starting with quantization in the noncovariant vector gauge, for which the timelike component b_0 of the gauge field is set equal to zero. The Hamiltonian used does not appear to satisfy the Schwinger covariance condition,[7,8] although formal covariance follows from the fact that a covariant prescription is finally obtained.

Quantization in the radiation gauge, with $\vec{\nabla}\cdot\vec{b}=0$ and a time-independent constraint fixing b_0, is also possible, using the Hamiltonian of Schwinger,[7] but is complicated by the presence of coincident discontinuities and δ functions at equal times. These are related to the presence of the Schwinger term t_φ, and are not serious in themselves, but make it difficult to relate to the standard form of Sec. II.

*Work done in part while a visitor at CERN, Geneva, Switzerland, and at Argonne National Laboratory, Argonne, Ill.

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Lorentz-Invariant Localization for Elementary Systems. IV. The Nonrelativistic Limit

A. J. Kálnay* and P. L. Torres
Departamento de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela
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In parts II and III of the present series of papers, the localization problem was studied for the relativistic nonzero-mass (spin 0 and $\frac{1}{2}$) and zero-mass (spin 0, $\frac{1}{2}$, and 1) cases. With a set of postulates of which the basic one was the imposition of Lorentz invariance of localization (which is a self-consistency requirement), it was possible to define the position operator uniquely (except for a constant for spin $\frac{1}{2}$, nonzero mass). However, the position operator has unusual properties, e.g., non-Hermiticity. This does not preclude its having physical meaning, but is such a strange result that in the present paper we test the validity of the postulates in the nonrelativistic case (spin 0 and $\frac{1}{2}$) where the position operator is well known. We use the same set of postulates, except that here we impose the Galilean invariance of localization. The result of the test is positive; the position operator found is the usual one. Moreover, with our set of postulates, we restate the notion of localization in a more physical basis, giving a precise definition of it without imposing avoidable requirements.

I. INTRODUCTION

A. General

The present series of papers studies the problem of localizability of elementary systems in relativistic quantum mechanics. In part I,[1] Philips's[2] results were discussed. In part II[3] (hereafter called II), the general consequences of imposing Lorentz invariance of localization (i.e., the physical consistency of the description of the localization by observers in different inertial frames) were derived and applied to nonzero-mass systems of spin 0 and $\frac{1}{2}$. In part III[4] (hereafter called III), the same procedure as in II was applied to zero-mass systems of spin 0, $\frac{1}{2}$, and 1.

The set of axioms which express the Lorentz invariance of localization for elementary systems in the most general way was stated in II. These axioms are restricted to such systems for which the localization with respect to a k axis has sense in a point.[5] In order not to introduce avoidable hypotheses, some standard assumptions were neither used nor rejected. For example, it can be recalled[3] that (a) Lorentz invariance of localization can be achieved without imposing manifest formal covariance (see II and the references there); (b) non-Hermitian operators can have a legitimate use in quantum mechanics[6,7]; (c) the possibility that components of the position do not commute cannot be rejected in an absolute way (as for instance in the case of the angular momentum), so it is not necessary to impose the existence of 3-localized states[3] (i.e., simultaneous eigenstates of the three components of position); instead, it is sufficient to consider 1-localized states[3] (i.e., eigenstates of only one component).

One of the results in II and III is that there is no Hermitian position operator. This implies that it is[3,4] certain that one of the following statements is the right one for the relativistic case:

(a) Position has no meaning at the quantum-relativistic level.

(b) The description of localization from different inertial frames is not self-consistent.

(c) Position is the only observable which cannot be represented by an operator.

(d) The position operator exists, but it is non-Hermitian.

(e) Some unusual interaction effects do not disappear when the interaction is switched off.

If the possibility (d) is considered, then we have in II as well as in III:

(i) The eigenvalues of the position operator are real in spite of the non-Hermiticity of the operators (see Secs. III–V of III and Refs. 22 and 24 of II).

(ii) The components of the position are compatible with each other for spin 0 and incompatible for spin $\frac{1}{2}$ and 1.

(iii) The commutation relations of the position and the linear momentum are the expected ones.

(iv) The velocity operator is Hermitian and has the expected form.

Some of the consequences of the self-consistency of localization in relativistic quantum mechanics are rather unusual. It would be prudent to test this method in that part of quantum mechanics where the localization problem has a well-known solution, i.e., in the nonrelativistic quantum mechanics of a point particle.[8,9] To do this is the basic purpose of the present paper. As a by-product we shall obtain what we expect is a better approach to the nonrelativistic localization.

B. Outline of the Argument

In Sec. II we state the postulates and discuss them in Sec. III. In Secs. IV and V the Galilean-invariant localization problems for spin 0 and $\frac{1}{2}$, respectively, are solved. The results are discussed in Sec. VI.

C. Conventions and Scalar Product to be Used

The dimension of a quantity A will be indicated by $[A]$. We shall only work in the p representation and in the Heisenberg picture of time evolution, and in the Schrödinger-type picture of spatial translations and pure Galilean transformations[10] of the one-particle state. The scalar product will be the Galilean-invariant one, which for any spin takes the form (see, e.g., Ref. 11)

$$(\psi, \phi) = \sum_{\xi} \int \psi^{\dagger}(\vec{p}, \xi)\, \phi(\vec{p}, \xi)\, d^3p. \qquad \text{(I.1)}$$

As in II, we denote by $\mathcal{E}^{(0)}$ the orbital-variable space and by $\mathcal{E}^{(s)}$ the spin-variable space, so that the state vector space is $\mathcal{E} = \mathcal{E}^{(0)} \otimes \mathcal{E}^{(s)}$.

II. POSTULATES

We make the following assumptions:

(1) Position makes sense for a particle.

(2) Each component of 3-position is represented by an operator X^k whose eigenvalues are its possible values. An eigenvalue of this component of position can be known with certainty if and only if the vector that represents the state is an eigenfunction of X^k corresponding to this eigenvalue.

(3) The set of the X^k $(k = 1, 2, 3)$ is a 3-vector operator.

(4) Localization is Galilean-invariant in real space-time[3] in the sense that if a particle looks localized in a region $\mathcal{R}$ of real space-time when seen from an inertial frame of reference, and if a homogeneous continous Galilean transformation is made in such a way that $\mathcal{R}$ is left invariant, then the particle also looks localized in $\mathcal{R}$ when seen from the new inertial frame. A region of real space-time is invariant under a homogeneous Galilean transformation if the region is invariant under the corresponding Galilean transformation of non-quantum Galilean relativity.

(5) The space-translation operators transform localized states into new localized states, but they only translate the real part of position, leaving invariant the pure imaginary part.

(6) For fixed $\vec{p}$, the set of eigenfunctions of a component X^k of position at a given point $a^k + ib^k$ contains a basis of $\mathcal{E}^{(s)}$.

III. REMARKS AND AUXILIARY FORMULAS (FOR SPIN 0 AND $\frac{1}{2}$)

The postulates stated in Sec. II are basically the same as those stated in II. The unique difference will be found in postulate (4), in which, here, we do not impose the Lorentz-invariance of the localized state but the Galilean invariance, as should be done for the nonrelativistic case. The physical discussion of the postulates is the same as in II and we do not reproduce it here.

It must be noted that in the usual treatment of nonrelativistic quantum mechanics, the notion of localization is not defined on a sound physical basis.

Further, one of the basic suppositions in that presentation, i.e., the commutation relation between the components of the position operator, $[X^i, X^k] = 0$, has no *a priori* justification. There does not exist experimental evidence, for small distances, for this relation,[12] and we know of 3-vectors whose components do not commute, e.g., the angular momentum components; furthermore, in relativistic quantum theories the position operators do not always have commuting components.[3,4,13-17] With our set of postulates, we restate the notion of localization on a more physical basis, giving a precise definition of it without imposing avoidable requirements.

Hamermesh[18] defined the position operator for nonrelativistic quantum mechanics in terms of certain transformation rules of mean values. He imposed stronger requirements for the position operator than those used in the present paper. He found a solution, which is the same as the one we obtained, but did not discuss its uniqueness.

Inönü and Wigner,[19] using another set of postulates for localization and the vectorial representations of the Galilei group, arrived at the conclusion that the position operator does not exist for them, and that the vectorial representations of the Galilei group are not the physical representations. Considering localization in regions, and using a set of postulates similar to those of Inönü and Wigner, together with the projective representations of the Galilei group, Wightman[20] found that the position operator exists for them and that it is unique. In contrast with our work, Inönü and Wigner assumed as did Wightman, that the components of the position operator commute. One of the basic axioms used in the Inönü-Wigner and Wightman papers can be expressed as the requirement of physical consistency of the description of localization by observers in frames related by a three-dimensional rotation. However, in a nonrelativistic theory a similar requirement should be satisfied in all inertial frames, and that is the content of our postulate (4).

Let $\vec{X}$ be the *position three-vector* and $\varphi_{ab\nu}$ a 1-*localized state,*

$$X^3\varphi_{ab\nu} = (a^3 + ib^3)\varphi_{ab\nu}, \quad a \equiv a^3, \quad b \equiv b^3 \qquad \text{(III.1)}$$

where a and b are real numbers and ν stands for the remaining degeneracy; in order to work in a more general case we assume complex eigenvalues.[3,6,21-23] As in II, all 1-localized states can be obtained from $\varphi_{ab\nu}$ by the action of the Galilei group, so it is sufficient to find $\varphi_{0b\nu}$.

In II it was shown that the components of the three-vector operator $\vec{X}$ can be written[24]

$$X^k = (i\hbar\partial_k + R^k), \qquad \text{(III.2)}$$

where $\vec{R}$ is a three-vector which only depends on $\vec{p}$, and on the matrices of the theory – that is, $\vec{R} = \vec{R}(\vec{p}, \vec{\Gamma})$, where the components of $\vec{\Gamma}$ are zero for the spin-0 case and the Pauli matrices for the spin-$\frac{1}{2}$ case; $\vec{R}$ can also depend on the constants of the theory.

As regards real space-time,[3] the original state $\varphi_{0b\nu}$ is localized in the two-dimensional plane $\mathcal{R}$ specified by making the third spatial coordinate and the time equal to zero. By postulate (4), $\mathcal{R}$ remains invariant under the pure Galilean transformation

$$\vec{x}' = \vec{x} - \vec{v}t, \quad t' = t. \qquad \text{(III.3)}$$

Using the physical, i.e., the projective representations of the Galilei group,[11] we know how the localized state $\varphi_{0b\nu}(\vec{p}, \xi)$ transforms under Eqs. (III.3)[11] into another localized state in the same region $\mathcal{R}$, that is,

$$\varphi_{0b\nu}(\vec{p}, \xi) \to \varphi_{0b\nu}(\vec{p} + m\vec{v}, \xi), \qquad \text{(III.4)}$$

where m is the mass of the system. Then, postulate (2) implies

$$X^3\varphi_{0b\nu}(\vec{p} + m\vec{v}, \xi) = (0 + ib_\lambda{}^3)\varphi_{0b\nu}(\vec{p} + m\vec{v}, \xi). \qquad \text{(III.5)}$$

λ is a real parameter; $b_\lambda{}^3$ is such a function of λ that $b_0{}^3 = b$ with $\lambda = \lambda(\vec{v})$. By Eqs. (III.2) and (III.5), and calling $\vec{u} = m\vec{v}$, we have

$$[i\hbar\partial_3 + R^3(\vec{p}, \vec{\Gamma})]\varphi_{0b\nu}(\vec{p} + \vec{u}, \xi) = ib_\lambda{}^3\varphi_{0b\nu}(\vec{p} + \vec{u}, \xi). \qquad \text{(III.6)}$$

We now premultiply Eq. (III.6) by $e^{-\vec{u}\cdot\vec{\nabla}_{\vec{p}}}$; since

$$\varphi_{0b\nu}(\vec{p} + \vec{u}, \xi) = e^{\vec{u}\cdot\vec{\nabla}_{\vec{p}}}\varphi_{0b\nu}(\vec{p}, \xi),$$

the equation is converted into

$$e^{-\vec{u}\cdot\vec{\nabla}_{\vec{p}}}[R^3(\vec{p}, \vec{\Gamma}) - ib_\lambda{}^3]\varphi_{0b\nu}(\vec{p} + \vec{u}, \xi) + i\hbar\partial_3\varphi_{0b\nu}(\vec{p}, \xi) = 0. \qquad \text{(III.7)}$$

Taking Eq. (III.7) and combining it with its particularization for $\vec{u} = 0$, we obtain

$$[R^3(\vec{p}, \vec{\Gamma}) - ib_\lambda{}^3 + ib^3 - R^3(\vec{p} + \vec{u}, \vec{\Gamma})]\varphi_{0b\nu}(\vec{p} + \vec{u}, \xi) = 0. \qquad \text{(III.8)}$$

A solution of the form $\varphi_{0b\nu} = \delta(\ldots)$ for Eq. (III.8) is not allowed because $\varphi_{0b\nu}$ is independent of $\vec{u}$; this implies

$$R^3(\vec{p}, \vec{\Gamma}) - ib_\lambda{}^3 + ib^3 - R^3(\vec{p} + \vec{u}, \vec{\Gamma}) = 0. \qquad \text{(III.9)}$$

All that has been said above is valid for spin 0 and $\frac{1}{2}$.

Let us consider separately the spin-$\frac{1}{2}$ case. We have then two vectors $\vec{V}_{(i)}$, $i = 1, 2$, that form the maximal set of linearly independent three-vectors which are functions of $\vec{p}$ and of the matrices of the theory, i.e., $\vec{p}$ and $\vec{p}\times\vec{r}$; then we can prove, as in III, that

$$\vec{R}=\sum_i \hbar C_i\,|\vec{p}|^{-2}\vec{V}_{(i)}\,,\quad [C_i]=1,\quad i=1,2. \tag{III.10}$$

Equation (III.10) is also valid for spin 0 because then $\vec{V}_{(2)}=\vec{p}\times\vec{r}=0$.

IV. SPIN ZERO

In this case, we have by Eq. (III.10)

$$\vec{R}=C\hbar\,|\vec{p}|^{-2}\vec{p}\;;\quad [C]=1. \tag{IV.1}$$

Substituting Eq. (IV.1) into Eq. (III.9), we obtain

$$\hbar C\,|\vec{p}|^{-2}p^3-\hbar C\,|\vec{p}+\vec{u}|^{-2}(p^3+u^3)=ib_\lambda{}^3-ib^3. \tag{IV.2}$$

By differentiating Eq. (IV.2) with respect to p^1, we find

$$C[(p^1+u^1)(p^3+u^3)|\vec{p}+\vec{u}|^{-4}-p^1p^3|\vec{p}|^{-4}]=0. \tag{IV.3}$$

By choosing $p^3=-u^3$, we deduce that

$$C=0. \tag{IV.4}$$

Then, the components of the position operator are uniquely defined and are the standard ones

$$X^k=i\hbar\partial_k\,. \tag{IV.5}$$

This operator is Hermitian with respect to the scalar product (I.1); then

$$b_\lambda{}^3=b=0. \tag{IV.6}$$

V. SPIN $\frac{1}{2}$

In this case, we have by Eq. (III.10)

$$\vec{R}=\hbar C_1|\vec{p}|^{-2}\vec{p}+\hbar C_2|\vec{p}|^{-2}(\vec{p}\times\vec{\sigma}). \tag{V.1}$$

Substituting Eq. (V.1) into Eq. (III.9), we obtain

$$\begin{aligned}\hbar C_1|\vec{p}|^{-2}p^3-ib_\lambda{}^3+ib^3-\hbar C_1|\vec{p}+\vec{u}|^{-2}(p^3+u^3)&\\ +\hbar C_2[|\vec{p}|^{-2}p^1-|\vec{p}+\vec{u}|^{-2}(p^1+u^1)]\sigma^2&\\ -\hbar C_2[|\vec{p}|^{-2}p^2-|\vec{p}+\vec{u}|^{-2}(p^2+u^2)]\sigma^1&=0.\end{aligned} \tag{V.2}$$

Setting

$$A=\hbar C_1|\vec{p}|^{-2}p^3-ib_\lambda{}^3+ib^3-\hbar C_1|\vec{p}+\vec{u}|^{-2}(p^3+u^3),$$

$$B_i=\hbar C_2[|\vec{p}|^{-2}p^i-|\vec{p}+\vec{u}|^{-2}(p^i+u^i)],\quad i=1,2 \tag{V.3}$$

we multiply Eq. (V.2) by the operator $(A-B_1\sigma^2+B_2\sigma^1)$, and obtain

$$A=\pm(B_1{}^2+B_2{}^2)^{1/2}\,. \tag{V.4}$$

By differentiation of Eq. (V.4) with respect to p^1, and making then $p^3=u^3=0$, $p^2=u^2=0$, we find by choosing $u^1=p^1$ that

$$C_2=0. \tag{V.5}$$

Equation (V.2) is the same as for the spin-0 case and we have

$$C_1=0. \tag{V.6}$$

Thus, the components of the position operator are again uniquely defined as

$$X^k=i\hbar\partial_k\,. \tag{V.7}$$

Again it follows by the Hermiticity of X^k that

$$b_\lambda{}^3=b^3=0. \tag{V.8}$$

VI. CONCLUSIONS

Using the consistency of the description of localization from different inertial frames in relativistic quantum mechanics as the only basic assumption, unusual properties of the position operators have been found,[3,4] e.g., their non-Hermiticities and their noncommuting components for certain spins. Here we have found that, with the corresponding basic assumption in the nonrelativistic case, the position operator is uniquely defined for the spin-0 and spin-$\frac{1}{2}$ cases and that this is the usual Hermitian, commuting-components operator of the nonrelativistic quantum theory. Furthermore, in contrast with the usual treatments, we have, with the postulates stated in Sec. II, a physical, precise definition of localization which avoids the standard hypothesis.

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*Present address: IVIC, Sección Física, Apartado 1827, Caracas, Venezuela.

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Simple Derivation of Seagull Terms for Propagator Functions*

D. A. Nutbrown
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 19 February 1971)

We demonstrate how a careful derivation of the Lehmann-Källén representation for the propagator function of any two operators, each with an arbitrary number of Lorentz indices, naturally gives rise to frame-dependent pieces. We verify (a) that the Bjorken limit is satisfied, (b) that a previously proposed partial differential equation connecting the seagull and Schwinger terms is also satisfied, and (c) that covariantized propagator functions may be defined such that Feynman's hypothesis is satisfied. Finally, we give some concrete examples.

Since the original observations[1-3] that the time-ordered product of two currents is not, in general, a covariant object, there have been many papers devoted to the construction of covariantized time-ordered products of several operators, together with the associated "seagull" and Schwinger terms. The investigations of some authors[4-6] have dealt with this problem from the field-theoretic standpoint. In this approach one constructs time-ordered products by considering the response of the system to perturbations of an external gauge field. On the other hand, Dashen and Lee[7] have constructed time-ordered products for any number of conserved currents without appeal to a gauge principle. More general studies[8] have also been carried out, for example, for the case of nonconserved currents.[9]

In this paper we consider the vacuum expectation value of the time-ordered product of any two Hermitian operators $A(x)$ and $B(x)$, each carrying an arbitrary number of Lorentz indices. We assume that both $A(x)$ and $B(x)$ may be obtained from some operator $\Theta(x)$ by taking divergences with respect to a subset of Lorentz indices and contracting over another disjoint subset. The advantage of this latter restriction, by virtue of the positive-definite spectral functions arising in the propagator function of Θ, lies in the fact that the seagull and Schwinger terms we find are necessarily nonzero.[10]

By inserting a complete set of intermediate states into the time-ordered product, we see how the frame-dependent seagull terms arise naturally, for essentially kinematical reasons.[3] The situation is reminiscent of the ambiguity involved in approximating form factors by sums over intermediate single-particle states, where the approximation obtained by means of a dispersion relation differs from that obtained by Feynman diagram methods by nonpole terms. Such nonpole terms correspond exactly to the seagull terms arising in propagator functions.

The analysis allows us, given the spectral representation of either the time-ordered product or one of its covariantized versions, to construct the seagull and Schwinger terms for operators $A(x)$ and $B(x)$ and their timelike derivatives. We should make clear at this point that, for convenience, we sometimes refer to the whole of the commutator of two operators as a Schwinger term. The piece