Note that

$$n^{\mu}S^{(0)}_{\mu\nu\alpha\beta}(p,n)=ip^{\mu}C^{(0)}_{\mu\nu\alpha\beta}(p,n),$$

thus satisfying Feynman's hypothesis. Equation (31) is just the result of Boulware and Deser[10] in momentum space. Finally we wish to emphasize that what we have presented here is a simple derivation of the propagator function of any two operators possessing properties (a)–(c). Hopefully, similar techniques could be employed to study more complicated functions, e.g., the time-ordered product of two operators between nonvacuum states.

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Singularity Structure of Equal-Time Commutators*

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On the basis of simple dimensional considerations, it is argued that equal-time commutators as calculated canonically are, in general, not sufficiently singular. This general result is then verified in a number of examples of interest by computing the vacuum expectation values of the commutators in perturbation theory. Specific cases discussed are the electromagnetic current in spinor, scalar, and Pauli electrodynamics and the "new, improved" stress-energy tensor for a free scalar field.

I. INTRODUCTION

Equal-time commutators (ETC's) contain considerable dynamical information. Symmetry principles provide some constraints on their form,[1] but the more singular parts of commutators have thus far remained inaccessible to computation on general grounds. In particular, it has long been known that, in *specific* instances, the application of the canonical formalism yields results which are inconsistent with the general principles of field theory.[2] More recently, it has become popular to define ETC's via perturbation theory,[3-7] the justification being that the most convincing successes of field theory have been in perturbative calculations.

In Sec. II, it is argued on the basis of simple dimensional considerations that under very general circumstances ETC's calculated canonically have the wrong singularity structure. The exceptions to this rule are catalogued. In Sec. III, it is verified explicitly in several examples of interest that ETC's as defined by perturbation theory are in fact more singular than those calculated canonically. An example is also given in which Schwinger terms of a given singularity structure appear canonically, but in which noncanonical Schwinger terms of the same structure must also appear.

II. GENERAL RESULT

Let T be any tensor of the form

$$T^{\{\sigma\}}=\sum_{n}\lambda_{n}M_{n}^{\{\sigma\}}\ , \tag{2.1}$$

where M_n is a monomial in the canonical variables $\{\phi\}$ and $\{\pi\}$ and their spatial derivatives, the λ_n are constants, and the superscripts denote Lorentz indices. The dimension (measured in units of mass) of each canonical variable satisfies

$$d(\phi) > 0, \quad d(\pi) > 0 . \tag{2.2}$$

This is always true canonically, i.e., for a free theory, because the form of the mass term implies that $d(\phi) = 1$ for bosons and $\frac{3}{2}$ for fermions while the canonical commutation relations give $d(\pi_\phi) = 3 - d(\phi)$. However, the dimensionality may differ from the canonical value.[8] We choose to work with the canonical dimensionality because, for our purposes, it is sufficient to deal in lowest-order perturbation theory, where the fields may be treated as free. That more singular terms may occur in commutators in the exact solution of the theory or even in higher orders in perturbation theory merely serves to strengthen our conclusions.

Dimensional analysis suggests that, barring cancellations, a term in T of the form $\lambda M^{\{\sigma\}}$ will give rise to a term in $[T^{\{\sigma\}}(\vec{x}), T^{\{\rho\}}(0)]$ which has the singularity structure

$$\partial^n \delta^3(\vec{x}) , \qquad n = d(M^{\{\sigma\}}) + d(M^{\{\rho\}}) - 3 . \tag{2.3}$$

This implies that if T is j^μ, the electromagnetic current in spinor (or scalar) electrodynamics, $[j^\mu, j^\nu]$ will contain $\partial^3\delta$ terms[5,9] because j^μ has dimension 3.[10] Similarly, $\Theta^{\mu\nu}$, the stress-energy tensor, has dimension 4, so that $[\Theta^{\mu\nu}, \Theta^{\sigma\rho}]$ should contain $\partial^5\delta$ terms.[11] (Of course, symmetry considerations will require that these terms vanish for some sets of indices.) We will demonstrate in Sec. III that terms of this singularity structure do arise in perturbation theory. If the M_n which appear in (2.1) do not all have the same dimension, then the most singular term in the commutator clearly comes from the M_n with the largest dimension.

It must be emphasized that the conclusions reached in perturbation theory are always subject to the condition that cancellations do not occur. It is difficult to imagine such cancellations taking place in a nonrenormalizable theory (where the coupling constant has a negative dimension) because the singularity structure worsens with each additional order. In fact, it is unlikely that the commutators in such a theory exist in any meaningful sense. These conclusions may have to be modified if the currents are not polynomials[12] in the canonical variables, so we will restrict ourselves to finite sums in (2.1). We will examine one nonrenormalizable theory, that of Pauli electrodynamics, but only in lowest order. If cancellations in a renormalizable theory do not follow from symmetry considerations, then they must be viewed as "careful planning," i.e., as an eigenvalue condition on the coupling constant. On the other hand, cancellations can easily take place in super-renormalizable theories, i.e., those in which the coupling constant has a positive dimension, where they may follow, for example, from the vanishing of a finite number of integrals.

We now proceed to compare the singularity structure expected on dimensional grounds, Eq. (2.3), with that derived by use of the canonical commutation relations. The most singular term in the canonical evaluation of $[T^{\{\sigma\}}, T^{\{\rho\}}]$ occurs when the $M^{\{\sigma\}}$ with the highest-order derivative (say, n_1) of some canonical coordinate ϕ is commuted with its "conjugate," the $\tilde{M}^{\{\rho\}}$ which has the highest-order derivative (say, n_2) of π_ϕ. A singularity of the form $\partial^{n_1+n_2}\delta$ results. Because of condition (2.2) and the fact that $d(\phi) + d(\pi_\phi) = 3$, it follows that

$$n_1 + n_2 \leq d(M^{\{\sigma\}}) + d(\tilde{M}^{\{\rho\}}) - 3 . \tag{2.4}$$

Equality holds in (2.4) if and only if

$$M^{\{\sigma\}} = \lambda\, \partial^{n_1}\phi, \quad \tilde{M}^{\{\rho\}} = \lambda' \partial^{n_2}\pi_\phi \tag{2.5}$$

because the presence of any other ϕ's or π's serves to reduce $n_1 + n_2$. But the right-hand side of (2.4) characterizes the singularity structure as computed by dimensional analysis, i.e., Eq. (2.3). Thus, the condition for agreement between the form of the leading canonical and "dimensional" singularities in the ETC of two monomials is that they are linear functions of two canonically conjugate variables or of their derivatives. Here, linear means that only one canonical variable appears and that it appears precisely once. For a general tensor T, the requirement that the canonical result is sufficiently singular is simply that among the terms in (2.1) with the largest dimension are a linear "conjugate" pair such as described above. Under any other circumstances, the canonical commutator will be less singular.

The electromagnetic current in spinor and scalar electrodynamics, the hadron currents in the quark model, and the stress-energy tensor for *any* theory do not contain terms linear in the canonical variables, so we expect the canonical calculations to be incorrect in all of these cases. The correct singularity structure *is* obtained canonically for the currents of the algebra of fields and, of course, for the canonical commutators themselves. The case of the algebra of fields is particularly amusing because $d(V^0) = 3$ and $d(V^i) = 1$, so that the expected singularity is simply a single derivative of a δ function and this is provided by the canonical commutation rules. Since the dynamical theory of currents,[13] the so-called Sugawara model, can be viewed as a limit of a Yang-Mills field,[14] its current commutators also have at worst $\partial\delta$ singularities. The canonical structure of this theory has been discussed by Deser.[15]

The argument that we have presented works equally well in a space with an arbitrary number of spatial dimensions. However, boson fields in two-dimensional space-time satisfy $d(\phi)=0$ so that condition (2.2) is violated (it remains valid for fermions). In this case, canonical commutators of tensors which are not linear functions of ϕ may have the correct structure. An example is $j^\mu = i\phi^\dagger \overleftrightarrow{\partial}^\mu \phi$ for a free scalar field; here, $[j^\mu, j^\nu]$ has a $\partial\delta$ singularity structure both canonically and dimensionally.

III. PERTURBATION-THEORY RESULTS

A. Calculational Techniques

Following Johnson and Low,[3] we define the ETC by

$$[A(\vec{x},0), B(0)] = \lim_{t\to 0^+} D(\vec{x},t)\,, \tag{3.1a}$$

where

$$D(\vec{x},t) = \tfrac{1}{2}\{A(\vec{x},t)B(0) - B(0)A(\vec{x},-t) + A(\vec{x},-t)B(0) - B(0)A(\vec{x},t)\}\,. \tag{3.1b}$$

For our purposes, it is more convenient to deal with the spatial Fourier transform of the vacuum expectation value of $D(\vec{x},t)$, which we denote by $D(\vec{q},t)$:

$$D(\vec{q},t) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \langle 0|D(\vec{x},t)|0\rangle = (2\pi)^3 \sum_n \cos E_n t\, [\langle 0|A|n(\vec{q})\rangle\langle n(\vec{q})|B|0\rangle - \langle 0|B|n(-\vec{q})\rangle\langle n(-\vec{q})|A|0\rangle]\,, \tag{3.2}$$

where $A \equiv A(0)$. Let us consider first the case in which A and B are, respectively, the time and space components of a vector current j^μ. Then use of parity invariance gives

$$D^i(\vec{q},t) = (2\pi)^3 \sum_n \cos E_n t [\langle 0|j^0|n(\vec{q})\rangle\langle n(\vec{q})|j^i|0\rangle + \langle 0|j^i|n(\vec{q})\rangle\langle n(\vec{q})|j^0|0\rangle]\,. \tag{3.3}$$

The combination $q_i D^i(q,t)$ has a particularly simple representation when j^μ is divergence-free, namely,

$$q_i D^i(\vec{q},t) = 2(2\pi)^3 \sum_n E_n \cos E_n t\, |\langle 0|j^0|n(\vec{q})\rangle|^2\,. \tag{3.4}$$

A formalism which is virtually identical to this has been applied by Chanowitz[7,16] to the study of $\langle 0|[j^0, j^i]|0\rangle$ in fermion electrodynamics. Note that even if $\partial_\mu j^\mu \neq 0$, the corrections to (3.4) will not affect our conclusions regarding the leading singularity, provided that $\partial_\mu j^\mu$ is of lower dimension than $\partial_0 j^0$; the success of PCAC (partial conservation of axial-vector current) suggests that this is the case for the axial-vector current. In any case, we will restrict our attention to the (conserved) electromagnetic current. By calculating the right-hand side of (3.4) we can recover $D^i(\vec{q},t)$ and hence the ETC because rotational invariance requires that $D^i \propto q^i$.

Before proceeding to the evaluation of (3.4) in specific examples, it should be pointed out that (3.1) is by no means a unique way of defining ETC's. A plausible alternative is to use the equal-time limit of the commutator function:

$$[A(\vec{x},0), B(0)] = \lim_{t\to 0^+} C(\vec{x},t)\,,\qquad C(\vec{x},t) = A(\vec{x},t)B(0) - B(0)A(\vec{x},t)\,. \tag{3.5}$$

In the special case under consideration, i.e., $A=j^0$, $B=j^i$, and $\partial_\mu j^\mu = 0$, it is easily shown that $q_i C^i = q_i D^i$, so that (3.1) and (3.5) are equivalent for our purposes.[17] This may be stated in another way: If one defines the vacuum expectation value of the ETC of components of a conserved current via (3.5), one discovers that the Bjorken-Johnson-Low[3,18] (BJL) form for the commutator is valid, because the latter follows from (3.1).[19] Chanowitz[7] has used (3.5) in his work and has compared his results to those obtained by yet another means of evaluating ETC's, the method of point splitting.

It is important to note that the limit $t\to 0$ is taken after the sum over intermediate states is computed. Performing these operations in the opposite order leads to incorrect results; see Ref. 7 for details. In fact, this way of proceeding would lead to the standard spectral (Källén-Lehmann) form[20] for the ETC's but computation of the spectral functions in the examples below shows that the spectral representation fails to converge.

In the following, we use Eq. (3.4) and its analog for the case of the stress-energy tensor to evaluate D^i in several examples of interest.

B. Fermion Electrodynamics

To lowest order in α, the intermediate states that contribute to (3.4) are electron-positron pairs, i.e.,

$$|n(\vec{q})\rangle = |e^-(\vec{p}), e^+(\vec{q}-\vec{p})\rangle\,. \tag{3.6}$$

Using $j^\mu = \bar{\psi}\gamma^\mu\psi$, (3.4) becomes, after summing over spins,

$$q_i D^i(\vec{q},t)=\frac{2}{(2\pi)^3}\int d^3p\,(\cos q_0 t)\frac{q_0}{p_0(q_0-p_0)}\times[p_0(q_0-2p_0)+\vec{q}\cdot\vec{p}]\,, \tag{3.7}$$

where $p_0=(\vec{p}^{\,2}+m^2)^{1/2}$ and $q_0=(\vec{p}^{\,2}+m^2)^{1/2}+((\vec{q}-\vec{q})^2+m^2)^{1/2}$. We are able to perform this integration only in the massless case, i.e., when $p^2=0$ and $(q-p)^2=0$. It is easily verified that the integrand is well defined over the whole range of integration for all values of the mass and that the large $|\vec{p}|$ behavior of the integrand is mass-independent; hence, the result for $m=0$ is a continuous limit of the result in the massive case. In fact, it is expected that the most singular terms are mass-independent.[8] In the massless case, the integral becomes, upon converting to spherical coordinates with $\mu=\cos(\hat{q}\cdot\hat{p})$ and introducing the variable $\xi=q_0/|\vec{q}|$ and the notation $\lambda=|\vec{q}|t$,

$$\begin{aligned}q_i D^i(\vec{q},t)&=\frac{\vec{q}^{\,4}}{2(2\pi)^2}\int_1^\infty d\xi\,\xi(\xi^2-1)^2(\cos\xi\lambda)\times\int_{-1}^{1}d\mu\,\frac{1-\mu^2}{(\xi-\mu)^4}\\&=\frac{2\vec{q}^{\,4}}{3(2\pi)^2}\int_1^\infty d\xi\,\xi\cos\xi\lambda\\&=-\frac{2\vec{q}^{\,4}}{3(2\pi)^2}\left[\frac{1}{\lambda^2}(\cos\lambda+\lambda\sin\lambda-1)\right]+\frac{2\vec{q}^{\,2}}{3(2\pi)^2}\int_0^\infty d\xi\,\xi\cos\xi t\,.\end{aligned} \tag{3.8}$$

Recall that the ETC is obtained from the limit as $t\to 0$ of this expression. The second term is singular in t and is evaluated in the Appendix; the important point for our considerations is that it behaves like $\vec{q}^{\,2}$ and hence corresponds to a $\partial\delta$ singularity in $\langle 0|[j^0,j^i]|0\rangle$. The factor in brackets is simply $\frac{1}{2}+O(t)$. Hence, we have

$$q_i D^i(\vec{q},t)=-\frac{\vec{q}^{\,4}}{3(2\pi)^2}+\vec{q}^{\,2}\times(\text{function singular in } t)+O(t)\,. \tag{3.9}$$

This corresponds to

$$\langle 0|[j^0(\vec{x}),j^i(0)]|0\rangle=\frac{i}{3(2\pi)^2}\,\partial^i\vec{\nabla}^2\delta^3(\vec{x})+\Lambda\partial^i\delta(\vec{x})\,, \tag{3.10}$$

where Λ is a divergent constant. This is the singularity structure expected from the arguments given in Sec. II and is to be compared with the (vanishing) canonical commutator. This result was first reported by Brandt[9] and has since been found by other authors.[5-7]

C. Scalar Electrodynamics

To lowest order, $j^\mu=i\phi^\dagger\overleftrightarrow{\partial}^\mu\phi$ and the intermediate states are again two-particle states. One finds

$$q_i D^i(\vec{q},t)=\frac{2}{(2\pi)^3}\int d^3p(\cos q_0 t)\frac{q_0}{p_0(q_0-p_0)}(2p_0-q_0)^2\,. \tag{3.11}$$

In the limit $m=0$, this integral (but not the integrand) is precisely one-half the analogous expression in the fermion case. Thus

$$\langle 0|[j^0(\vec{x}),j^i(0)]|0\rangle=\frac{i}{6(2\pi)^2}\,\partial^i\vec{\nabla}^2\delta^3(\vec{x})+\tfrac{1}{2}\Lambda\partial^i\delta^3(\vec{x})\,. \tag{3.12}$$

Canonically, the result is

$$[j^0(\vec{x}),j^i(0)]=2i\phi^\dagger\phi(0)\partial^i\delta^3(\vec{x})\,. \tag{3.13}$$

D. Pauli Electrodynamics

Canonical Schwinger terms also appear in spinor electrodynamics when a Pauli moment term is added. To this end, we consider

$$\mathcal{L}=\bar{\psi}(i\gamma^\mu\partial_\mu-e\gamma^\mu A_\mu-m)\psi-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+e\kappa\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}\,, \tag{3.14}$$

where $F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$, $\sigma^{\mu\nu}=\frac{1}{2}i[\gamma^\mu,\gamma^\nu]$, and κ is an arbitrary constant.[21] The electromagnetic current is given by

$$\begin{aligned}j^\mu&=\bar{\psi}\gamma^\mu\psi-2\kappa\partial_\nu(\bar{\psi}\sigma^{\mu\nu}\psi)\\&=j^\mu_{\mathrm{M}}+j^\mu_{\mathrm{NM}},\end{aligned} \tag{3.15}$$

the subscripts M and NM denoting, respectively, the minimal and nonminimal contributions. In Coulomb gauge, the canonical commutation relations are conveniently summarized by

$$\begin{aligned}&\{\psi_\alpha(\vec{x}),\psi^\dagger_\beta(0)\}=\delta_{\alpha\beta}\delta^3(\vec{x})\,,\\&[A_i(\vec{x}),F_{0j}(0)]=i\left(\delta_{ij}-\frac{\partial_i\partial_j}{\vec{\nabla}^2}\right)\delta^3(\vec{x})\,,\end{aligned} \tag{3.16a}$$

$$[\psi,A_i]=[\psi^\dagger,A_i]=0\,;$$

$$\begin{aligned}&[\psi(\vec{x}),F_{0i}(0)]=2ie\kappa\gamma_i\psi\delta^3(\vec{x})\,,\\&[\psi^\dagger(\vec{x}),F_{0i}(0)]=-2ie\kappa\psi^\dagger\gamma_i\delta^3(\vec{x})\,,\end{aligned} \tag{3.16b}$$

where (3.16b) shows the effect of the Pauli interaction on the canonical structure. The use of Coulomb gauge is for convenience only; we are interested in $[j^\mu,j^\nu]$ and j^μ is gauge-invariant. To facilitate the computation of ETC's involving j^μ, we use the field equations to rewrite (3.15) as

$$j^0 = \psi^\dagger\psi - 2i\kappa\partial_k(\psi^\dagger\gamma^k\psi)\,,$$

$$j^i = \bar\psi\gamma^i\psi + 2i\kappa(\bar\psi\overleftrightarrow{\partial}^i\psi) - 2\kappa\bar\psi(2eA^i + 2m\gamma^i - e\kappa\{\sigma^{\mu\nu},\gamma^i\}F_{\mu\nu})\psi\,. \tag{3.17}$$

One then finds

$$\begin{aligned}
&[j^0_{\rm M}, j^\mu_{\rm M}] = [j^0_{\rm M}, j^0_{\rm NM}] = 0,\\
&[j^0_{\rm M}(\vec x), j^i_{\rm NM}(0)] = [j^0_{\rm NM}(\vec x), j^i_{\rm M}(0)] = 4i\kappa\bar\psi\psi(0)\partial^i\delta^3(\vec x)\,,\\
&[j^0_{\rm NM}(\vec x), j^0_{\rm NM}(0)] = 8i\kappa^2\partial_k(\psi^\dagger\sigma^{jk}\psi)(0)\partial_j\delta^3(\vec x),\\
&[j^0_{\rm NM}(\vec x), j^i_{\rm NM}(0)] = 8\kappa^2\{\bar\psi[\gamma^j(\overleftrightarrow{\partial}^i + 2ieA^i) + 2im\eta^{ij} + 2ie\kappa(-\eta^{ij}\sigma^{\mu\nu}F_{\mu\nu} + 2\sigma^{i\lambda}F^j{}_\lambda)]\psi\}(0)\partial_j\delta^3(\vec x)\,.
\end{aligned} \tag{3.18}$$

Collecting terms gives

$$\begin{aligned}
&[j^0(\vec x), j^0(0)] = 8i\kappa^2[\partial_k(\psi^\dagger\sigma^{jk}\psi)](0)\partial_j\delta^3(\vec x)\,,\\
&[j^0(\vec x), j^i(0)] = 8i\kappa\bar\psi\psi(0)\partial^i\delta^3(\vec x) - 8\kappa^2\{\bar\psi[\gamma^j(\overleftrightarrow{\partial}^i + 2ieA^i) + 2im\eta + 2ie\kappa(-\eta^{ij}\sigma^{\mu\nu}F_{\mu\nu} + 2\sigma^{i\lambda}F^j{}_\lambda)]\psi\}(0)\\
&\qquad\times\partial_j\delta^3(\vec x).
\end{aligned} \tag{3.19}$$

The occurrence of a Schwinger term in $[j^0, j^0]$ is of some interest. Note that it may be rewritten in the form

$$[j^0(\vec x), j^0(0)] = Y\delta^3(\vec x) + \epsilon_{ijk}\partial^j X^k(0)\partial^i\delta^3(\vec x)\,, \tag{3.20}$$

with $Y=0$ and $X^k = 4i\kappa^2\epsilon^{klm}\psi^\dagger\sigma_{lm}\psi$. Equation (3.20) has been shown by Gross and Jackiw[22] to be a necessary and sufficient condition for the validity of Feynman's conjecture in a given theory, viz., that it is possible to choose a seagull term consistent with the constraints imposed by Lorentz invariance such that the Schwinger terms and the divergence of the seagull cancel in the derivation of the Ward-Takahashi identities. It is perhaps amusing to note that if one generates a current algebra using for the neutral currents the j^μ of this theory and for the charged currents operators of the quark type, i.e., $V^\mu = \bar\psi_a\gamma^\mu\psi_b$, then Eq. (3.20) is *not* satisfied for $[j^0, V^0]$; for here the Schwinger term has the form $\kappa\psi_a^\dagger\gamma_i\psi_b\partial^i\delta$. Such a theory would thus not be a useful current algebra because low-energy theorems would involve the Schwinger terms.

Since we have been arguing that canonical reasoning fails to give the most singular terms in the commutator, one may well ask whether the Schwinger terms that we have found have any significance. In fact, Geren[23] has shown that the matrix element of $[j^0, j^\mu]$ between single-particle states calculated to order e via the BJL procedure does agree completely with Eq. (3.19). It is not surprising that the canonical and BJL calculations agree because there are no (even superficially) divergent diagrams to this order and hence one can take the limit $q_0\to\infty$ inside the momentum integrals.

To lowest order in α, j^μ has dimension 4 because κ has the dimension of a length.[24] Thus we expect $[j^0, j^i]$ to have $\partial^5\delta$ singularities in this order. Using the perturbation theory techniques that we have discussed gives ($\lambda\equiv|\vec q|t$)

$$\begin{aligned}
q_i D^i(\vec q, t) = {}& \frac{4\vec q^{\,6}\kappa^2}{3(2\pi)^2}\left\{\frac{1}{\lambda^4}[(-2\lambda^2+6)\cos\lambda + 6\lambda\sin\lambda - \lambda^2 - 6]\right\}\\
&-\frac{2\vec q^{\,4}}{3(2\pi)^2}\left\{\frac{1}{\lambda^2}(\cos\lambda + \lambda\sin\lambda - 1)\right\}\\
&+\frac{2\vec q^{\,2}}{3(2\pi)^2}\left\{(1-2\kappa^2\vec q^{\,2})\int_0^\infty d\xi\,\xi\cos\xi t + 2\kappa^2\int_0^\infty d\xi\,\xi^3\cos\xi t\right\}.
\end{aligned} \tag{3.21}$$

The expressions in brackets equal $\frac14 + O(t)$ and $\frac12 + O(t)$, respectively. The first term then generates a term in $\langle 0|[j^0, j^i]|0\rangle$ of the form

$$\frac{-i\kappa^2}{3(2\pi)^2}\,\partial^i(\vec\nabla^2)^2\delta^3(\vec x)\,,$$

The last term has only $\vec q^{\,2}$ and $\vec q^{\,4}$ contributions and hence corresponds to less singular commutator structures. The integrals appearing in (3.21) are discussed in the Appendix. Setting $\kappa=0$ in (3.21) simply reproduces the result for fermion electrodynamics, as it must. Thus it is seen that the coefficient of $\partial^i\delta$ in $\langle 0|[j^0, j^i]|0\rangle$ has a κ-independent part as well as κ-dependent part. This is to be compared with the canonical result, which has no κ-independent part. Thus, even when canonical Schwinger terms appear, they may not be the only ones of that structure.

E. "New, Improved" Energy-Momentum Tensor

Callan, Coleman, and Jackiw[25] (CCJ) have defined the stress-energy tensor for a free scalar field to be

$$\Theta_{\mu\nu} = T_{\mu\nu} - \lambda\Delta_{\mu\nu}\,, \tag{3.22a}$$

where

$$\begin{aligned}
&T_{\mu\nu} = \partial_\nu\phi\partial_\nu\phi + \tfrac12\eta_{\mu\nu}(m^2\phi^2 - \partial^\sigma\phi\partial_\sigma\phi)\,,\\
&\Delta_{\mu\nu} = (\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\phi^2\,.
\end{aligned} \tag{3.22b}$$

The standard definition of $\Theta_{\mu\nu}$ as the response of the action to a change in the metric gives $\Theta_{\mu\nu} = T_{\mu\nu}$, i.e., $\lambda=0$. However, the generators of the Poincaré group are integrals or first moments of $\Theta_{\mu\nu}$ and are hence independent of λ; thus for nongravitational phenomena, all $\Theta_{\mu\nu}$ of the form (3.22) are equivalent. In the generalization of (3.22) to (renormalizable) interacting theories, it is possible to choose a λ such that matrix elements of $\Theta_{\mu\nu}$ are well defined, at least to low orders in perturbation theory. Furthermore, this "new, improved" $\Theta_{\mu\nu}$ is related to the transformation properties of the theory under scale and conformal transformations.[26,27] The significant point for our considerations is the observation of CCJ that use of the canonical commutation relations gives rise to a $\partial^3\delta$ term in $[\Theta^{00},\Theta^{0k}]$, the presence of which is required[28] by general principles of field theory. In fact,

$$[\Theta^{00}(\vec{x}),\Theta^{0i}(0)]=4i\lambda^2\phi^2(0)\partial^i\vec{\nabla}^2\delta^3(\vec{x}) + \text{less singular terms}. \tag{3.23}$$

No $\partial^3\delta$ terms appear canonically in any renormalizable theory when the standard stress-energy tensor is used. However, we concluded in Sec. II that, in general, there should be $\partial^5\delta$ terms in such a commutator. We next proceed to use the techniques of Sec. IIIA to calculate this commutator.[29]

We adopt as our conserved current $\Theta^{0\mu}$. Then, defining

$$\Theta^i(\vec{q},t)=\int d^3\vec{x}\, e^{-i\vec{q}\cdot\vec{x}} \times\langle 0|\tfrac{1}{2}\{\Theta^{00}(\vec{x},t)\Theta^{0i}(0)-\Theta^{0i}(0)\Theta^{00}(\vec{x},-t) +\Theta^{00}(\vec{x},-t)\Theta^{0i}(0)-\Theta^{0i}(0)\Theta^{00}(\vec{x},t)\}|0\rangle, \tag{3.24}$$

we have

$$q_i\Theta^i(\vec{q},t)=2(2\pi)^3\sum_n E_n\cos E_n t\,|\langle 0|\Theta^{00}|n\rangle|^2. \tag{3.25}$$

The evaluation of this is tedious, even in the massless case, so we merely schematically indicate the result[30]:

$$q_i\Theta^i(\vec{q},t)=-\frac{\lambda^2}{4(2\pi)^2}\,\vec{q}^{\,6}+f(\lambda)\vec{q}^{\,4}+g(\lambda)\vec{q}^{\,2}. \tag{3.26}$$

This disagrees with the canonical calculation not only in the appearance of a $\partial^5\delta$ term but also in the form of the $\partial^3\delta$ singularity. For, we find that $f(\lambda)=a\lambda^2+b\lambda+c$ with a, b, and c nonvanishing (in fact, all are singular in the limit $t\to 0$) whereas the canonical procedure gave only a λ^2 term. It is easy to see that there must be a λ-independent $\partial^3\delta$ term, for it follows from Ref. 28 that

$$[T^{00},T^{0i}]=[\Theta^{00}-\lambda\Theta^{00},\Theta^{0i}-\lambda\Theta^{0i}] \tag{3.27}$$

must have a (by necessity λ-independent) $\partial^3\delta$ term. We conclude that no significance can be attributed to the canonical $\partial^3\delta$ term of CCJ and that the appearance of such a term is not an argument in favor of the use of the "new, improved" $\Theta_{\mu\nu}$.

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APPENDIX

We examine here certain integrals which arise in the perturbation-theory calculations, namely,

$$I_n(t)=\int_0^\infty d\xi\,\xi^n\cos\xi t. \tag{A1}$$

In order to legitimize the mathematical operations which we perform, it is necessary to smear I_n with a test function in t (to be consistent, this should have been done throughout all the derivations in Sec. III):

$$\begin{aligned}I_{2n+1}(f)&=\int_{-\infty}^\infty dt\,f(t)\int_0^\infty d\xi\,\xi^{2n+1}\cos\xi t\\&=(-1)^{n+1}\int_{-\infty}^\infty dt\,f^{(2n+1)}(t)\int_0^\infty d\xi\,\sin\xi t\\&=(-1)^{n+1}\int_{-\infty}^\infty dt\,f^{(2n+1)}(t)\times\text{Im}\lim_{\delta\to 0}\int_0^\infty d\xi\,e^{i\xi(t+i\epsilon)},\end{aligned} \tag{A2}$$

where

$$f^{(m)}(t)\equiv\frac{d^m}{dt^m}f(t).$$

Then we have

$$I_{2n+1}(f)=(-1)^{n+1}P\int_{-\infty}^\infty dt\,\frac{f^{(2n+1)}(t)}{t}, \tag{A3}$$

where P denotes the principal part. Taking f to be a function of compact support and expanding it in a power series leads to

$$I_{2n+1}(f)=(-1)^{n+1}(2n+1)!\,P\int_{-\infty}^\infty dt\left\{t^{-2(n+1)}\times\left[f(t)-\sum_{m=0}^{2n}\frac{t^m}{m!}f^{(m)}(0)\right]+R(t)\right\}. \tag{A4}$$

The remainder term $R(t)$ vanishes at $t=0$ and hence does not contribute to the commutator. With the understanding that such terms are to be dropped, we tabulate the integrals that appear in Sec. III:

$$I_1(f) = -P\int_{-\infty}^{\infty} dt\, \frac{f(t)-f(0)}{t^2}\,,$$
$$I_3(f) = 6P\int_{-\infty}^{\infty} dt\, \frac{f(t)-f(0)-\frac{1}{2}t^2 f^{(2)}(0)}{t^4}\,. \qquad \text{(A5)}$$

Equation (A3) is a more compact way of expressing the result.

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[16]This reference contains several errors. Equation (3) should read

$$C^i(\vec{q},t) = 2q^k \sum_n (E_n)^{-1}\{e^{-iE_n t}\langle 0|j^k|n(\vec{q})\rangle\langle n(\vec{q})|j^i|0\rangle + e^{iE_n t}\langle 0|j^i|n(\vec{q})\rangle\langle n(\vec{q})|j^k|0\rangle\},$$

and a similar modification is needed in Eq. (4). Further, we have adopted a somewhat different approach in that, instead of taking the limit $t\to 0$ of $D(\vec{q},t)$, we have exhibited the singularities at $t\to 0$ by treating $D(\vec{q},t)$ as a distribution in t (see Appendix). These approaches give results differing only in the form of the (poorly defined) coefficients of nonleading (less singular) terms and hence do not affect any of our conclusions.

[17]The general relation between (3.1) and (3.5) is

$$\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\langle\alpha|D(\vec{x},t)|\beta\rangle = \mathrm{Re}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\langle\alpha|C(\vec{x},t)|\beta\rangle\,.$$

Definition (3.1) is preferable because expressions such as $\ln(-q_0^2)$ can arise by use of (3.5); see Ref. 3 for such an example.

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