Splitting of the Maxwell Tensor. II. Sources*

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The sources of the bound tensor $T_{\rm I}^{\mu\nu}$ and the emitted tensor $T_{\rm II}^{\mu\nu}$ introduced by the author in his treatment of radiation reaction without advanced fields are investigated. It is found that both parts of the energy tensor have a δ-function-type source on the world line of the charged particle. The strength of the source is, in each case, the rate of change of the corresponding four-momentum. This is in sharp contrast with the situation in the advanced-field approach where the radiation field is source free and one has emission but not emitter. A very simple derivation of the Lorentz-Dirac equation based on the above results is presented.

IN 1938 Dirac[1] introduced advanced fields to study the problem of radiation reaction in classical electrodynamics. One of the main reasons for doing so was "...providing a definite value for the field of radiation throughout space-time, which is an advantage in giving us a meaning for the radiation field close to the electron. (The usual theory gives this field inextricably mixed with the Coulomb field.) . . . ." However, the advanced-field approach has serious difficulties of physical interpretation. As noticed by Dirac himself, "... it gives a meaning for the radiation field before its time of emission, when it can have no physical significance." This is not the only trouble. The radiation field to which reference is made in the above quotations does not have a source on the world line. Consequently, the part of the energy tensor describing radiation does not have a source on the particle either. One is faced with the paradoxical situation of having emission but no emitter.[2]

Later the advanced fields reached their highest expression in the beautiful action-at-a-distance theory of Wheeler and Feynman[3] where the above-mentioned difficulties do not appear. The source-free field becomes an external field (field of the absorber). The problem of having no emitter does not arise because there is no emission either. The physical picture of "something leaving the charge" is completely abandoned. There is no energy density in the field and consequently there is no energy-momentum tensor.

However, the difficulties in question can also be solved in the framework of classical, "orthodox," retarded electrodynamics.[4] In fact the velocity and acceleration fields are not as mixed as it could seem at first sight. If the retarded Liénard-Wiechert field is decomposed into a velocity part $F_{\rm I}^{\mu\nu}$ and an acceleration part $F_{\rm II}^{\mu\nu}$ (both retarded), the following equations were found[5] in I to hold off the world line of the charge[6]:

$$\partial^\lambda F_{\rm I,II}^{\mu\nu}+\partial^\mu F_{\rm I,II}^{\nu\lambda}+\partial^\nu F_{\rm I,II}^{\lambda\mu}=0\,, \qquad (1)$$

$$\partial_\mu F_{\rm I}^{\mu\nu}(x)=-\partial_\mu F_{\rm II}^{\mu\nu}(x)=(2e/\rho^3)a_r r^\nu. \qquad (2)$$

Equation (1) shows that the I and II fields are only partially mixed because there are separate four-potentials for each of them. But if the attention is focused on the energy and momentum contained in the fields more than on the fields themselves, the dynamical disentanglement becomes complete. The energy-momentum tensor of the Liénard-Wiechert field splits as

$$T^{\mu\nu}=T_{\rm I}^{\mu\nu}+T_{\rm II}^{\mu\nu}\,, \qquad (3)$$

with

$$\partial_\nu T_{\rm I}^{\mu\nu}=0\,,\quad \partial_\nu T_{\rm II}^{\mu\nu}=0 \qquad (4)$$

off the world line. The tensor $T_{\rm II}^{\mu\nu}$, which is the energy-momentum tensor of the acceleration field only, was identified in I without any ambiguity whatsoever as corresponding to what must be called energy-momentum emitted by the charge. This provides a definition of the energy and momentum of radiation (which is what is physically important) as close to the charge as desired. There are no problems with fields defined before their instant of emission because everything is retarded. On the other hand $T_{\rm I}^{\mu\nu}$, which contains the energy of the pure I field as well as the energy of interference between the I and II fields, corresponds to the part of the electromagnetic energy that is not emitted by, but remains bound to, the charge.

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[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).
[2] There is also another problem, closely related to this one. The energy-momentum balance is obscure because the "radiation damping" term in the equation of motion differs from the negative of the emission rate by the so-called Schott term. It has been shown in Ref. 5 how this problem can be solved in terms of retarded fields only.
[3] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 15 (1945); **21**, 425 (1949).
[4] With the exception of the self-energy problem that does not appear in an action-at-a-distance theory but unavoidably arises in any approach totally based on an energy-momentum tensor.
[5] C. Teitelboim, Phys. Rev. D **1**, 1572 (1970); **2**, 1763 (E) (1970), hereafter referred to as I. The same conventions and notation used in I are adopted in this note.
[6] As we said in I, Eq. (1) can be proved straightforwardly by the standard techniques of retarded differentiation. However, the proof is slightly cumbersome, and it is easier to verify that the fields $F_{\rm I}^{\mu\nu}$ and $F_{\rm II}^{\mu\nu}$ can be obtained as the curl of the following four-potentials: $A_{\rm I}^\mu=er^\mu/\rho^2$ and $A_{\rm II}^\mu=e\{\rho[v^\mu]-r^\mu\}/\rho^2$. It is curious to notice that the four-potential $A_{\rm I}^\mu$ of the bound field is a null vector whereas the four-potential $A_{\rm II}^\mu$ of the emitted field is spacelike and orthogonal to the retarded four-velocity. These two four-potentials are not divergence-free. In fact it is easily seen that $\partial_\mu A_{\rm I}^\mu=-\partial_\mu A_{\rm II}^\mu=e/\rho^2$. On the other hand, the total four-potential $A^\mu=A_{\rm I}^\mu+A_{\rm II}^\mu=e[v^\mu]/\rho$ is timelike and satisfies the Lorentz condition $\partial_\mu A^\mu=0$.

Equations (4) are valid only off the world line of the charge. If the difficulty of having emission without emitter is going to be absent from our theory, the tensor $T_{\rm II}{}^{\mu\nu}$ should have a *nonvanishing divergence on the world line*. To find the source (in order to test its divergence), we apply Gauss's integral theorem in Minkowski space to the region enclosed inside a cylindrical tube that surrounds the part of the world line between proper times τ and $\tau+d\tau$, and let the radius of the tube go to zero. We have then

$$\int_{\substack{\text{four-volume}\\ \text{inside tube}}} \partial_\nu T_{\rm II}{}^{\mu\nu} d^4x = \int_{\substack{\text{surface}\\ \text{of tube}}} T_{\rm II}{}^{\mu\nu} u_\nu d^3\sigma\,, \qquad (5)$$

where u^ν is the normal pointing out of the tube. But it was shown in I that

$$\int_{\substack{\text{surface}\\ \text{of tube}}} T_{\rm II}{}^{\mu\nu} u_\nu d^3\sigma = -\tfrac{2}{3}e^2a^2(\tau)v^\mu(\tau)d\tau\,, \qquad (6)$$

where terms vanishing with the radius of the tube have been dropped. Introducing (6) into (5), we obtain

$$\partial_\nu T_{\rm II}{}^{\mu\nu} d^3x dz^0 = -\tfrac{2}{3}e^2a^2(\tau)v^\mu(\tau)d\tau\,,$$

which implies

$$\partial_\nu T_{\rm II}{}^{\mu\nu}(x) = -\delta^{(3)}(\mathbf{x}-\mathbf{z}(\tau))\tfrac{2}{3}e^2a^2(\tau)v^\mu(\tau)/v^0(\tau)\,,$$

with $z^0(\tau)=x^0$. This equation can be cast into a manifestly covariant form in the same way as the current $j^\mu(x)=e\delta^{(3)}(\mathbf{x}-\mathbf{z})v^\mu/v^0$, namely,

$$\partial_\nu T_{\rm II}{}^{\mu\nu}(x) = -\int_{-\infty}^{+\infty} d\tau\; \delta^{(4)}(x-z(\tau))\tfrac{2}{3}e^2a^2(\tau)v^\mu(\tau)\,. \qquad (7)$$

It is seen that the emitted tensor $T_{\rm II}{}^{\mu\nu}$ has a δ-function source on the world line as it should be for a point emitter.[7] Moreover, the strength of the source is precisely the energy-momentum emission rate. The emitted tensor is source free if and only if the charge is not radiating. This result is completely satisfactory from a physical point of view.

A similar analysis applied to the bound tensor $T_{\rm I}{}^{\mu\nu}$ gives the result

$$\partial_\nu T_{\rm I}{}^{\mu\nu} = -\int_{-\infty}^{+\infty} d\tau\; \delta^{(4)}(x-z(\tau))\big(m_{\rm Coul}a^\mu(\tau)-\tfrac{2}{3}e^2\dot{a}^\mu(\tau)\big)\,, \qquad (8)$$

where $m_{\rm Coul}=\lim_{\epsilon\to 0}e^2/2\epsilon$ is the divergent "Coulomb mass" of the point charge. The bound tensor $T_{\rm I}{}^{\mu\nu}$ also has, therefore, a δ-function source on the world line whose strength is precisely the rate of change of the bound electromagnetic energy and momentum.

It thus becomes apparent that even on the world line the emitted and bound parts of the energy tensor are far from being mixed. Their sources are obviously disentangled and the physical meaning of the source terms is clear.

As a last point, it is interesting to see how easily the equation of motion can be derived using the above results. The infinite mass in (8) can be handled by the usual classical mass-renormalization procedure. Thus the energy-momentum tensor $t^{\mu\nu}$ of the particle is written as the sum of the bound electromagnetic tensor $T_{\rm I}{}^{\mu\nu}$, and the mechanical or "bare" tensor

$$T_{\rm bare}{}^{\mu\nu} = -m_{\rm bare}\int_{-\infty}^{+\infty} d\tau\; \delta^{(4)}(x-z(\tau))v^\mu(\tau)v^\nu(\tau)\,,$$

with $m_{\rm bare}+m_{\rm Coul}=m$, the observed mass of the particle. It is easily verified that

$$\partial_\nu T_{\rm bare}{}^{\mu\nu} = -\int_{-\infty}^{+\infty} d\tau\; \delta^{(4)}(x-z(\tau))m_{\rm bare}a^\mu(\tau)$$

and, consequently, $t^{\mu\nu}=T_{\rm I}{}^{\mu\nu}+T_{\rm bare}{}^{\mu\nu}$, satisfies

$$\partial_\nu t^{\mu\nu} = -\int_{-\infty}^{+\infty} d\tau\; \delta^{(4)}(x-z(\tau))\big(ma^\mu(\tau)-\tfrac{2}{3}e^2\dot{a}^\mu(\tau)\big)\,. \qquad (9)$$

If the particle is under the action of an external four-force F^μ, the corresponding force density is

$$f^\mu(x) = \int_{-\infty}^{+\infty} d\tau\; \delta^{(4)}(x-z(\tau))F^\mu(\tau)$$

and the dynamics of the particle is governed by

$$\partial_\nu t^{\mu\nu} = -\partial_\nu T_{\rm II}{}^{\mu\nu} - f^\mu\,, \qquad (10)$$

which is equivalent to

$$\int_{-\infty}^{+\infty} d\tau\; \delta^{(4)}(x-z(\tau))\big(ma^\mu(\tau)-\tfrac{2}{3}e^2\dot{a}^\mu(\tau) + \tfrac{2}{3}e^2a^2(\tau)v^\mu(\tau) - F^\mu(\tau)\big) = 0\,.$$

The last equality holds *if and only if*

$$ma^\mu - \tfrac{2}{3}e^2\dot{a}^\mu = -\tfrac{2}{3}e^2a^2v^\mu + F^\mu\,,$$

which is the Lorentz-Dirac equation.

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[7] The minus signs in (7) and (8) as well as the one in front of f^μ in (10) are due to the fact that, with our conventions, $T^{\mu 0}$ is the negative of the energy-momentum density.