [M^2, X_k]] = -\delta_{jk}\Lambda, \tag{4.7}$$

which can be used to calculate a hadron mass spectrum.

In order to put this equation in group-theoretical terms, use is made of definition (3.21):

$$[M^2, X_k] \equiv -iZ_k, \tag{4.8}$$

where

$$Z_k^{\ 3} \equiv Z_k.$$

Equation (4.7) becomes

$$[X_j, Z_k] = -i\delta_{jk}\Lambda, \tag{4.9}$$

and use of Eq. (3.23) leads to the commutator

$$[X_k, \Lambda] = -iZ_k. \tag{4.10}$$

So the matrices Λ and Z_k form a four-matrix Γ_μ ($\Gamma_0 \equiv -\Lambda$, $\Gamma_k \equiv -Z_k$) which transforms exactly as a four-vector under the iso-Lorentz group transformations. This result leads us to the conclusion that the mass-squared matrix M^2, given by

$$M^2 = m_0^{\ 2} + \Gamma_0, \tag{4.11}$$

behaves as a sum of a scalar m_0^2 and the fourth component Γ_0 of a four-vector Γ_μ.

The masses of the hadrons, classified according to unitary representations of the iso-Lorentz group, can now be written as functions of their spins. This calculation will first be done by utilizing the final result given by Eq. (4.11). This approach exemplifies the group-theoretical method and is generally described in Sec. III B. Alternatively Eq. (4.7) can be used to derive a system of difference equations for the hadron mass spectrum. The latter calculation is the more direct one and is discussed in Sec. III A. Needless to say, both approaches are entirely equivalent.

A. Group-Theoretical Calculation

Unitary irreducible representations of the iso-Lorentz group generated by the Lie algebra (4.6) are specified by two numbers j_0 and λ which are given by eigenvalue equations of the following form:

$$(\vec{S}^2 - \vec{X}^2)|j_0, \lambda\rangle = (j_0^2 - \lambda^2 - 1)|j_0, \lambda\rangle \tag{4.12a}$$

and

$$\vec{S}\cdot\vec{X}|j_0, \lambda\rangle = j_0\lambda|j_0, \lambda\rangle. \tag{4.12b}$$

Here j_0 may be any integer or half-integer while λ is an arbitrary real number (the principal series), or $j_0 = 0$ and λ is a pure imaginary number fulfilling the restriction $|\lambda| \leq 1$ (the supplementary series).[13,24]

A state within a unitary irreducible representation is denoted by $|j_0, \lambda; SS_z\rangle$, where S and S_z denote the spin and its third component, respectively, a hadron state belonging to the representation characterized by j_0 and λ. The normalization of the states is given by

$$\langle j_0', \lambda'; S'S_z | j_0, \lambda; SS_z\rangle = \delta_{SS'}\delta_{S_z S_z'}\delta_{j_0 j_0'}\delta(\lambda - \lambda')(\lambda^2 + j_0^2)^{-1}. \tag{4.13}$$

The spin S and its third component S_z can have the following discrete values:

$$S = j_0, j_0 + 1, \ldots, \infty, \qquad -S \leq S_z \leq S. \tag{4.14}$$

This simply implies that the particle states accommodated in a single unitary irreducible representaton of the iso-Lorentz group form an infinite tower of spins, starting with the lowest spin value $S = j_0$ and going up in integral steps to infinity for each set of intrinsic quantum numbers such as third component of isospin, hypercharge, and parity.

Recall that the final result given by Eq. (4.11) specifies the mass-squared matrix M^2 as the sum of the scalar m_0^2 and the fourth component Γ_0 of the four-vector Γ_μ. It is a well-known fact that on the unitary representations of $SO(3,1) \approx SL(2,C)$ one can define a unique algebraic four-vector Γ_μ for only three different classes of the representations of the group in question.[20,24] Therefore the requirement that a nondegenerate mass spectrum of hadrons exists restricts the physically interesting representations of the iso-Lorentz group to the following types:

$$(1)\ j_0 = 0, \quad \lambda = i\tfrac{1}{2}, \tag{4.15}$$

which is suitable for accommodating mesons;

$$(2)\ j_0 = \tfrac{1}{2}, \quad \lambda = 0, \tag{4.16}$$

which can be used for fermions; and

$$(3)\ j_0 = \tfrac{1}{2}, \quad \lambda = \text{arbitrary real number}, \tag{4.17a}$$

where the states within this class of the representations must be defined as

$$|\lambda; SS_z \pm\rangle = \tfrac{1}{2}\sqrt{2}\,(|\tfrac{1}{2}, \lambda; SS_z\rangle \pm |\tfrac{1}{2}, -\lambda; SS_z\rangle) \tag{4.17b}$$

and are obviously suitable for fermions. The representations of the first and second class (4.15) and (4.16), respectively, are irreducible, while the representations belonging to the third class are reducible.

In order to determine the mass spectrum of hadrons it is sufficient to know the explicit form of the matrix Γ_0 and this is given in the literature[20,24,25] as

$$\Gamma_0 | j_0\lambda; SS_z\rangle = (S + \tfrac{1}{2})\rho(j_0, \lambda)|j_0, \lambda; SS_z\rangle \tag{4.18}$$

for the unitary irreducible representations of the classes described by (4.15) and (4.16). The symbol $\rho(j_0, \lambda)$ stands for a reduced matrix element. The form of the matrix Γ_0 for the unitary reducible representations of the third class (4.17) is given by[20,24,25]

$$\Gamma_0|\lambda; SS_z \pm\rangle = \pm(S + \tfrac{1}{2})\rho(\lambda)|\lambda; SS_z \pm\rangle, \tag{4.19}$$

where $\rho(\lambda)$ is again the reduced matrix element. Equations (4.11) and (4.18) imply that hadron states accommodated in the unitary irreducible representations (4.15) and (4.16) of the iso-Lorentz group must have a mass spectrum linear in spin, namely

$$M^2(S) = m_0^2 + (S + \tfrac{1}{2})\rho(j_0, \lambda). \tag{4.20}$$

However, as has been mentioned above, the hadron states of fixed third component of isospin, hypercharge, and parity, but with various spins, may also be accommodated in unitary reducible representations of the iso-Lorentz group. These hadron states can be formed out of the representations belonging to the class (4.17). So, a physical fermion particle state $|S, S_z\rangle$ of definite spin S and third component S_z can be written as a sum of the representations (4.17),

$$|SS_z\rangle = \int_{-\infty}^{\infty} d\lambda\, \alpha(\lambda)\,|\lambda; SS_z+\rangle + \int_{-\infty}^{\infty} d\lambda\, \beta(\lambda)\,|\lambda; SS_z-\rangle, \tag{4.21}$$

where $\alpha(\lambda)$ and $\beta(\lambda)$ are mixing angles. The normalization of the states $\langle S'S'_z|SS_z\rangle = \delta_{SS'}\delta_{S_zS_z'}$ requires the restriction

$$\int_{-\infty}^{\infty} (4\lambda^2+1)^{-1}[|\alpha(\lambda)|^2 + |\beta(\lambda)|^2]d\lambda = \tfrac{1}{4} \tag{4.22}$$

on the mixing angles $\alpha(\lambda)$ and $\beta(\lambda)$. Making use of Eqs. (4.11), (4.19), and (4.21) the mass-squared matrix M^2 for the hadron states (4.21) can be written as follows:

$$M^2(S) = m_0{}^2 + k(S+\tfrac{1}{2}), \tag{4.23}$$

where k is a constant defined by

$$k = 4\int_{-\infty}^{\infty} (4\lambda^2+1)^{-1}[|\alpha(\lambda)|^2 - |\beta(\lambda)|^2]\rho(\lambda)d\lambda. \tag{4.24}$$

This result implies that the mass squared of hadrons of fixed intrinsic quantum numbers must be a linear function of spin. This form of mass spectrum has been phenomenologically assumed in the recent Regge phenomenology associated with the celebrated Veneziano model.[26] It has been shown here that this result is an exact consequence of the derived algebraic relations. This same result is now obtained using the "algebraic approach."

B. Algebraic Approach

Consider the matrix relation (4.7). It can be written down in a spherical basis as follows:

$$[X_a, [M^2, X_b]] = \sqrt{3}\begin{pmatrix}1 & 1 & 0\\ a & b & 0\end{pmatrix}\Lambda, \tag{4.25}$$

where X_a are the generator matrices of the iso-Lorentz group (4.6) written in the spherical basis, $a, b = +, -, 0$, and the symbol $\left(\begin{smallmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{smallmatrix}\right)$ stands for the Clebsch-Gordan coefficient of the group $SU(2)$. Multiplying Eq. (4.25) by the Clebsch-Gordan coefficient

$$\begin{pmatrix}1 & 1 & 2\\ a & b & c\end{pmatrix}$$

and summing over the range of the a and b indices gives

$$\sum_{ab}\begin{pmatrix}1 & 1 & 2\\ a & b & c\end{pmatrix}[X_a, [M^2, X_b]] = 0. \tag{4.26}$$

The matrix $G^s_{s'}$ is now defined as follows:

$$\langle s's'_z|X_a|ss_z\rangle = G^s_{s'}\begin{pmatrix}s & 1 & s'\\ s_z & a & s_z\end{pmatrix}. \tag{4.27}$$

The unitarity of the representations requires that

$$G^s_{s'} = (-1)^{s-s'}\left(\frac{2s+1}{2s'+1}\right)^{1/2}\bar{G}^{s'}_s, \tag{4.28}$$

where $\bar{G}$ means complex conjugate of G. The matrices $G^s_{s'}$ for the unitary irreducible representations of the Lorentz group can be found in the literature.[24]

The mass spectrum condition (4.26) can now be written down in terms of the hadron masses $M^2(s)$ and matrices $G^s_{s'}$. By taking the matrix element in the s'th row and sth column of the matrix (4.26) one obtains, using the orthogonality properties of the Clebsch-Gordan coefficients and their relations to the 6-j symbols[27] of the group $SU(2)$, the equation

$$\sum \bar{G}^{s'}_J G^s_J[2M^2(J) - M^2(s) - M^2(s')](2J+1)(-1)^J\begin{Bmatrix}1 & 1 & 2\\ s & s' & J\end{Bmatrix} = 0, \tag{4.29}$$

where $\left\{\begin{smallmatrix}a & b & c\\ d & e & f\end{smallmatrix}\right\}$ is the 6-j symbol of the group $SU(2)$. Consider the transition $s \to s-1$ and $s' \to s+1$ in Eq. (4.29). The summation range over J is then reduced to only one term with $J = s$ and (4.29) becomes

$$\bar{G}^{s+1}_s G^{s-1}_s[2M^2(s) - M^2(s+1) - M^2(s-1)] = 0. \tag{4.30}$$

The most general solution to the above difference equation is

$$M^2(s) = m_0{}^2 + k(s+\tfrac{1}{2}), \tag{4.31}$$

where $m_0{}^2$ and k are integration constants.

This last equation implies that the mass squared $M^2(s)$ is a linear function of spin irrespective of the representations of the group under consideration. The same result was obtained in the previous subsection on the basis of group theory. It can also be verified easily that the application of Eq. (4.29) for the remaining set of values s' and s places a strong restriction on the representations for which mass splitting is allowed. In fact, inserting the known expressions for the $G^s_{s'}$ matrices,[24] along with the permissible mass spectrum (4.31), in Eq. (4.29) gives a mass spectrum of hadrons which is not degenerate if and only if hadron states are associated with the unitary representations of the iso-Lorentz group defined by (4.15)–(4.17) and by (4.21). It is also interesting to note that only these representations allow the hadron states to couple to the electromagnetic field, as was pointed out by Barut and Kleinert in Ref. 18.

V. CONCLUDING REMARKS

Recently Capps has derived certain consistency conditions, which the hadron-hadron coupling matrices must satisfy, by saturating forward dispersion-relation sum rules with single-particle states,[28] or by utilizing the Veneziano formula.[29] The form of these consistency conditions leads to the conclusion that hadron states must be regarded as unitary representations of certain Lie groups. However, this scheme does not determine the mass spectrum of the hadrons.

Another algebraic approach to meson-hadron dynamics is that of Sugawara.[30] In this scheme consistency conditions relating the meson-hadron coupling matrices and the hadron masses are derived by saturating unsubtracted dispersion relations of meson-hadron form factors with one-particle states. However, a unique solution to these conditions is hard to find.

More recently, the most attractive scheme for determining pion-hadron transition amplitudes and hadron mass spectra has been that of Weinberg.[1] An attempt has been made here to further this work. It has been shown, under fairly general assumptions, that the dependence of the axial-vector coupling matrices is entirely determined by the Lie algebra of the group $SO(4,3)$ and, furthermore, that the mass-squared value of hadrons of fixed intrinsic quantum numbers must be a linear function of spin.

Other models[20,31] do lead to linear spin dependence for the hadron mass spectra. However, these use somewhat broader assumptions than have been used here.

It hardly needs to be emphasized that the group-theoretical properties of the physical observables encountered here have not been hypothesized – either on the basis of a quark model, or by a free act of intuition as is common in the consideration of dynamical groups – but have been derived from the usually accepted assumptions about the equal-time commutator algebra of axial charges and axial divergences.

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APPENDIX

The purpose of this appendix is to prove the tensorial character of the mass-squared matrix M^2. Consider the matrix relation (3.22), which is of the form

$$[X_\mu^\alpha, Z_\nu^\beta] = i\delta^{\alpha\beta} g_{\mu\nu}\Lambda + i\epsilon^{\alpha\beta\gamma} A_{\mu\nu}^\gamma . \tag{A1}$$

The matrices Λ and $A_{\mu\nu}^\gamma$ are now rewritten in terms of X_μ^α and Z_ν^β, and the following results are obtained:

$$\Lambda = -\tfrac{1}{12} i g^{\mu\nu}[X_\mu^\alpha, Z_\nu^\alpha] \tag{A2}$$

and

$$A_{\mu\nu}^\gamma = -\tfrac{1}{2} i\epsilon^{\alpha\beta\gamma}[X_\mu^\alpha, Z_\nu^\beta]. \tag{A3}$$

The commutators $[X_\rho^\beta, A_{\mu\nu}^\gamma]$ can then be written down by making use of the Jacobi identity as

$$[X_\rho^\beta, \Lambda] = \tfrac{1}{12} i g^{\mu\nu}\{[Z_\nu^\alpha, [X_\rho^\beta, X_\mu^\alpha]] + [X_\mu^\alpha, [Z_\nu^\alpha, X_\rho^\beta]]\} \tag{A4}$$

and

$$[X_\rho^\beta, A_{\mu\nu}^\gamma] = \tfrac{1}{2} i\epsilon^{\alpha\delta\gamma}\{[Z_\nu^\delta, [X_\rho^\beta, X_\mu^\alpha]] + [X_\mu^\alpha, [Z_\nu^\delta, X_\rho^\beta]]\}. \tag{A5}$$

Carrying out the algebraic reduction using the commutation relations (3.1)–(3.6), the following intermediate results are obtained:

$$11[X_\rho^\beta, \Lambda] = -5iZ_\rho^\beta + \epsilon^{\beta\alpha\gamma} g^{\mu\nu}[X_\mu^\alpha, A_{\rho\nu}^\gamma] \tag{A6}$$

and

$$2[X_\rho^\beta, A_{\mu\nu}^\gamma] = i\epsilon^{\beta\gamma\alpha}(g_{\rho\mu} Z_\nu^\alpha - g_{\rho\nu} Z_\mu^\alpha + g_{\mu\nu} Z_\rho^\alpha - g_{\nu\rho} Z_\mu^\alpha) + [X_\mu^\beta, A_{\rho\nu}^\gamma] - \delta^{\gamma\beta}[X_\mu^\alpha, A_{\rho\nu}^\alpha]. \tag{A7}$$

The above equation (A7) is then used to determine the second term on the right-hand side of Eq. (A6). This yields the result

$$\epsilon^{\beta\alpha\gamma} g^{\mu\nu}[X_\mu^\alpha, A_{\rho\nu}^\gamma] = -2iZ_\rho^\beta + 4[X_\rho^\beta, \Lambda], \tag{A8}$$

which, when inserted into Eq. (A6), gives the final form for the commutator in question, namely

$$[X_\rho^\beta, \Lambda] = -iZ_\rho^\beta. \tag{A9}$$

Summing over β and γ indices in Eq. (A7), the following relation is obtained:

$$[X_\rho^\alpha, A_{\mu\nu}^\alpha] = -[X_\mu^\alpha, A_{\rho\nu}^\alpha], \tag{A10}$$

which implies that a matrix $R_{\rho\mu\nu}$ defined by

$$R_{\rho\mu\nu} \equiv -\tfrac{1}{3} i[X_\rho^\alpha, A_{\mu\nu}^\alpha] \tag{A11}$$

behaves as an isoscalar, totally antisymmetric Lorentz tensor of the third rank. Making use of the definition (A11) along with the relation (A7), the commutator $[X_\mu^\beta, A_{\rho\nu}^\gamma]$ is calculated to give

$$2[X_\mu^\beta, A_{\rho\nu}^\gamma] = i\epsilon^{\beta\gamma\alpha}(g_{\mu\rho} Z_\nu^\alpha - g_{\mu\nu} Z_\rho^\alpha + g_{\rho\nu} Z_\mu^\alpha - g_{\nu\mu} Z_\rho^\alpha) + [X_\rho^\beta, A_{\mu\nu}^\gamma] - 3i\delta^{\beta\gamma} R_{\rho\mu\nu}. \tag{A12}$$

Inserting this into Eq. (A7) yields the result

$$[X_\rho^\alpha, A_{\mu\nu}^\beta] = i\epsilon^{\alpha\beta\gamma}(g_{\rho\mu}Z_\nu^\gamma - g_{\rho\nu}Z_\mu^\gamma) + i\delta^{\alpha\beta}R_{\rho\mu\nu}, \tag{A13}$$

which has been used in Sec. III B. The value of the commutator

$$[X_\sigma^\beta, R_{\rho\mu\nu}] \equiv -\tfrac{1}{3}i[X_\sigma^\beta, [X_\rho^\alpha, A_{\mu\nu}^\alpha]] \tag{A14}$$

can now be evaluated by using the Jacobi identity for the double commutator and Eqs. (3.1)–(3.6) and (3.29b), and result is

$$[X_\sigma^\beta, R_{\rho\mu\nu}] = -i(g_{\sigma\rho}A_{\mu\nu}^\beta - g_{\sigma\mu}A_{\rho\nu}^\beta + g_{\sigma\nu}A_{\rho\mu}^\beta). \tag{A15}$$

The commutators (A1), (A9), (A13), and (A15) have been used in Sec. III B to prove that the 35 matrices Λ, $Z_\mu^\alpha, A_{\mu\nu}^\alpha$, and $R_{\rho\mu\nu}$ transform as components of the 35-dimensional totally antisymmetric tensor of the third rank under the group $SO(4,3)$ transformations.

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