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[12]This was suggested by Professor Mandelstam.

[13]We can solve for X in terms of t (in the frame $y_{\alpha+1} = x_1 = \infty$, $x_2 = 0$, $y_{\alpha-1} = 1$); from Eqs. (2.34a)-(2.34c), we get

$$X = \frac{1}{2(1-t)}\{1 \pm [1 - 4(1-t)ty_{\alpha-1}]^{1/2}\}.$$

The minus branch gives $0 \le X < 1$, whereas the plus branch gives $1 \le X \le \infty$.

[14]This was demonstrated by Professor Mandelstam.

[15]This factor also comes out naturally, if we neglect the spurious problem. In this case,

$$X = tP_\alpha(A+2) = tP(\alpha, \alpha+1, \alpha-1, \alpha+2)$$
$$= t\frac{(y_\alpha - y_{\alpha-1})(y_{\alpha+1} - y_{\alpha+2})}{(y_\alpha - y_{\alpha+2})(y_{\alpha+1} - y_{\alpha-1})},$$

and $x_1 = y_{\alpha+1}$, $x_2 = y_\alpha$; hence from the factor $(1-t)^{\alpha_0 - 1}$ one gets this factor.

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PHYSICAL REVIEW D VOLUME 3, NUMBER 12 15 JUNE 1971

Unitarization of the Dual-Resonance Amplitude. II. The Nonplanar N-Loop Amplitude*

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Following our previous paper on the planar N-loop Veneziano amplitude, we derive the nonplanar N-loop formula in this paper. The calculation is performed by tracing over both the multiply factorized tree and the Sciuto three-Reggeon vertex functions.

I. INTRODUCTION

This paper is the second of three articles devoted to calculating all multiloop amplitudes in the dual-resonance model. In the first paper,[1] we presented the planar N-loop amplitude; we discussed at length the principal-axes method, the infinite-cancellation technique, the Kikkawa-Sakita-Virasoro interpretation, the Jacobian calculation, and the range of integration. Because the planar and nonplanar loop calculations are similar, we present the nonplanar amplitudes in this paper without many of these details. In the third paper, we will present rules for writing down arbitrary planar, nonplanar, overlapping, and nonorientable loop amplitudes.[2]

The nonplanar amplitude differs from the planar one in three major ways:

(a) The linear-dependence correction is $(1-X)^2$ for each loop, not $(1-X)$, where X is the multiplier of each projective transformation.

(b) The factors raised to the $\frac{1}{2}m^2 - 1$ power differ slightly, to reflect the different quark topology [see Eqs. (2.26) and (3.13) below].

(c) There are variables of integration which lie between the invariant points of each projective transformation.

II. MULTIPLE-FACTORIZATION FORMULATION OF NONPLANAR MULTIPLE LOOPS

As in the previous paper,[1] we first consider the nonplanar single-loop[3] amplitude, expressed in a general projective frame, and then apply the method with modification to the nonplanar multiloop diagrams.

A. Nonplanar Single-Loop Amplitude

We first write down[4] the following doubly factorized tree formula for the amplitude corresponding to Fig. 1:

$$G^{(2)}_{(Y)}(a^\alpha, a^\beta) = \int \prod_i dy_i \{Y_{S+2}\} \exp\left[\sum_{\substack{i=0\\(i\neq\alpha)}}^{S+1} (a^\alpha | P_\alpha(i) | k_i) + \sum_{\substack{i=0\\(i\neq\beta)}}^{S+1} (a^\beta | P_\beta(i) | k_i) + (a^\alpha | P_\alpha(\beta) M_- P(\alpha+1, \beta-1, \alpha, \beta) M_-^T P_\beta(\alpha) | a^\beta) \right], \tag{2.1}$$

where

$$P_\alpha(i) = P(\alpha, \alpha+1, \alpha-1, i) \equiv \frac{(y_\alpha - y_{\alpha-1})(y_{\alpha+1} - y_i)}{(y_{\alpha+1} - y_{\alpha-1})(y_\alpha - y_i)}, \tag{2.2a}$$

$$P_\beta(i) = P(\beta, \beta-1, \beta+1, i). \tag{2.2b}$$

Applying the sewing prescriptions[1] on the excited a^α, a^β legs and using the principal-axes technique,[1] we obtain, from Eq. (2.1), the nonplanar single-loop amplitude (Fig. 2); call it $F_{\rm nl}(1)$:

$$F_{\rm nl}(1) = \int d^4k_\alpha \int_0^1 dt\, t^{-l(k\alpha)-1}(1-t)^{\alpha_0 - 1 + \frac{1}{2}k_\alpha^2} \times \int \prod_i dy_i \{Y_{S+2}\} I, \tag{2.3}$$

where

$$I = \frac{1}{(\det[\Delta])^{1/2}} \exp\left[\tfrac{1}{2} \sum_{n=0}^{\infty} ((E|, (F|)[GH]^n \begin{pmatrix} |F) \\ |E) \end{pmatrix} \right] \tag{2.4}$$

and

$$[\Delta] = \begin{pmatrix} 0 & [I] - [\bar{C}]^T \\ [I] - [\bar{C}] & 0 \end{pmatrix}, \tag{2.5}$$

$$[GH] = \begin{pmatrix} [\bar{C}] & 0 \\ 0 & [\bar{C}]^T \end{pmatrix}, \tag{2.6}$$

with

$$[\bar{C}] = M_-^T\left(\frac{t}{t-1}\right) P_\alpha(\beta) M_- P(\alpha+1, \beta-1, \alpha, \beta) M_-^T P_\beta(\alpha), \tag{2.7a}$$

$$|F) = \sum_{\substack{i=0\\(i\neq\alpha,\beta)}}^{S+1} M_-^T\left(\frac{t}{t-1}\right) P_\alpha \begin{bmatrix} t \\ \beta \\ \alpha+1 \end{bmatrix} \begin{vmatrix} k_i \\ k_\beta \\ k_\alpha \end{pmatrix}, \tag{2.7b}$$

$$|E) = \sum_{\substack{i=0\\(i\neq\alpha,\beta)}}^{S+1} P_\beta \begin{bmatrix} i \\ \alpha \\ \beta+1 \end{bmatrix} \begin{vmatrix} k_i \\ k_\alpha \\ k_\beta \end{pmatrix}. \tag{2.7c}$$

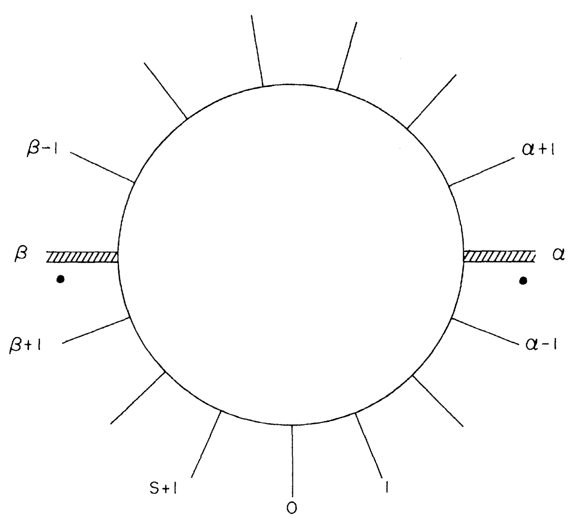

FIG. 1. Doubly factorized tree diagram (nonplanar).

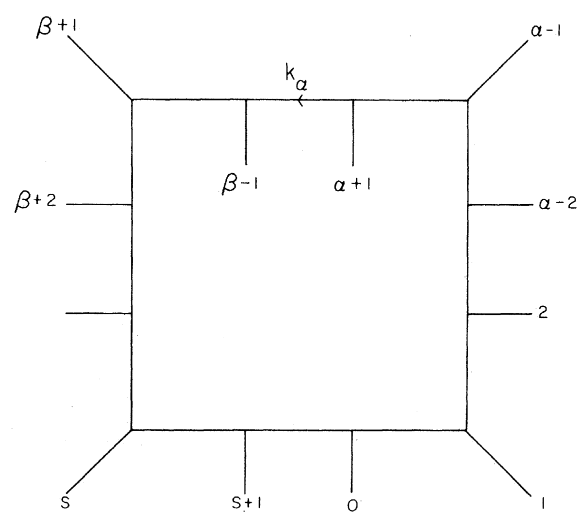

FIG. 2. Nonplanar single-loop diagram.

We then calculate Eq. (2.4), order by order in the $[GH]$ matrix, by defining the projective operator

$$Q(x)=\left[\left(\frac{t}{t-1}\right)\left(1-\frac{1}{x}\right)\right]^{-1},\qquad Q^{-1}(x)=\frac{1}{1-(t-1)/tx} \tag{2.8}$$

and the projective operator corresponding to encircling the loop

$$R_{\beta\alpha}\equiv R_{\beta\alpha}{}^{\dagger}\equiv P_\alpha^{-1}Q\hat{P}_\beta,\quad R_{\beta\alpha}^{-1}\equiv\hat{P}_\beta^{-1}Q^{-1}P_\alpha, \tag{2.9}$$

where

$$\hat{P}_\beta(x)=\frac{1}{P_\beta(x)}=P(\beta-1,\beta,\beta+1,x), \tag{2.10a}$$

$$\hat{P}_\beta^{-1}(x)=y_{\beta-1}-\frac{y_{\beta-1}-y_\beta}{1-x(y_\beta-y_{\beta+1})/(y_{\beta-1}-y_{\beta+1})}, \tag{2.10b}$$

$$\hat{P}_\beta^{-1}(x)=P_\beta^{-1}(1/x). \tag{2.10c}$$

From Eq. (2.9), we have two identities,

$$R_{\beta\alpha}^{-1}(y_\alpha)=y_{\beta+1}, \tag{2.11a}$$

$$R_{\beta\alpha}(y_\beta)=y_{\alpha+1}. \tag{2.11b}$$

These two identities, Eqs. (2.11a) and (2.11b), enable us to get the "invariant points" of $R_{\beta\alpha}$.[1]

We find, after tedious calculation, the expression for I:

$$I\equiv\frac{1}{(\det[\Delta])^{1/2}}\prod_{n=0}^{\infty}\prod_{\substack{i,j=0\\(i,j\neq\alpha,\beta)}}^{S+1}[y_i-R_{\beta\alpha}^{\pm(n+1)}(y_j)]^{-\frac{1}{2}k_i\cdot k_j}\prod_{\substack{i=0\\(i\neq\alpha,\beta)}}^{S+1}\left(\frac{y_i-x_2}{y_i-x_1}\right)^{-k_i\cdot k_\alpha}\left(\frac{y_i-y_\beta}{y_i-y_\alpha}\right)^{-k_i\cdot k_\alpha}\times\left[\frac{(y_\alpha-y_\beta)}{(y_\alpha-x_1)}\frac{R_{\beta\alpha}^{-1}(y_\alpha)-x_1}{R_{\beta\alpha}^{-1}(y_\alpha)-y_\beta}\right]^{-k_\alpha\cdot k_\alpha}. \tag{2.12}$$

We also separate out, in the factor $\{Y_{S+2}\}$ of Eq. (2.3), all factors containing y_α, y_β, and combine them with Eq. (2.12); we get, finally,

$$\begin{aligned}\{Y_{S+2}\}I=&\frac{1}{(\det[\Delta])^{1/2}}\prod_{n=0}^{\infty}\prod_{\substack{i,j=0\\(i,j\neq\alpha,\beta)\\(n=0,i\neq j)}}^{S+1}[y_i-R_{\beta\alpha}^{\pm(n)}(y_j)]^{-\frac{1}{2}k_i\cdot k_j}\prod_{\substack{i=0\\(i\neq\alpha,\beta)}}^{S+1}\left\{\frac{y_i-x_2}{y_i-x_1}\right\}^{-k_i\cdot k_\alpha}\prod_{\substack{i=0\\(i\neq\alpha,\alpha-1,\beta,\beta-1)}}^{S+1}(y_i-y_{i+1})^{\alpha_0-1}\\&\times\left\{\left[\frac{R_{\beta\alpha}^{-1}(y_\alpha)-x_1}{(y_\alpha-x_1)[R_{\beta\alpha}^{-1}(y_\alpha)-y_\beta]}\right]^{-k_\alpha^2}\left[\frac{(y_{\alpha-1}-y_\alpha)(y_\alpha-y_{\alpha+1})(y_{\beta-1}-y_\beta)(y_\beta-y_{\beta+1})}{(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})}\right]^{-\frac{1}{2}k_\alpha^2-1}\right.\\&\left.\times[(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})]^{\alpha_0-1}(y_a-y_b)(y_b-y_c)(y_c-y_a)\right\}.\end{aligned} \tag{2.13}$$

We now express our final answer in a projectively invariant form by transforming the set of variables $(t, y_\alpha\ y_\beta)$ into the new set of variables (X, x_1, x_2). We first extract out all factors containing t, y_α, y_β in Eq. (2.3). From Eqs. (2.3) and (2.13), they are

$$\begin{aligned}&dt\,dy_\alpha dy_\beta t^{-l(k_\alpha)-1}(1-t)^{\alpha_0-1+\frac{1}{2}k_\alpha^2}\left\{\left[\frac{R_{\beta\alpha}^{-1}(y_\alpha)-x_1}{(y_\alpha-x_1)[R_{\beta\alpha}^{-1}(y_\alpha)-y_\beta]}\right]^{-k_\alpha^2}\right.\\&\left.\times\left[\frac{(y_{\alpha-1}-y_\alpha)(y_\alpha-y_{\alpha+1})(y_{\beta-1}-y_\beta)(y_\beta-y_{\beta+1})}{(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})}\right]^{-\frac{1}{2}k_\alpha^2-1}[(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})]^{\alpha_0-1}(y_a-y_b)(y_b-y_c)(y_c-y_a)\right\}.\end{aligned} \tag{2.14}$$

The Jacobian calculation is quite involved, and details can be found in the Appendix. We merely quote the result here. In the frame $x_1=\infty,\ x_2=0$, the expression (2.14) is equal to

$$dX[dx_1][dx_2](1-X)^2X^{-l(k_\alpha)-1}[(y_{\alpha-1}-Xy_{\beta+1})(y_{\alpha+1}-Xy_{\beta-1})]^{\alpha_0-1}. \tag{2.15}$$

The unique projective generalization of the expression (2.15) is exactly similar to that found in the previous paper[1]; it is

$$dX\,dx_1dx_2X^{-l(k_\alpha)-1}(1-X)^2\frac{(y_a-y_b)(y_b-y_c)(y_c-y_a)}{(x_1-x_2)^2}\left\{\frac{[y_{\alpha-1}-R_{\beta\alpha}(y_{\beta+1})](x_1-y_{\beta+1})}{x_1-R_{\beta\alpha}(y_{\beta+1})}\right\}^{\alpha_0-1}$$
$$\times\left\{\frac{[y_{\alpha+1}-R_{\beta\alpha}(y_{\beta-1})](x_1-y_{\beta-1})}{x_1-R_{\beta\alpha}(y_{\beta-1})}\right\}^{\alpha_0-1}. \tag{2.16}$$

Now we are ready to write down the nonplanar single-loop formula. By combining Eqs. (2.16) and (2.13) with (2.3), we obtain the final form:

$$F_{\rm nl}(1)=\int d^4k_\alpha\int dX\,X^{-l(k_\alpha)-1}(1-X)^2\int\prod_{\substack{i=0\\(i\neq\alpha,\beta,a,b,c)}}^{S+1}dy_i\,dx_1dx_2[dy_a][dy_b][dy_c]\frac{(y_a-y_b)(y_b-y_c)(y_c-y_a)}{(x_1-x_2)^2}$$
$$\times\prod_{\substack{i=0\\(i\neq\alpha,\alpha-1,\beta,\beta-1)}}^{S+1}(y_i-y_{i+1})^{\alpha_0-1}\left\{\frac{[y_{\alpha-1}-R_{\beta\alpha}(y_{\beta+1})](x_1-y_{\beta+1})}{x_1-R_{\beta\alpha}(y_{\beta+1})}\right\}^{\alpha_0-1}\left\{\frac{[y_{\alpha+1}-R_{\beta\alpha}(y_{\beta-1})](x_1-y_{\beta-1})}{x_1-R_{\beta\alpha}(y_{\beta-1})}\right\}^{\alpha_0-1}$$
$$\times\prod_{n=1}^{\infty}\frac{1}{(1-X^n)^4}\prod_{\substack{i=0\\(i\neq\alpha,\beta)}}^{S+1}\left(\frac{y_i-x_2}{y_i-x_1}\right)^{-k_i\cdot k_\alpha}\prod_{\substack{i,j=0\\(i,j\neq\alpha,\beta)}}^{S+1}\prod_{\substack{n=0\\(n=0,i\neq j)}}^{\infty}[y_i-R_{\beta\alpha}^{\pm(n)}(y_j)]^{-\frac{1}{2}k_i\cdot k_j}, \tag{2.17}$$

where

$$(\det[\Delta])^{-1/2}=\prod_{n=1}^{\infty}(1-X^n)^{-4}. \tag{2.18}$$

The ordering of y_i ($i=0,1,\ldots,S+1$, $i\neq\alpha,\beta$) and x_1, x_2 will now be discussed. The variables of the multiply factorized tree, before sewing, had the ordering

$$y_0\geq y_1\geq\cdots\geq y_{\alpha-1}\geq y_\alpha\geq y_{\alpha+1}\geq\cdots\geq y_{\beta-1}\geq y_\beta\geq y_{\beta+1}\geq\cdots\geq y_{S+1}.$$

It is sufficient to specialize to the frame $x_1=\infty$, $x_2=0$, and consider the case $0\leq X<1$. After sewing, Eq. (2.11) gives the relations

$$y_\alpha=Xy_{\beta+1}<y_{\beta+1}, \tag{2.19a}$$
$$y_\beta=X^{-1}y_{\alpha+1}>y_{\alpha+1}. \tag{2.19b}$$

These two relations imply two inequalities similar to Eq. (2.43) of paper I:

$$X^{-1}y_{\alpha-1}>y_{\beta+1}\geq y_{\beta+2}\geq\ldots\geq y_{S+1}\geq y_0\geq\ldots y_{\alpha-1}>Xy_{\beta+1}, \tag{2.20a}$$
$$X^{-1}y_{\alpha+1}>y_{\beta-1}\geq y_{\beta-2}\geq\ldots\geq y_{\alpha+1}>Xy_{\beta-1}. \tag{2.20b}$$

Equations (2.20a) and (2.20b) force us to put x_1 between $y_{\beta+1}$ and $y_{\beta-1}$, and to put x_2 between $y_{\alpha-1}$ and $y_{\alpha+1}$. Therefore we conclude that the ordering is

$$y_0\geq y_1\geq\cdots\geq y_{\alpha-1}\geq x_2\geq y_{\alpha+1}\geq\cdots\geq y_{\beta-1}\geq x_1\geq y_{\beta+1}\geq\cdots\geq y_{S+1}. \tag{2.20c}$$

One observes that the nonplanar single-loop formula, Eq. (2.17), is essentially the product of two planar single-loop formulas, one with external legs outside the loop, and the other with external legs inside the loop. The interpretation of various factors is exactly parallel to the interpretations discussed in paper I; we will not repeat them here.

We see that the nonplanar single-loop formula, Eq. (2.17), is hardly different from the planar single-loop formula in Ref. 1, and as we will see further, the nonplanar N-loop formula again is very similar to the nonplanar single-loop formula.

B. The Nonplanar N-Loop Amplitude

In this subsection, we apply the techniques of the previous subsection to the nonplanar multiply factorized tree diagram (Fig. 3). Each loop is labeled by two indices, e.g., the ($\alpha\beta$) loop is obtained by sewing the excited α leg with the β leg. We adopt the convention that the first index (e.g., α) of each loop [e.g., the ($\alpha\beta$) loop] corresponds to the complex parameter (λ_α^*|.

We now write down[4] the $2N$th factorized tree amplitude corresponding to Fig. 3:

$$G_{(Y)}^{(2N)}(a^\alpha, a^\beta; a^\gamma, a^\delta; \ldots; a^\sigma, a^\lambda) = \int \prod_i dy_i \{Y_{S+2}\} \exp \Bigg\{ \sum_{\alpha\in\{\mathcal{L}\}} \sum_{\substack{i=0\\(i\neq\alpha)}}^{S+1} (a^\alpha | P_\alpha(i) | k_i) + \sum_{\beta\in\{\mathcal{L}\}} \sum_{\substack{i=0\\(i\neq\beta)}}^{S+1} (a^\beta | P_\beta(i) | k_i)$$

$$+ \frac{1}{2} \sum_{\substack{\alpha,\,\gamma\in\{\mathcal{L}\}\\(\alpha\neq\gamma)}} (a^\alpha | P_\alpha(\gamma) M_- P(\alpha+1, \gamma+1, \alpha, \gamma) M_-{}^T P_\gamma(\alpha) | a^\gamma)$$

$$+ \frac{1}{2} \sum_{\substack{\beta,\,\delta\in\{\mathcal{L}\}\\(\beta\neq\delta)}} (a^\beta | P_\beta(\delta) M_- P(\beta-1, \delta-1, \beta, \delta) M_-{}^T P_\delta(\beta) | a^\delta)$$

$$+ \sum_{\alpha,\,\delta\in\{\mathcal{L}\}} (a^\alpha | P_\alpha(\delta) M_- P(\alpha+1, \delta-1, \alpha, \delta) M_-{}^T P_\delta(\alpha) | a^\delta) \Bigg\}, \tag{2.21}$$

where

$$P_\alpha(i) = P(\alpha, \alpha+1, \alpha-1, i), \tag{2.22a}$$

$$P_\beta(i) = P(\beta, \beta-1, \beta+1, i), \tag{2.22b}$$

$$P_\gamma(i) = P(\gamma, \gamma+1, \gamma-1, i), \tag{2.22c}$$

$$P_\delta(i) = P(\delta, \delta-1, \delta+1, i). \tag{2.22d}$$

The sum $\sum_{\{\mathcal{L}\}}$ is over one index from each pair $(\alpha\beta), (\gamma\delta), \ldots, (\sigma\lambda)$; the total number of pairs is N. We will use $\mathcal{L}^*$ to denote the second index in the pair $(\alpha\beta)$.

The variable $t_{\alpha\beta}$ corresponds to the propagator which joins the α leg to the β leg. We first apply the sewing prescriptions[1] simultaneously on the N pairs of excited legs a^α, a^β, $(\alpha\beta) = \{\mathcal{L}\}$; then we use the principal-axes technique[1]; then we define the projective operator $R_{\beta\alpha}$ responsible for circling the $(\alpha\beta)$ loop; then we use Eq. (2.11) to facilitate the infinite number of cancellations[5] leading to the invariant points $x_{\alpha\beta}^{(1)}$, $x_{\alpha\beta}^{(2)}$ of $R_{\beta\alpha}$; and finally we obtain the nonplanar N-loop amplitude (Fig. 4):

$$F_{\rm nl}(N) = \int \prod_{\alpha\in\{\mathcal{L}\}} d^4k_\alpha \int_0^1 \prod_{(\alpha\beta)\in\{\mathcal{L}\}} dt_{\alpha\beta}\, t_{\alpha\beta}{}^{-l(k_\alpha)-1} (1-t_{\alpha\beta})^{\alpha_0 - 1 + \frac{1}{2}k_\alpha{}^2} \prod_i dy_i \{Y_{S+2}\} I, \tag{2.23}$$

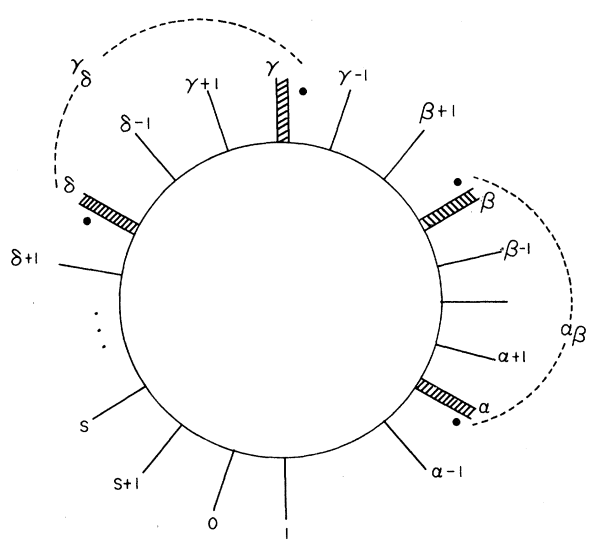

FIG. 3. 2Nth factorized tree diagram.

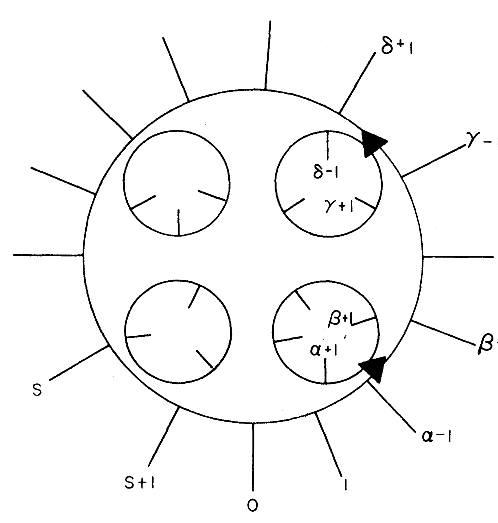

FIG. 4. Nonplanar N-loop diagram (rubber band).

where

$$I=\frac{1}{(\det[\Delta])^{1/2}}\prod_{n=0}^{\infty}I_n=\frac{1}{(\det[\Delta])^{1/2}}\prod_{\substack{(\alpha\beta),\dots,(\gamma\delta)\\ \in\{\mathfrak{L}\}}}\prod_{\substack{i,j=0\\ (i,j\neq\{\mathfrak{L}\})}}\prod_{n=0}^{\infty}\{y_i-[R^{\pm}]^{(n+1)}_{\beta\alpha,\delta\gamma}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times\prod_{\substack{(\alpha\beta),\dots,(\sigma\lambda),\\ (\gamma\delta)\in\{\mathfrak{L}\}\\ (\sigma\lambda)\neq(\gamma\delta)}}\prod_{\substack{i=0\\ (i\neq\{\mathfrak{L}^*,\mathfrak{L}\})}}^{S+1}\prod_{n=0}^{\infty}\left\{\frac{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(1)}_{\gamma\delta})}{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(2)}_{\gamma\delta})}\right\}^{+k_i\cdot k_\gamma}\left\{\frac{y_i-y_\delta}{y_i-y_\gamma}\right\}^{-k_i\cdot k_\gamma}$$

$$\times\prod_{\substack{(\alpha\beta),(\alpha'\beta'),\dots,(\sigma\lambda),(\gamma\delta)\in\{\mathfrak{L}\}\\ (\alpha\beta)\neq(\alpha'\beta');\,(\sigma\lambda)\neq(\gamma\delta)}}\prod_{n=0}^{\infty}\left\{\frac{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(1)}_{\gamma\delta})}{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(1)}_{\gamma\delta})}\,\frac{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(2)}_{\gamma\delta})}{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(2)}_{\gamma\delta})}\right\}^{-\frac{1}{2}k_\alpha\cdot k_\gamma}$$

$$\times\prod_{\substack{(\alpha\beta),(\gamma\delta)\in\{\mathfrak{L}\}\\ (\alpha\beta)\neq(\gamma\delta)}}\left\{\frac{(y_\alpha-y_\delta)(y_\beta-y_\gamma)}{(y_\beta-y_\delta)(y_\alpha-y_\gamma)}\right\}^{-\frac{1}{2}k_\alpha\cdot k_\gamma}$$

$$\times\prod_{(\alpha\beta)\in\{\mathfrak{L}\}}\left\{\frac{R^{-1}_{\beta\alpha}(y_\alpha)-R^{-1}_{\beta\alpha}(y_\beta)}{y_\alpha-R_{\beta\alpha}(y_\beta)}\,\frac{R_{\beta\alpha}(y_\alpha)-R_{\beta\alpha}(y_\beta)}{y_\beta-R^{-1}_{\beta\alpha}(y_\alpha)}\right\}^{-\frac{1}{2}k_\alpha\cdot k_\alpha}. \tag{2.24}$$

Again, separating out all Koba-Nielsen variables $y_\alpha, y_\beta (\alpha\beta)\in\{\mathfrak{L}\}$ in $\{Y_{S+2}\}$ in Eq. (2.23) and combining it with I of Eq. (2.24), we get

$$\{Y_{S+2}\}I=\frac{1}{(\det[\Delta])^{1/2}}\prod_{(\alpha\beta),\dots,(\gamma\delta)\in\{\mathfrak{L}\}}\prod_{\substack{i,j=0\\ (i,j\notin\{\mathfrak{L}^*,\mathfrak{L}\})}}^{S+1}\prod_{n=0}^{\infty}\{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\delta\gamma}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times\prod_{\substack{(\alpha\beta),\dots,(\sigma\lambda),(\gamma\delta)\in\{\mathfrak{L}\}\\ (\sigma\lambda)\neq(\gamma\delta)}}\prod_{\substack{i=0\\ (i\notin\{\mathfrak{L}^*,\mathfrak{L}\})}}^{S+1}\prod_{n=0}^{\infty}\left\{\frac{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(1)}_{\gamma\delta})}{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(2)}_{\gamma\delta})}\right\}^{+k_i\cdot k_\gamma}\prod_{\substack{i=0\\ (i\notin\mathfrak{L}^*,\,\mathfrak{L}^*-1,\,\mathfrak{L},\,\mathfrak{L}-1)}}^{S+1}(y_i-y_{i+1})^{\alpha_0-1}$$

$$\times\prod_{\substack{(\alpha\beta),(\alpha'\beta'),\dots,(\sigma\lambda),(\gamma\delta)\in\{\mathfrak{L}\}\\ (\alpha\beta)\neq(\alpha'\beta');\,(\sigma\lambda)\neq(\gamma\delta)}}\prod_{n=0}^{\infty}\left\{\frac{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(1)}_{\gamma\delta})}{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(1)}_{\gamma\delta})}\,\frac{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(2)}_{\gamma\delta})}{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(2)}_{\gamma\delta})}\right\}^{-\frac{1}{2}k_\alpha\cdot k_\gamma}$$

$$\times\Bigg\{\prod_{(\alpha\beta)\in\{\mathfrak{L}\}}(y_\alpha-y_\beta)^{k_\alpha^2}\left[\frac{R^{-1}_{\beta\alpha}(y_\alpha)-R^{-1}_{\beta\alpha}(y_\beta)}{y_\alpha-R_{\beta\alpha}(y_\beta)}\,\frac{R_{\beta\alpha}(y_\alpha)-R_{\beta\alpha}(y_\beta)}{y_\beta-R^{-1}_{\beta\alpha}(y_\alpha)}\right]^{-\frac{1}{2}k_\alpha^2}$$

$$\times\prod_{(\alpha\beta)\in\{\mathfrak{L}\}}\left[\frac{(y_{\alpha-1}-y_\alpha)(y_\alpha-y_{\alpha+1})(y_{\beta-1}-y_\beta)(y_\beta-y_{\beta+1})}{(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})}\right]^{-\frac{1}{2}k_\alpha^2-1}$$

$$\times\prod_{(\alpha\beta)\in\{\mathfrak{L}\}}[(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})]^{\alpha_0-1}(y_a-y_b)(y_b-y_c)(y_c-y_a)\Bigg\}. \tag{2.25}$$

We note that the factors in the last brace in Eq. (2.25) are not identical to the analogous factors in the nonplanar single-loop case, Eq. (2.13). However, in the frame $x^{(1)}_{\alpha\beta}=\infty$, $x^{(2)}_{\alpha\beta}=0$, in which $R^{\pm}_{\beta\alpha}$ reduces to its multiplier $X^{\pm}_{\alpha\beta}$, they are fortunately identical, and this is enough for our purpose. We can transform the set of variables $[t_{\alpha\beta}, y_\alpha, y_\beta; (\alpha\beta)\in\{\mathfrak{L}\}]$ into the new set of variables $[X_{\alpha\beta}, x^{(1)}_{\alpha\beta}, x^{(2)}_{\alpha\beta}; (\alpha\beta)\in\{\mathfrak{L}\}]$ by performing the same calculation as in the one-loop case, i.e., Eqs. (2.14), (2.15), and (2.16). Each time we pick out a particular frame $x^{(1)}_{\alpha\beta}=\infty$, $x^{(2)}_{\alpha\beta}=0$, we find a linear dependence factor $(1-X_{\alpha\beta})^2$ for the $(\alpha\beta)$ loop, and obtain an expression similar to (2.16). We then repeat the calculation for the $(\gamma\delta)$ loop, etc. Therefore, on combining Eqs. (2.25), (2.23), and (2.16), we finally obtain the projectively invariant nonplanar N-loop formula:

$$F_{nl}(N)=\int\prod_{\alpha\in\{\mathcal{L}\}}d^4k_\alpha\int\prod_{(\alpha\beta)\in\{\mathcal{L}\}}dX_{\alpha\beta}\,X_{\alpha\beta}{}^{-l(k_\alpha)-1}(1-X_{\alpha\beta})^2\prod_{\{\bar{R}\}}[1-X_{\bar{R}}]^{-4}$$

$$\times\int\prod_{\substack{i=0\\(i\notin\{\mathcal{L},\mathcal{L}^*,a,b,c\})}}^{S+1}dy_i[dy_a][dy_b][dy_c]\prod_{(\alpha\beta)\in\{\mathcal{L}\}}dx^{(1)}_{\alpha\beta}\,dx^{(2)}_{\alpha\beta}\frac{(y_a-y_b)(y_b-y_c)(y_c-y_a)}{\prod_{(\alpha\beta)\in\{\mathcal{L}\}}[x^{(1)}_{\alpha\beta}-x^{(2)}_{\alpha\beta}]^2}$$

$$\times\prod_{\substack{i=0\\ i\notin\{\mathcal{L}^*,\mathcal{L}^*-1;\mathcal{L},\mathcal{L}-1\}}}^{S+1}(y_i-y_{i+1})^{\alpha_0-1}\prod_{(\alpha\beta)\in\{\mathcal{L}\}}\left\{\frac{[y_{\alpha-1}-R_{\beta\alpha}(y_{\beta+1})][x^{(1)}_{\alpha\beta}-y_{\beta+1}]}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta+1})]}\right\}^{\alpha_0-1}$$

$$\times\left\{\frac{[y_{\alpha+1}-R_{\beta\alpha}(y_{\beta-1})][x^{(1)}_{\alpha\beta}-y_{\beta-1}]}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta-1})]}\right\}^{\alpha_0-1}\prod_{\substack{i,j=0\\(i,j\notin\{\mathcal{L}^*,\mathcal{L}\})}}^{S+1}\;\prod_{(\alpha\beta),\ldots,(\gamma\delta)\in\{\mathcal{L}\}}\;\prod_{n=0}^{\infty}\{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\delta\gamma}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times\prod_{\substack{i=0\\ i\notin\{\mathcal{L}^*,\mathcal{L}\}}}^{S+1}\;\prod_{\substack{(\alpha\beta),\ldots,(\sigma\lambda),(\gamma\delta)\in\{\mathcal{L}\}\\(\sigma\lambda)\neq(\gamma\delta)}}\;\prod_{n=0}^{\infty}\left\{\frac{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(2)}_{\gamma\delta})}{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(1)}_{\gamma\delta})}\right\}^{-k_i\cdot k_\gamma}$$

$$\times\prod_{\substack{(\alpha\beta),(\alpha'\beta'),\ldots,(\sigma\lambda),(\gamma\delta)\in\{\mathcal{L}\}\\(\alpha\beta)\neq(\alpha'\beta');(\sigma\lambda)\neq(\gamma\delta)}}\;\prod_{n=0}\left\{\frac{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(1)}_{\gamma\delta})}{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(1)}_{\gamma\delta})}\;\frac{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(2)}_{\gamma\delta})}{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}(x^{(2)}_{\gamma\delta})}\right\}^{-\frac{1}{2}k_\alpha\cdot k_\gamma},\qquad(2.26)$$

where

$$R^{\pm}_{\beta\alpha}(z)=\frac{z[x^{(2)}_{\alpha\beta}-X^{\pm}_{\alpha\beta}x^{(1)}_{\alpha\beta}]-x^{(1)}_{\alpha\beta}x^{(2)}_{\alpha\beta}(1-X^{\pm}_{\alpha\beta})}{z(1-X^{\pm}_{\alpha\beta})+x^{(2)}_{\alpha\beta}X^{\pm}_{\alpha\beta}-x^{(1)}_{\alpha\beta}}\qquad(2.27)$$

and

$$(\det[\Delta])^{-1/2}=\prod_{\{\bar{R}\}}(1-X_{\bar{R}})^{-4}.\qquad(2.28)$$

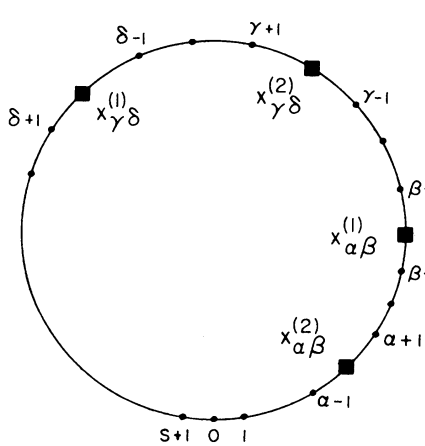

FIG. 5. Ordering of the $S+2$ variables y_i, $i=0,1,\ldots,S+1$, $i\notin\{\mathcal{L}^*,\mathcal{L}\}$, and $x^{(1)}_{\alpha\beta}$, $x^{(2)}_{\alpha\beta}$, $(\alpha\beta)\in\{\mathcal{L}\}$.

The ordering of y_i, $x^{(1)}_{\mathcal{L}}$, $x^{(2)}_{\mathcal{L}}$ can be seen in Eqs. (2.19) and (2.20), and the result is shown in Fig. 5 or Fig. 6.

The region of integration and periodicities are fully explained in Sec. III [see Eq. (3.16) below].

We see that the nonplanar N-loop formula is little different from the product of planar loop formulas.[1] The interpretation of various factors in Eq. (2.26) is again parallel to paper I.[1]

III. THE N-LOOP AMPLITUDE IN THE FORMULATION OF SCIUTO

The nonplanar N-loop amplitude can also be calculated with the three-Reggeon vertex introduced by Sciuto. These vertex functions are inserted in a scalar multiperipheral tree, as shown in Fig. 7. We insert a complete set of intermediate states $|\lambda_{\alpha\beta}\rangle\langle\lambda_{\alpha\beta}|$ in the upper portion of each loop:

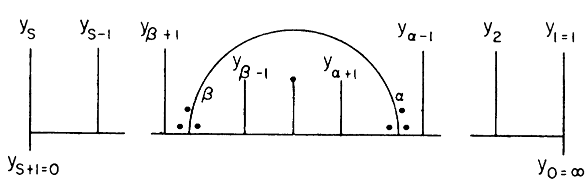

FIG. 6. Ordering of the external legs y_i relative to the loops. There is no y_i between any two adjacent loops.

$$F_{nl}(N)=\prod_{\alpha\in\{\mathcal{L}\}}\int d^4k_\alpha\langle 0|_a V_s^a D_s^a V_{s-1}^a\cdots V_{\beta+1}^a D_{\beta+1}^a L_{\alpha\beta} D_\alpha^a V_{\alpha-1}^a\cdots V_2^a D_2^a V_1^a|0\rangle_a\,, \tag{3.1}$$

where

$$D_i^a\equiv\int_0^1 dx_i\, x_i^{R_a-\alpha(k_i)-1}(1-x_i)^{-c}\,,$$

$$L_{\alpha\beta}\equiv\langle 0|_0 W_\beta^{ab} D_{\alpha\beta}^b D_\beta^a V_{\beta-1}^a\cdots V_{\alpha+1}^a D_{\alpha+1}^a \overline{W}_\alpha^{ab}|0\rangle_b\,,$$

$$W_\beta^{ab}\equiv\exp(a^\dagger|k_\beta)\exp(a^\dagger,b)_+\exp(a|k_\beta)\exp(a,b)_-\exp(b|-\pi_{\beta+1})\,,$$

$$V_i^a\equiv\exp(k_i|a^\dagger)\exp(k_i|a)\,,$$

$$\overline{W}_\alpha^{ab}\equiv\exp(a^\dagger|k_\alpha)\exp(a^\dagger,b^\dagger)_-\exp(a|k_\alpha)\exp(a,b^\dagger)_+\exp(b^\dagger|\pi_\alpha)\,,$$

$$D_{\alpha\beta}^b\equiv\int_0^1 du_{\alpha\beta}\,u_{\alpha\beta}^{R_b-\alpha(k_\alpha)-1}(1-u_{\alpha\beta})^{-c}\,.$$

[Notice that, for the moment, we have omitted the linear-dependence correction factor and the $(1-z)^R$ factor associated with the Sciuto vertex.]

We will use the identities

$$\langle 0|_b W_\beta^{ab} D_{\alpha\beta}^b|\lambda_{\alpha\beta}\rangle=\int_0^1 du_{\alpha\beta}u_{\alpha\beta}^{-\alpha(k_\alpha)-1}(1-u_\alpha)^{-c}\exp(a^\dagger|k_\beta+M_+u_{\alpha\beta}\lambda_{\alpha\beta})\exp(a|k_\beta+M_-u_{\alpha\beta}\lambda_{\alpha\beta})\exp(-\pi_{\beta+1}|u_{\alpha\beta}\lambda_{\alpha\beta}) \tag{3.2}$$

and

$$\langle\lambda_{\alpha\beta}|\overline{W}_\alpha^{ab}|0\rangle_b=\exp(a^\dagger|k_a+M_-\lambda_{\alpha\beta}^*)\exp(a|k_\alpha+M_+\lambda_{\alpha\beta}^*)\exp(\pi_\alpha|\lambda_{\alpha\beta}^*). \tag{3.3}$$

Using the techniques given in Ref. 1, we now contract over a oscillators and find

$$F_{nl}(N)=\int\prod_{\alpha\in\{\mathcal{L}\}}d^4k_\alpha\int_0^1\prod_{(\alpha\beta)\in\{\mathcal{L}\}}du_{\alpha\beta}\int\prod_{i=1}^s dx_i\int\prod_{(\alpha\beta)\in\{\mathcal{L}\}}d\left(\frac{\lambda_{\alpha\beta}}{\sqrt{2}}\right)d\left(\frac{\lambda_{\alpha\beta}^*}{\sqrt{2}}\right)$$
$$\times u_{\alpha\beta}^{-\alpha(k_\alpha)-1}(1-u_{\alpha\beta})^{-c}x_i^{-\alpha(\pi_i)-1}(1-x_i)^{-c}\exp\left\{\sum_{i>j}^{s}(k_i|x_{j+1,i}|k_j)\right\}$$
$$\times\exp\left\{(\underline{\lambda}|\underline{\lambda}^*)+(\underline{\lambda}^*|[C]|\underline{\lambda})+(\underline{\lambda}|[B]|\underline{\lambda}^*)+(\underline{\lambda}^*|[D]|\underline{\lambda}^*)+(\underline{\lambda}|[A]|\underline{\lambda})+(\underline{\lambda}^*|\underline{F})+(\underline{\lambda}|\underline{E})\right\}, \tag{3.4}$$

where

$$A_{\alpha\beta,\gamma\delta}=u_{\alpha\beta}M_+y_\delta y_\beta^{-1}M_-u_{\gamma\delta}\quad(\text{for }\alpha<\beta<\gamma<\delta)$$
$$=0\quad(\text{otherwise}),$$

$$C_{\alpha\beta,\gamma\delta}=M_-^Ty_\delta y_\alpha^{-1}M_-u_{\gamma\delta}\qquad(\alpha<\beta<\gamma<\delta)$$
$$\text{or }(\delta=\beta\text{ and }\gamma=\alpha)$$
$$=0\quad(\text{otherwise}),$$

$$B_{\alpha\beta,\gamma\delta}=u_{\alpha\beta}M_+y_\gamma y_\beta^{-1}M_+\quad(\alpha<\beta<\gamma<\delta)$$
$$=0\quad(\text{otherwise}),$$

$$D_{\alpha\beta,\gamma\delta}=M_-^Ty_\gamma y_\alpha^{-1}M_+\quad(\alpha<\beta<\gamma<\delta)$$
$$=0\quad(\text{otherwise}),$$

$$|E_{\alpha\beta})=\sum_{j=1}^{\beta-1}u_{\alpha\beta}M_-^Ty_\beta y_j^{-1}|k_j)$$
$$+\sum_{j=\beta+1}^{s}u_{\alpha\beta}M_+y_jy_\beta^{-1}|k_j)-u_{\alpha\beta}\pi_{B+1}\,,$$

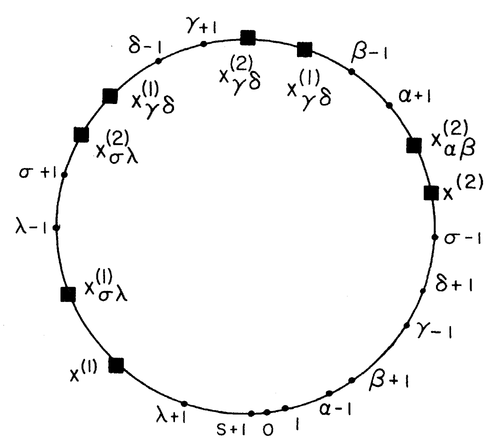

FIG. 7. Nonplanar configuration via Sciuto vertex.

$$|F_{\alpha\beta}) = \sum_{j=1}^{\alpha-1} M_+ y_\alpha y_j^{-1}|k_j) + \sum_{j=\alpha+1}^{S} M_-^T y_j y_\alpha^{-1}|k_j) + |\pi_\alpha)$$

($\alpha < \beta$ always) ($\gamma < \delta$ always). We shall symmetrize as follows:

$$[\overline{A}] = [A] + [A]^T, \quad [\overline{D}] = [D] + [D]^T, \quad [\overline{C}] = [C] + [B]^T. \tag{3.5}$$

We now perform the integration over λ. Then we get

$$F_{nl}(N) = \prod_{(\alpha\beta)\in\{\mathcal{L}\}} \int_0^1 du_{\alpha\beta} u_{\alpha\beta}^{-\alpha(k_\alpha)-1}(1-u_{\alpha\beta})^{-c} \int \prod_{\alpha=\{\mathcal{L}\}} d^4k_\alpha \prod_{i=1}^{S} \int_0^1 dx_i x_i^{-\alpha(\pi_i)-1}(1-x_i)^{-c}$$

$$\times (\det[\Delta])^{1/2} \exp\left\{ \tfrac{1}{2} \sum_{n=0}^{\infty} \left((\underline{E}| (\underline{F}| \right) [GH]^n \begin{pmatrix} |\underline{F}) \\ |\underline{E}) \end{pmatrix} \right\}, \tag{3.6}$$

where we have used

$$[G] \equiv \begin{pmatrix} 0 & [I] \\ [I] & 0 \end{pmatrix},$$

$$[H] \equiv \begin{pmatrix} [\overline{A}] & [\overline{C}]^T \\ [\overline{C}] & [\overline{D}] \end{pmatrix}, \tag{3.7}$$

$$[\Delta] = [G] - [H].$$

At this point, we will find it useful to introduce the following projective operator:

$$R_{\beta\alpha} = y_\alpha P_{\beta\alpha} y_\alpha^{-1},$$

$$P_{\beta\alpha} = \begin{pmatrix} 1 & -y_\beta y_\alpha^{-1} \\ 1 & -y_\beta y_\alpha^{-1}(1-u_{\alpha\beta}) \end{pmatrix}. \tag{3.8}$$

With this projective operator, we can reexpress all matrices as follows:

$$|E_{\alpha\beta}) = \sum_{j=0}^{S+1} K P_{\beta\alpha} y_\alpha^{-1} y_j |k_j), \qquad (E_{\alpha\beta}| = \sum_{j=0}^{S+1} \frac{(k_j|}{K^{-1}(P_{\beta\alpha} y_\alpha^{-1} y_j)^{-1}},$$

$$|F_{\alpha\beta}) = \sum_{j=0}^{S+1} K^{-1} y_\alpha y_j^{-1} |k_j), \qquad (F_{\alpha\beta}| = \sum_{j=0}^{S+1} \frac{(k_j|}{K y_\alpha^{-1} y_j}, \tag{3.9}$$

$$\overline{A}_{\alpha\beta,\gamma\delta} = K P_{\beta\alpha} y_\alpha^{-1} y_\gamma P_{\delta\gamma}^{-1} \frac{1}{K(\;)}, \qquad \overline{D}_{\alpha\beta,\gamma\delta} = K^{-1} \frac{1}{y_\alpha^{-1} y_\gamma K^{-1}(\;)},$$

$$\overline{C}_{\alpha\beta,\gamma\delta} = K^{-1} \frac{1}{y_\alpha^{-1} y_\gamma P_{\delta\gamma}^{-1}[K(\;)]^{-1}} \qquad \overline{C}^T_{\alpha\beta,\gamma\delta} = K P_{\beta\alpha} y_\alpha^{-1} y_\gamma K^{-1}(\;),$$

where $K(z) = 1 - 1/z$, $K^{-1}(z) = 1/(1-z)$, $K^{-1} \neq 1/K$. Notice that we have assumed momentum conservation in order to derive projective relations for $\overline{A}$, $\overline{D}$, and $\overline{C}$. When expressed in this fashion, all K's in $[GH]^n$ neatly cancel. (Also: $y_0 \equiv \infty$, $y_1 \equiv 1$, $y_{S+1} \equiv 0$.)

If we assume momentum conservation among the k's, then we can contract over harmonic-oscillator states and projectively manipulate these expressions:

$$\tfrac{1}{2}(\underline{F}|\underline{E}) = \prod_{i,j} \prod_{\alpha\beta\in\{\mathcal{L}\}} [y_j - R_{\beta\alpha} y_i]^{-\frac{1}{2}k_i k_j},$$

$$\tfrac{1}{2}(\underline{F}|[\overline{C}]^T|\underline{E}) = \prod_{i,j} \prod_{(\alpha\beta),(\gamma\delta)\in\{\mathcal{L}\}} [y_j - R_{\beta\alpha} R_{\delta\gamma} y_i]^{-\frac{1}{2}k_i k_j}, \tag{3.10}$$

etc.

Notice that we have imposed conservation of momentum everywhere, which allows us to ignore "residue" terms which arise from binomial contractions, i.e., $(M_+)_{nm} x^m \cong 1/(1-x)^n - 1$. One disturbing fact is that y_α and y_β are *not* the invariant points of $R_{\alpha\beta}$ (as was found earlier). When the binomial "residue terms" are added in, we get an infinite set of cancellations,[5] which replaces y_α and y_β with the invariant points of

$R_{\alpha\beta}$. (The cancellation is *exactly* as in the planar case, and hence is not presented here.) We merely state the result:

$$\exp\left\{\sum_{i>j}^{S}(k_i\,|x_{j+1,i}|\,k_j)\right\}\exp\left\{\left((\underline{E}|(\underline{F}|)[\Delta]^{-1}\begin{pmatrix}|\underline{E})\\|\underline{F})\end{pmatrix}\right)\right\}$$

$$=\prod_{n=0}^{\infty}\prod_{(\alpha\beta),\cdots,(\sigma\lambda)\in\{\mathfrak{L}\}}\prod_{\substack{i,j=0\\ i\neq\alpha,\beta;j\neq\lambda,\sigma\\ i\neq j \text{ if } n=0}}[w_i-(R^{\pm})^{(n)}_{\beta\alpha,\lambda\sigma}w_j]^{-\frac{1}{2}k_ik_j}$$

$$\times(x^{(2)}_{\alpha\beta}-y_\alpha)^{-\frac{1}{2}k_\alpha{}^2}(x^{(1)}_{\alpha\beta}-y_\beta)^{-\frac{1}{2}k_\beta{}^2}\left(\frac{y_\alpha}{x^{(1)}_{\alpha\beta}}\right)^{+\frac{1}{2}k_\alpha{}^2}\prod_{i>j}^{S+1}(-y_i)^{-k_ik_j}, \tag{3.11}$$

where

$$x^{(2)}_{\alpha\beta}\equiv R^{\infty}_{\alpha\beta}(z_1),\qquad z_1\neq x^{(1)}_{\alpha\beta},$$

$$x^{(1)}_{\alpha\beta}\equiv R^{-\infty}_{\alpha\beta}(z_2),\qquad z_2\neq x^{(2)}_{\alpha\beta}$$

and

$$w_i\equiv y_i \text{ if } i\notin\{\mathfrak{L}\},$$

$$w_\alpha\equiv x^{(2)}_{\alpha\beta}=\text{invariant point},$$

$$w_\beta\equiv x^{(1)}_{\alpha\beta}=\text{invariant point}.$$

Now that we have all the tools to derive the answer, we are ready to put in all $(1-z)^R$ factors (appearing in each Sciuto vertex) and the linear-dependence correction,[6]

$$\left(1-\frac{(1-x_\alpha)u_{\alpha\beta}x_{\alpha+1}x_{\beta-1}}{(1-x_{\alpha+1})(1-x_{\beta-1})}\right)^{-c}. \tag{3.12}$$

The linear-dependence correction to nonplanar and overlapping loops is a simple c number.[6] The planar loops, however, have modified propagators. We have

$$F_{nl}(N)$$

$$=\prod_{(\alpha\beta)\in\{\mathfrak{L}\}}\int d^4k_\alpha\int_0^1 du_{\alpha\beta}u_{\alpha\beta}{}^{-\alpha(k_\alpha)-1}(1-u_{\alpha\beta})^{-c}(1-x_\alpha)^{-\alpha(k_\alpha)}\prod_{i=2}^{S}\int_0^1 dx_ix_i{}^{-\alpha(\pi_i)-1}$$

$$\times(1-x_i)^{-c}(1-u_{\alpha\beta})^{-\alpha(\pi_\beta+1)}\left(1-\frac{(1-x_\alpha)u_{\alpha\beta}x_{\alpha+1}x_{\beta-1}}{(1-x_{\alpha+1})(1-x_{\beta-1})}\right)^{-c}\exp\left\{\sum_{i>j}^{S}(k_i\,|x_{j+1,i}|\,k_j)\right\}\exp\left\{\tfrac{1}{2}\left((\underline{E}|(\underline{F}|)[\Delta]^{-1}\begin{pmatrix}\underline{E})\\ \underline{F})\end{pmatrix}\right)\right\}$$

$$=\int\prod_{\alpha\in\{\mathfrak{L}\}}d^4k_\alpha\int\prod_{(\alpha\beta)\in\{\mathfrak{L}\}}\int\prod_{i=0}^{S+1}dw_i(dw_adw_bdw_c)^{-1}dX_{\alpha\beta}X_{\alpha\beta}{}^{-\alpha(k_\alpha)-1}(1-X_{\alpha\beta})^2\prod_{\{\bar{R}\}}(1-X_{\bar{R}})^{-4}$$

$$\times\prod_{n=0}^{\infty}\prod_{\substack{i,j=0\\ i\neq\alpha,\beta\\ j\neq\lambda,\sigma}}^{S+1}\prod_{(\alpha\beta),\cdots,(\sigma\lambda)\in\{\mathfrak{L}\}}[w_i-(R^{\pm})^{(n)}_{\beta\alpha,\lambda\sigma}y_j]^{-\frac{1}{2}k_ik_j}(w_a-w_b)(w_b-w_c)(w_c-w_a)\prod_{\substack{i=0\\ (i\notin\{\mathfrak{L},\mathfrak{L}-1,\mathfrak{L}^*,\mathfrak{L}^*-1\})}}^{S+1}(w_i-w_{i+1})^{\alpha_0-1}$$

$$\times\prod_{(\alpha\beta)\in\{\mathfrak{L}\}}(x^{(1)}_{\alpha\beta}-x^{(2)}_{\alpha\beta})^{-2}\left\{\frac{(x^{(1)}_{\alpha\beta}-w_{\beta+1})[w_{\alpha-1}-R_{\beta\alpha}(w_{\beta+1})]\,(x^{(1)}_{\alpha\beta}-w_{\beta-1})[w_{\alpha+1}-R_{\beta\alpha}(w_{\beta-1})]}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(w_{\beta+1})]\,[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(w_{\beta-1})]}\right\}^{\alpha_0-1}, \tag{3.13}$$

where we have

$$x^{(1)}_{\alpha\beta}\equiv w_\beta\equiv R^{\infty}_{\beta\alpha}(z_1),\qquad z_1\neq x^{(2)}_{\alpha\beta}$$

$$x^{(2)}_{\alpha\beta}\equiv w_\alpha\equiv R^{-\infty}_{\beta\alpha}(z_2),\qquad z_2\neq x^{(1)}_{\alpha\beta}$$

$$X_{\alpha\beta}\equiv\text{multiplier of } R_{\beta\alpha}=\tfrac{1}{2}[\Phi_{\alpha\beta}{}^2-2\pm\Phi_{\alpha\beta}(\Phi_{\alpha\beta}{}^2-4)^{1/2}],$$

where

$$\Phi_{\alpha\beta}=\frac{\mathrm{Tr}(R_{\beta\alpha})}{[\det(R_{\beta\alpha})]^{1/2}},$$

$$w_{S+2}=w_0,$$

$$w_a,\ w_b,\ \text{and } w_c=\text{fixed points}, \tag{3.14}$$

$$J_{\alpha\beta}=\frac{\partial(y_{\alpha+1},y_{\alpha+2},\cdots,y_{\beta-1},x^{(1)}_{\alpha\beta},x^{(2)}_{\alpha\beta},X_{\alpha\beta})}{\partial(x_\alpha,x_{\alpha+1},\cdots,x_\beta,u_{\alpha\beta})}=-\frac{y_\beta{}^2(1-x_\alpha)D_{\alpha\beta}}{y_\alpha{}^2y_\beta(1-t_{\alpha\beta})^3t_{\alpha\beta}},$$

where

$$D_{\alpha\beta}\equiv1-t_{\alpha\beta}[1-u_{\alpha\beta}(1-x_\alpha)]\ \text{and}\ y_\beta/y_\alpha=t_{\alpha\beta}(1-t_{\alpha\beta})D_{\alpha\beta}{}^{-1},$$

$$P_{\beta\alpha}\equiv\begin{pmatrix}1 & -y_\beta y_\alpha{}^{-1}\\ 1 & -y_\beta y_\alpha{}^{-1}[1-u_{\alpha\beta}(1-x_\alpha)]\end{pmatrix}, \tag{3.15}$$

$$R_{\beta\alpha}\equiv y_\alpha P_{\beta\alpha}y_\alpha{}^{-1}.$$

Notice that $P_{\beta\alpha}$ changes by a factor of $1-x_\alpha$ when the $(1-z)^R$ Sciuto factor is correctly inserted; notice also that $t_{\alpha\beta}$ is defined implicitly:

$$x^{(1)}_{\alpha\beta}=y_\alpha t_{\alpha\beta},\quad x^{(2)}_{\alpha\beta}=y_\beta t_{\alpha\beta}{}^{-1}.$$

When the calculation is actually performed, the region of integration is actually larger than what was found earlier (e.g., the multiplier ranges from 0 to ∞). As in the planar case, we take the branch where the multiplier is between zero and one. (When the multiplier is equal to one, the invariant points are equal to each other.)

We recover the usual single-loop nonplanar amplitude if we let

$$U_1=(0,1),\quad \{\mathcal{L}\}=\{\alpha\},\quad \{\mathcal{L}^*\}=\{\beta\},\quad R_{\beta\alpha}=X_{\beta\alpha},$$

$$w_a=x^{(2)}_{\alpha\beta}=0,\quad x^{(1)}_{\alpha\beta}=w_c=\pm\infty,\quad w_b=w_{\beta+1}=1,$$

$$U_2=(x^{(1)}_{\alpha\beta}=-\infty<w_{\beta-1}\le w_{\beta-2}\le\cdots\le w_{\alpha+1}\le x^{(2)}_{\alpha\beta}=0\le w_{\alpha-1}\le\cdots w_1\le w_0\le w_{S+2}\le\cdots\le w_{\beta+2}\le w_{\beta+1}=1<\infty=x^{(1)}_{\alpha\beta}).$$

Conveniently, we find that the cyclic ordering of the Koba-Nielsen variables mimics the ordering in Fig. 3 if we let w_α and w_β be the invariant points.

We are free to move external lines past loops, as required by rubber-band duality, because

$$w_{\beta+1}\le w_\beta\le\cdots\le w_\alpha\to w_\beta\le\cdots w_\alpha\le R_{\beta\alpha}(w_{\beta+1})$$

and

$$w_\beta\le w_{\beta-1}\le\cdots\le w_{\alpha+1}\le w_\alpha\to w_\beta\le\cdots\le w_{\alpha+1}\le R_{\beta\alpha}(w_{\beta-1})\le w_\alpha.$$

Notice that variables trapped between w_α and w_β always remain trapped, while variables located between the invariant points of different, adjacent loops are free to move past these points.

(In the planar case, no variables are allowed between w_α and w_β.)

In studying these periodicity properties, we will find it convenient to move these latter lines completely away from the region occupied by the invariant points. A simple renumbering yields

$$(w_\alpha\le w_{S+1}\cdots\le w_1\le w_0\le w_\lambda\cdots\le w_\gamma\le w_\beta\le\cdots\le w_{\alpha+1}\le w_\alpha).$$

[Notice that the factors in the braces in (3.13) change slightly, depending on the quark topology.]

Since the operator $R_{\beta\alpha}$ flips these latter lines across the $(\alpha\beta)$ loop, the operator $(R_{\beta\alpha}\cdots R_{\lambda\sigma})$ flips these lines completely around the diagram. The regions occupied by these "rotated" lines are disjoint from previously rotated lines. As we rotate these lines an infinite number of times, they asymptotically approach the invariant points $x^{(1)}$ and $x^{(2)}$ of $(R_{\beta\alpha}\cdots R_{\lambda\sigma})^{-1}$. These points $x^{(1)}$ and $x^{(2)}$ separate the region occupied by the invariant points from the region occupied by these rotated lines. Likewise, the lines lying between w_α and w_β are rotated by the action of $R_{\beta\alpha}$. We summarize these statements as follows:

$$U_2=[x^{(1)}\le R_{\beta\alpha}\cdots R_{\lambda\sigma}(w_0)\le w_{S+1}\le w_S\le\cdots\le w_0\le x^{(2)}\le w_\lambda\le w_{\lambda-1}\le\cdots\le w_{\sigma+1}\le R_{\lambda\sigma}(w_{\lambda-1})\le w_\sigma\le\cdots\le w_\gamma\le w_\beta$$
$$\le w_{\beta-1}\le\cdots\le w_{\alpha+1}\le R_{\beta\alpha}(w_{\beta-1})\le w_\alpha\le x^{(1)}]. \tag{3.16}$$

We subtract out periodicities by constraining *one* variable in each set to lie between y_0 and $R(y_0)$, where y_0 is arbitrary, i.e.,

$[R_{\alpha\beta}\cdots R_{\lambda\sigma}(y_0)\leqslant w_0\leqslant y_0\leqslant x^{(2)}]$

and

$[w_\beta\leqslant y_\beta\leqslant w_{\beta-1}\leqslant R_{\beta\alpha}(y_{\beta-1})]$

for each $(\alpha\beta)$ in $\{\mathfrak{L}\}$. Notice the complete symmetry between the $R_{\beta\alpha}$'s and $(R_{\alpha\beta}\cdots R_{\lambda\sigma})^{-1}$, meaning that the distinction between outer and inner quark loops disappears. In each case, external lines belonging to each quark loop are confined to lie between the invariant points of that loop. (In the planar case, we only have outer quark lines, i.e., the lines between w_β and w_α are missing.)

These constraints are enough to determine U_1 uniquely. (All multipliers range from 0 to 1, but now they are no longer independent.)

We understand that Lovelace and Alessandrini have obtained similar results.[7]

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APPENDIX: THE JACOBIAN CALCULATION

We show how the variables t, y_α, y_β in the expression (2.14) are eliminated and transformed into the variables X, x_1, x_2 in Eq. (2.15). We first find a set of identities that relate t, y_α, y_β to X, x_1, x_2. Using Eqs. (2.8). (2.9), (2.10), and (2.2), we can express the projective operator $R^{-1}_{\beta\alpha}$, defined in Eq. (2.9) as

$$R^{-1}_{\beta\alpha}(z)=\frac{z(y_\beta-ay_{\beta+1})-(y_\alpha y_\beta-y_{\alpha+1}y_{\beta+1}a)}{z(1-a)+(ay_{\alpha+1}-y_\alpha)}, \qquad \text{(A1)}$$

with

$$a\equiv\frac{t}{t-1}d,\quad t=\frac{a}{a-d}, \qquad \text{(A2a)}$$

$$d\equiv\frac{(y_\alpha-y_{\alpha-1})(y_\beta-y_{\beta-1})}{(y_{\alpha+1}-y_{\alpha-1})(y_{\beta+1}-y_{\beta-1})}. \qquad \text{(A2b)}$$

On comparison of Eq. (A1) with the standard form in paper I, we find the set of identities

$$(1-a)=l(1-X^{-1}), \qquad \text{(A2c)}$$

$$y_\beta-ay_{\beta+1}=l(x_2-X^{-1}x_1), \qquad \text{(A2d)}$$

$$ay_{\alpha+1}-y_\alpha=l(x_2X^{-1}-x_1), \qquad \text{(A2e)}$$

$$y_\alpha y_\beta-y_{\alpha+1}y_{\beta+1}a=lx_1x_2(1-X^{-1}), \qquad \text{(A2f)}$$

$$l=\left[\frac{a(y_\alpha-y_{\alpha+1})(y_{\beta+1}-y_\beta)}{X^{-1}(x_1-x_2)^2}\right]^{1/2}. \qquad \text{(A2g)}$$

From Eqs. (A2d) and (A2e), we can derive the identity

$$a=\frac{y_\beta+y_\alpha(x_2-X^{-1}x_1)/(x_2X^{-1}-x_1)}{y_{\beta+1}+y_{\alpha+1}(x_2-X^{-1}x_1)/(x_2X^{-1}-x_1)}. \qquad \text{(A3a)}$$

With further identities

$$y_\alpha=R_{\beta\alpha}(y_{\beta+1})=\frac{y_{\beta+1}(x_2-x_1X)-x_1x_2(1-X)}{y_{\beta+1}(1-X)+x_2X-x_1}, \qquad \text{(A3b)}$$

$$y_\beta=R^{-1}_{\beta\alpha}(y_{\alpha+1})=\frac{y_{\alpha+1}(x_2-x_1X^{-1})-x_1x_2(1-X^{-1})}{y_{\alpha+1}(1-X^{-1})+x_2X^{-1}-x_1}, \qquad \text{(A3c)}$$

$$R^{\pm}_{\beta\alpha}(z)=\frac{z(x_2-X^{\pm}x_1)-x_1x_2(1-X^{\pm})}{z(1-X^{\pm})+x_2X^{\pm}-x_1}, \qquad \text{(A3d)}$$

$$\frac{R^{\pm}_{\beta\alpha}(z)-x_2}{R^{\pm}_{\beta\alpha}(z)-x_1}=X^{\pm}\frac{z-x_2}{z-x_1}, \qquad \text{(A3e)}$$

$$\frac{R^{-1}_{\beta\alpha}(y_\alpha)-x_1}{(y_\alpha-x_1)[R^{-1}_{\beta\alpha}(y_\alpha)-y_\beta]}=\frac{R_{\beta\alpha}(y_\beta)-x_2}{[y_\alpha-R_{\beta\alpha}(y_\beta)](y_\beta-x_2)}, \qquad \text{(A3f)}$$

one then can show that the expression (2.14) is equal to

$$dXdx_1dx_2\left|\frac{\partial(t,y_\alpha,y_\beta)}{\partial(X,x_1,x_2)}\right|a^{-\alpha_0-\frac{1}{2}k\alpha^2}\frac{(a-d)^2}{ad}[(y_\alpha-y_{\alpha-1})(y_\beta-y_{\beta-1})]^{\alpha_0-1}$$
$$\times\left\{\frac{[R^{-1}_{\beta\alpha}(y_\alpha)-x_1][R_{\beta\alpha}(y_\beta)-x_2]}{(y_\alpha-x_1)(y_\beta-x_2)}\right\}^{-\frac{1}{2}k\alpha^2}\left\{\frac{(y_a-y_b)(y_b-y_c)(y_c-y_a)}{(y_{\alpha+1}-y_a)(y_{\beta+1}-y_\beta)}\right\}. \qquad \text{(A4)}$$

Now we specialize to the frame $x_1=\infty$, $x_2=0$. Then $R^{\pm}_{\beta\alpha}\to X^{\pm}$, and $y_\alpha=x_1$, $y_b=1$, $y_c=x_2$, so that

$$
\begin{aligned}
&a \to 1\\
&l \to \frac{y_{\beta+1} - X^{-1} y_{\alpha+1}}{x_1 X^{-1}},\\
&y_\alpha \to X y_{\beta+1},\\
&y_\beta \to X^{-1} y_{\alpha+1}.
\end{aligned}
\tag{A5}
$$

Hence the expression (A4) reduces to

$$
dX[dx_1][dx_2][J]\Big|_{x_1=\infty;x_2=0}[(y_{\alpha-1} - Xy_{\beta+1})(y_{\alpha+1} - Xy_{\beta-1})]^{\alpha_0-1}X^{-l(k_\alpha)+1}\frac{(a-d)^2}{l^2 dX^{-1}}. \tag{A6}
$$

The calculation of the Jacobian factor

$$
J \equiv \frac{\partial(t, y_\alpha, y_\beta)}{\partial(X, x_1 x_2)} \tag{A7}
$$

is rather complicated. Fortunately, it gives

$$
\frac{l^2 d}{(a-d)^2}\frac{(1-X)^2}{X^3}. \tag{A8}
$$

Proof: From Eq (A2a), taking derivatives of t with respect to X, x_1, x_2 and using Eq. (A2b), we get

$$
J = \frac{d}{(a-d)^2}\left|\frac{\partial(a, y_\alpha, y_\beta)}{\partial(X, x_1, x_2)}\right|. \tag{A9}
$$

In deriving Eq. (A9), we have used the theorem that the determinant vanishes when two rows are identical. We now use Eq. (A3a) to take derivatives of a with respect to X, x_1, x_2 and evaluate in the frame $x_1 = \infty$, $x_2 = 0$; we get

$$
J = \frac{d}{(a-d)^2}\frac{X^{-1}(y_{\beta+1} - y_{\alpha+1}X^{-1})}{(y_{\beta+1} + y_{\alpha+1}X^{-1})}
\times\begin{vmatrix}\partial y_\alpha/\partial x_1 & \partial y_\alpha/\partial x_2\\ \partial y_\beta/\partial x_1 & \partial y_\beta/\partial x_2\end{vmatrix}. \tag{A10}
$$

We then calculate, from Eqs. (A3b) and (A3c), the derivatives of y_α, y_β with respect to x_1, x_2 (evaluate in the frame $x_1 = \infty$, $x_2 = 0$); we finally get

$$
\begin{aligned}
J &= \frac{d}{(a-d)^2}\frac{(1-X)^2}{X^3}\left[\frac{(y_{\beta+1} - X^{-1}y_{\alpha+1})^2}{x_1^2 X^{-2}}\right]\\
&= \frac{dl^2}{(a-d)^2}\frac{(1-X)^2}{X^3}. \qquad \text{Q.E.D.}
\end{aligned}
\tag{A11}
$$

Substituting Eq. (A11) in Eq. (A6), we obtain the expression (2.15):

$$
dX[dx_1][dx_2](1-X)^2 X^{-l(k_\alpha)-1}
\times[(y_{\alpha-1} - Xy_{\beta+1})(y_{\alpha+1} - Xy_{\beta-1})]^{\alpha_0-1}.
$$

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