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Space-Time Symmetries and the Spontaneous Breakdown of Dilation Invariance*

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The assumption that the Poincaré generators are the space integrals of local conserved currents is used to derive restrictions on the spectral function of certain vacuum expectation values. Then an argument is given that these restrictions and the vanishing of the trace of the stress-energy tensor (conservation of dilation current) imply that all single-particle states have zero mass, even when the conservation of the current is implemented by a Goldstone boson.

I. INTRODUCTION

The purpose of this paper is to study two consequences of the existence of a local energy-momentum tensor, $\Theta_{\mu\nu}(x)$, in a relativistic quantum field theory.[1] First, we study in what sense the infinitesimal generators P_μ and $M_{\mu\nu}$ of the Poincaré group are given as space integrals of suitable components and some particular first moments of $\Theta_{\mu\nu}(x)$. This question is relevant. Normally one assumes that P_μ is the space integral of $\Theta_{0\mu}(x)$.[2] That this cannot be true in the sense of operator convergence is already known from the work of Reeh,[3] who has shown that, barring the trivial case $\Theta_{\mu\nu}(x)=0$, the norm of the vector $\Theta_{0\mu}(x)|0\rangle$ diverges as R, for R large. It will be shown in this paper that P_μ and $M_{\mu\nu}$ can be written as space integrals of corresponding densities in the sense of equality of matrix elements over a dense set of states. This can be proved from the assumed properties of $\Theta_{\mu\nu}(x)$, provided no zero-mass particles are present in the spectrum. If the latter are present, then for the equality to be still true, a number of further restrictions, in addition to the properties of $\Theta_{\mu\nu}(x)$, must be satisfied. These additional conditions are given as restrictions on certain "spectral functions" that occur in the representation of the vacuum expectation values of the commutator of $\Theta_{\mu\nu}(x)$ with certain polynomials in the smeared fields. Secondly, we study the dilation symmetry, introduced through a conserved dilation current [or, equivalently, via the tracelessness property of $\Theta_{\mu\nu}(x)$] and its spontaneous breaking. It is found that Goldstone's theorem is valid in this case. Spontaneous breakdown of dilation implies existence of zero-mass bosons. Arguments are then presented that spontaneous breakdown of dilation does not admit a situation in which some particle states remain massive, provided that translational invariance is not broken.

Recently there have been many discussions, for internal symmetries, about the connection between (global) "charge" operators and the space integral of corresponding densities.[3-6] A review of these results is given by Swieca.[7] We have found it possible to apply the techniques detailed in this last reference, *in toto*, to the present investigation.

Let us now make some introductory remarks concerning the nature of spontaneous breakdown of internal (nongeometric) symmetries. One introduces the symmetry through the local conservation $\partial^\mu J_\mu(x)=0$ of a current $J_\mu(x)$. One now defines

$$Q_R(x_0)=\int_{R>|\vec{x}|} d^3x\, J_0(x, x_0)$$

and assumes there exists some local Wightman polynomial A, such that

$$\lim_{R\to\infty}\langle 0|[Q_R(x_0), A]|0\rangle = B.$$

It is easily proved that B is a constant. The broken-symmetry condition is then $B\neq 0$.[8,9] We should emphasize that one uses the displacement property of $J_\mu(x)$, i.e.,

$$J_\mu(x)=e^{iP_\nu x^\nu}J_\mu(0)e^{-iP_\nu x^\nu}.$$

This generally is not the transformation property of a current that generates geometrical transformations. It should also be noted that whenever a symmetry is spontaneously broken, the corresponding "charge" is not a well-defined operator, i.e., $\lim_{R\to\infty}Q_R$ does not exist.[9]

In Sec. II we discuss the construction of Poincaré-group generators as space integrals of components of an energy-momentum tensor and its suitable moments. In Sec. III we discuss dilation symmetry and its breaking. In Sec. IV we make concluding remarks.

II. ENERGY-MOMENTUM TENSOR AND POINCARÉ-GROUP GENERATORS

In this section we consider the problem of constructing the infinitesimal generators of the Poincare group P_μ and $M_{\mu\nu}$ as integrals over the components of a local energy-momentum tensor and its suitable moments. This problem will be discussed within the framework of a local field theory satisfying the Wightman axioms.[1] The postulates of the latter theory include the existence of the operators P_μ, $M_{\mu\nu}$. What happens if we additionally require that P_μ and $M_{\mu\nu}$ be given as integrals over the components of an energy-momentum tensor $\Theta_{\mu\nu}(x)$ and its moments? As we will see in the following, a number of consistency conditions must be satisfied in order that this be possible. In the remainder of this section we will derive these conditions.

The energy-momentum tensor $\Theta_{\mu\nu}(x)$ will be required to have the following properties:

(i) Symmetry: $\Theta_{\mu\nu}(x)=\Theta_{\nu\mu}(x)$.

(ii) Divergenceless: $\partial^\mu\Theta_{\mu\nu}(x)=0$.

(iii) $\Theta_{\mu\nu}(x)$ is local with respect to the (local) basic fields $\Phi_i(x)$: $[\Theta_{\mu\nu}(x),\Phi_i(y)]=0$, $(x-y)^2<0$.

Let us consider a local Wightman polynomial A associated with a finite space-time region Θ:

$$A=\sum_{n=0}^{m}\int f_n(x_1,x_2,\ldots,x_n)\Phi_{i_1}(x_1)\times\Phi_{i_2}(x_2)\cdots\Phi_{i_n}(x_n)d^4x_1d^4x_2\cdots d^4x_n\,, \tag{1}$$

where $f_n(x_1,\ldots,x_n)$ are infinitely differentiable functions of compact support with support in Θ. The assumed locality of $\Theta_{\mu\nu}(x)$ relative to $\Phi_i(x)$ enables us to use a representation due to Araki, Hepp, and Ruelle[10] for the vacuum expectation value of the commutator

$$\langle 0|[\Theta_{0\nu}(x),A]|0\rangle=\int_0^\infty d\mu^2\int d^3y\,\Delta(x-y,x_0,\mu^2)\rho_{1\nu}(\mu^2,y)+\int_0^\infty d\mu^2\int d^3y\frac{\partial}{\partial x_0}\Delta(x-y,x_0,\mu^2)\rho_{2\nu}(\mu^2,y). \tag{2}$$

In Eq. (2), $\Delta(x,x_0,\mu^2)$ is the Pauli-Jordan function for a scalar field of mass μ, and $\rho_{1\nu}$ and $\rho_{2\nu}$ are measures in μ^2 having compact support in y as a consequence of local commutativity. Following Ezawa and Swieca[11] we write

$$\left.\begin{aligned}\rho_{1\nu}(\mu^2,y)&=\bar\rho_{1\nu}(\mu^2)\delta^3(y)+\frac{\partial}{\partial y_i}\sigma^i_{1\nu}(\mu^2,y),\\ \rho_{2\nu}(\mu^2,y)&=\bar\rho_{2\nu}(\mu^2)\delta^3(y)+\frac{\partial}{\partial y_i}\sigma^i_{2\nu}(\mu^2,y),\end{aligned}\right\}\; i=1,2,3 \tag{3}$$

where $\sigma^i_{1\nu}$ and $\sigma^i_{2\nu}$ are also functions of compact support in y and repeated indices imply summation. The divergenceless condition $\partial^\mu\Theta_{\mu\nu}(x)=0$ implies

$$\frac{d}{dx_0}\left\langle 0\left|\left[\int_{R>|\vec{x}|}d^3x\,\Theta_{0\nu}(x),A\right]\right|0\right\rangle=0 \quad \text{for } R>\text{some } R_0. \tag{4}$$

From Eqs. (2), (3), and (4) we get

$$\int_0^\infty d\mu^2\bar\rho_{1\nu}(\mu^2)\cos(x_0\mu)=0, \tag{5a}$$

$$\int_0^\infty d\mu^2\bar\rho_{2\nu}(\mu^2)\mu\sin(x_0\mu)=0, \tag{5b}$$

which implies

$$\bar\rho_{1\nu}(\mu^2)=0,\quad \bar\rho_{2\nu}(\mu^2)=\lambda_\nu\delta(\mu^2). \tag{6}$$

No further restrictions follow from the assumed properties of $\Theta_{\mu\nu}(x)$.

We are now in a position to investigate the problem stated in the beginning of this section. We start with the problem of constructing P_μ out of $\Theta_{0\mu}(x)$. First, let

$$\Theta_{0\mu}(f_R)=\int\Theta_{0\mu}(x)f_R(x)d^3x \tag{7}$$

with $f_R(x)$ a smooth function satisfying

$$\begin{aligned}f_R(x)&=1,\quad |\vec{x}|<R\\ f_R(x)&=0,\quad |\vec{x}|>R+\epsilon.\end{aligned} \tag{8}$$

In which sense now is $P_\mu=-\lim_{R\to\infty}\Theta_{0\mu}(f_R)$? Following the argument discussed by Swieca,[7] one can define an operator P_μ over a dense set of states obtained by applying A on vacuum by means of the relation

$$P_\mu A|0\rangle=-[\Theta_{0\mu}(f_R),A]|0\rangle,\quad R>R_0. \tag{9}$$

In order that P_μ defined above be identifiable with the energy-momentum operator, it is necessary that the following condition be satisfied:

$$\lim_{R\to\infty}\langle 0|\Theta_{0\mu}(f_R)A|0\rangle=0. \tag{10}$$

To see the validity of the above statement in more detail, we note first that Eq. (10) implies

$$\lim_{R\to\infty}\langle 0|[\Theta_{0\mu}(f_R),A]|0\rangle=0, \tag{11}$$

which is, in view of Eq. (9), an obvious requirement (displacement invariance of vacuum) on the energy-momentum operator. Secondly, Eqs. (9) and (10) imply

$$\begin{aligned}\langle 0|B\Theta_{0\mu}(f_R)A|0\rangle&=\langle 0|B[\Theta_{0\mu}(f_R),A]|0\rangle\\ &=-\langle 0|BP_\mu A|0\rangle\end{aligned} \tag{12}$$

for arbitrary local polynomials A and B. Thus the equality

$$P_\mu = -\lim_{R\to\infty} \Theta_{0\mu}(f_R)$$

is true in the sense of matrix elements taken between states formed by the application of local polynomials on vacuum (local states). When can Eq. (10) be true? First we note from Eqs. (2), (6), and (8) that

$$\lim_{R\to\infty}\langle 0|[\Theta_{0\nu}(f_R), A]|0\rangle = \frac{\lambda_\nu}{(2\pi)^3 i}\int d^3x \int_0^\infty d\mu^2 \delta(\mu^2)\frac{\partial}{\partial x_0}\Delta(x, x_0, \mu^2)$$

$$= \lambda_\nu \int_0^\infty d\mu^2 \delta(\mu^2)\cos(x_0\mu) = \lambda_\nu. \tag{13}$$

Hence, Eq. (11), which is a consequence of Eq. (10), requires that

$$\lambda_\nu = 0. \tag{14}$$

Further, from Eqs. (2), (10), and (14) it now follows that

$$\int d^3x \int_0^\infty d\mu^2 \int d^3y\, \Delta^{(+)}(x-y, x_0, \mu^2)\partial_i \sigma^i_{1\nu}(\mu^2, y) + \int d^3x \int_0^\infty d\mu^2 \int d^3y \frac{\partial}{\partial x_0}\Delta^{(+)}(x-y, x_0, \mu^2)\partial_i \sigma^i_{2\nu}(\mu^2, y) = 0. \tag{15}$$

In Eq. (15), $\Delta^{(+)}(x, x_0, \mu^2)$ is the positive-frequency part of the Pauli-Jordan function. By partial integration Eq. (15) can be reduced to

$$\int d^3x \int_0^\infty d\mu^2 \int d^3y\, \sigma^i_{1\nu}(\mu^2, y)\frac{\partial}{\partial y_i}\Delta^{(+)}(x-y, x_0, \mu^2) + \int d^3x \int_0^\infty d\mu^2 \int d^3y\, \sigma^i_{2\nu}(\mu^2, y)\frac{\partial}{\partial x_0}\frac{\partial}{\partial y_i}\Delta^{(+)}(x-y, x_0, \mu^2) = 0. \tag{16}$$

Introducing the Fourier transform of $\sigma^i_{a\nu}(\mu^2, y)$,

$$\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{y}}\sigma^i_{a\nu}(\mu^2, y)d^3y = \tilde{\sigma}^i_{a\nu}(\mu^2, k), \quad a = 1, 2 \tag{17}$$

Eq. (16) can be written as

$$\int_0^\infty d\mu^2 \int d^3k\, k_i \delta^3(k)\tilde{\sigma}^i_{1\nu}(\mu^2, k)\int_{-\infty}^{+\infty} dk_0 e^{ik_0x_0}\theta(k_0)\delta(k_0^2 - |\vec{k}|^2 - \mu^2)$$

$$+ i\int_0^\infty d\mu^2 \int d^3k\, k_i \delta^3(k)\tilde{\sigma}^i_{2\nu}(\mu^2, k)\int_{-\infty}^{+\infty} dk_0 e^{ik_0x_0}k_0\theta(k_0)\delta(k_0^2 - |\vec{k}|^2 - \mu^2) = 0. \tag{18}$$

Carrying out the k_0 integration, Eq. (18) becomes

$$\int_0^\infty d\mu^2 \int d^3k\, k_i \tilde{\sigma}^i_{1\nu}(\mu^2, k)\delta^3(k)\frac{1}{(|\vec{k}|^2 + \mu^2)^{1/2}}\exp[ix_0(|\vec{k}|^2 + \mu^2)^{1/2}]$$

$$+ i\int_0^\infty d\mu^2 \int d^3k\, k_i \tilde{\sigma}^i_{2\nu}(\mu^2, k)\delta^3(k)\exp[ix_0(|\vec{k}|^2 + \mu^2)^{1/2}] = 0. \tag{19}$$

The second term on the left-hand side of Eq. (19) is clearly zero, while the first term behaves as

$$\int_0^\infty d\mu^2 e^{ix_0\mu}\left[\tilde{\sigma}^i_{1\nu}(\mu^2, k)\frac{k_i}{(|\vec{k}|^2 + \mu^2)^{1/2}}\right]_{\vec{k}=0}.$$

The expression inside the square bracket is zero for $\mu \neq 0$, but not necessarily so otherwise. Hence from Eqs. (2), (6), and Eqs. (9)–(19) we conclude the following: (1) If there are no zero-mass neutral particles present in the theory, then Eq. (10) is automatically satisfied due to properties of $\Theta_{\mu\nu}(x)$, and consequently P_μ can be constructed as space integral of $-\Theta_{0\mu}(x)$ in the sense of Eq. (12). (2) If zero-mass neutral particles are present in the spectrum, then the assumed properties of $\Theta_{\mu\nu}(x)$ are not sufficient to guarantee Eq. (12) and one now further needs the restrictions

$$\lambda_\nu = 0, \quad \tilde{\sigma}^i_{1\nu}(\mu^2, 0) = 0. \tag{20a}$$

However, it will be proved later that $\tilde{\sigma}^i_{10}(\mu^2, 0) = 0$ irrespective of the presence of zero-mass neutral particles [see Eq. (33) below]. Thus, in case zero-mass neutral particles are present, one need only to assume

$$\tilde{\sigma}^i_{1j}(\mu^2, 0) = 0, \quad j = 1, 2, 3 \tag{20b}$$

which must be treated as an additional postulate of the theory.

Let us now consider the homogeneous Lorentz-group operators. Start with space-rotation operators and let

$$M_{ij}(f_R)=\int[x_i\Theta_{0j}(x)-x_j\Theta_{0i}(x)]f_R(x)d^3x,\quad i,j=1,2,3. \tag{21}$$

From the properties of $\Theta_{\mu\nu}(x)$ we have

$$\frac{d}{dx_0}\langle 0|[M_{ij}(f_R),A]|0\rangle=0,\quad R>R_0. \tag{22}$$

Using Eqs. (2), (6), (20a), and (22), we get

$$\int_0^\infty d\mu^2\sin(x_0\mu)\,\mu[\tilde\sigma^i_{2j}(\mu^2,0)-\tilde\sigma^j_{2i}(\mu^2,0)]=0. \tag{23}$$

The solution of the above equation is

$$\tilde\sigma^i_{2j}(\mu^2,0)=\beta^i_{2j}(\mu^2)+c^i_j\delta(\mu^2),\quad \beta^i_{2j}(\mu^2)=\beta^j_{2i}(\mu^2). \tag{24}$$

As before, we define

$$M_{ij}A|0\rangle=[M_{ij}(f_R),A]|0\rangle. \tag{25}$$

Repeating the argument of the previous paragraphs, we can now establish from Eqs. (23) and (24) that, in the absence of zero-mass particles in the spectrum,

$$\langle\Phi|M_{ij}|\Psi\rangle=\lim_{R\to\infty}\langle\Phi|M_{ij}(f_R)|\Psi\rangle \tag{26}$$

for $|\Phi\rangle$ and $|\Psi\rangle$ arbitrary local states. If zero-mass particles are present, then in order for Eq. (26) to be true one needs, in addition, the following conditions:

$$c^i_j=c^j_i, \tag{27}$$

$$\lim_{k\to 0}[\tilde\sigma^i_{1j}(\mu^2,k)-\tilde\sigma^j_{1j}(\mu^2,k)]\to k^n,\quad n>1,$$
$$\left(\frac{\partial\tilde\sigma^l_{1i}}{\partial k^j}\right)_{\vec k=0}=\left(\frac{\partial\tilde\sigma^l_{1j}}{\partial k^i}\right)_{\vec k=0}. \tag{28}$$

Turning our attention now to the remaining HLG operators, let us first define

$$M_{0i}(x)=x_0\Theta_{0i}(x)-x_i\Theta_{00}(x) \tag{29}$$

and let $M_{0i}(f_R)$ denote

$$M_{0i}(f_R)=\int M_{0i}(x)f_R(x)d^3x. \tag{30}$$

Now

$$\frac{d}{dx_0}\langle 0|[M_{0i}(f_R),A]|0\rangle=2\langle 0|[\Theta_{0i}(f_R),A]|0\rangle$$
$$-\left\langle 0\left|\left[\int f_R(x)\frac{\partial}{\partial x_j}[x_i\Theta_{0j}(x)]d^3x,A\right]\right|0\right\rangle$$
$$=2\langle 0|[\Theta_{0i}(f_R),A]|0\rangle,\quad R>R_0.$$

Hence, if we assume Eq. (14), then

$$\frac{d}{dx_0}\langle 0|[M_{0i}(f_R),A]|0\rangle=0,\quad R>R_0. \tag{31}$$

Equation (31), in conjunction with Eqs. (2) and (6), yields

$$\int_0^\infty d\mu^2\tilde\sigma^i_{10}(\mu^2,0)\cos(x_0\mu)=0, \tag{32a}$$

$$\int_0^\infty d\mu^2\tilde\sigma^i_{20}\mu\sin(x_0\mu)=0. \tag{32b}$$

From the above we get

$$\tilde\sigma^i_{10}(\mu^2,0)=0, \tag{33}$$

$$\tilde\sigma^i_{20}(\mu^2,0)=d^i\delta(\mu^2). \tag{34}$$

Thus we have seen that Eq. (33) is true, irrespective of any requirement on the spectrum, as was asserted before. We now define, as before,

$$M_{0i}A|0\rangle=[M_{0i}(f_R),A]|0\rangle,\quad R>R_0. \tag{35}$$

Once again we see from Eqs. (2), (6), (33), and (34) that, in the absence of zero-mass neutral particles in the theory,

$$\langle\Phi|M_{0i}|\Psi\rangle=\lim_{R\to\infty}\langle\Phi|M_{0i}(f_R)|\Psi\rangle \tag{36}$$

for arbitrary local states $|\Phi\rangle$ and $|\Psi\rangle$. If zero-mass particles are present, then for Eq. (36) to be true, one needs the following additional restrictions:

$$d^i=0, \tag{37}$$

$$\lim_{k\to 0}\tilde\sigma^i_{10}(\mu^2,k)\to k^n,\quad n>1 \tag{38}$$

$$\left.\frac{\partial\tilde\sigma^j_{10}(\mu^2,k)}{\partial k^i}\right|_{\vec k=0}=0. \tag{39}$$

To summarize: We have proved that the generators of the Poincaré group can be written as space integrals of corresponding densities in the sense of equality of matrix elements between a dense set of local states, in the absence of zero-mass neutral particles. If the latter are present, one needs additional restrictions. Three of these, namely Eqs. (14), (27), and (37), guarantee that

$$\langle 0|[\Theta_{0\mu}(f_R),A]|0\rangle=0,\quad \langle 0|[M_{\mu\nu}(f_R),A]|0\rangle=0,\quad R>R_0, \tag{40}$$

and hence are very reasonable from physical viewpoints. They merely state that the Poincaré-group symmetry is *not* being spontaneously broken. The remaining restrictions, viz., Eqs. (20b), (28), (38), and (39) are more technical in nature. In fact, we cannot be absolutely sure at this stage that these latter can always be consistently imposed. In any case, one can now extend the validity of the above results for local states given by Eqs. (12), (26), and (36) to more general states formed by the application of quasilocal Wightman polynomials to vacuum (quasilocal states). Let us consider quasilocal polynomials, i.e., those of the form

$$P=\sum_{n=0}^{m}\int g_n(x_1,\ldots,x_n)\Phi_{i_1}(x_1)\cdots\Phi_{i_n}(x_n)d^4x_1\cdots d^4x_n\,,$$

where $g_n(x_1,\ldots,x_n)$ are infinitely differentiable functions which, together with their derivatives, approach zero at infinity faster than any power of the Euclidean distance (s-class functions). Since for any P there exists an A_R such that, for any N,

$$\|P|0\rangle-A_R|0\rangle\|<\alpha|R^N, \tag{41}$$

and since it is further known[3] that

$$\|\Theta_{0\mu}(f_R)|0\rangle\|<\beta R^{3/2} \tag{42}$$

as $R\to\infty$, it follows that

$$\lim_{R\to\infty}\langle 0|(P-A_R)\Theta_{0\mu}(f_R)|0\rangle=0. \tag{43}$$

Hence, from Eq. (10) it follows that

$$\lim_{R\to\infty}\langle 0|P\Theta_{0\mu}(f_R)|0\rangle=0. \tag{44}$$

From Eq. (44) it follows, by a repetition of the argument made earlier for local polynomials, that Eq. (12) is true for quasilocal states. In an analogous manner one proves the same for Eqs. (26) and (36).

III. DILATION INVARIANCE AND ITS SPONTANEOUS BREAKDOWN

We can formulate dilation invariance in terms of a local conservation law,[12,13]

$$\partial^\mu J_\mu(x)=0,\quad J_\mu(x)=x^\nu\Theta_{\mu\nu}(x). \tag{45}$$

Obviously Eq. (45) implies the further restriction that the energy-momentum tensor be traceless,

$$\Theta_\mu{}^\mu(x)=0. \tag{46}$$

Let

$$J_0(f_R)=\int f_R(x)J_0(x)d^3x. \tag{47}$$

Then it follows from Eq. (45) that

$$\frac{d}{dx_0}\langle 0|[J_0(f_R),A]|0\rangle=0,\quad R>R_0. \tag{48}$$

What can we say about spontaneous breakdown of dilation invariance if we assume Eq. (10), i.e., that P_μ can be written as space integral of corresponding densities over local states? This section will be devoted to answering this question. Equation (48), together with Eqs. (2), (6), and (20a), yields

$$\int_0^\infty d\mu^2\tilde\sigma^i_{2i}(\mu^2,0)\mu\sin(x_0\mu)=0. \tag{49}$$

Hence,

$$\tilde\sigma^i_{2i}(\mu^2,0)=c\,\delta(\mu^2). \tag{50}$$

In the above, repeated indices imply summation: $\tilde\sigma^i_{2i}=\tilde\sigma^1_{21}+\tilde\sigma^2_{22}+\tilde\sigma^3_{23}$. From Eqs. (2), (6), (20a), and (50) we see that

$$\lim_{R\to\infty}\langle 0|[J_0(f_R),A]|0\rangle=c. \tag{51}$$

Thus if $c\neq 0$, then $\langle 0|[J_0(f_R),A]|0\rangle\neq 0$ for $R>R_0$, and we have a spontaneous breakdown of dilation symmetry. Equation (50) now is the statement of the Goldstone theorem for dilation breakdown. In other words, if we have spontaneous breakdown of dilation $c\neq 0$, then zero-mass particles are necessarily present in the spectrum and the current $J_\mu(x)$ connects these particles to the vacuum.[14] It has been stated in the literature that the Goldstone boson corresponding to dilation breakdown is *massive*.[15] This last statement is false in a relativistic field theory with local commutativity.[16] We also note that because of $c\neq 0$, the method used in the previous section to construct global operators as space integrals of corresponding operator densities will not work. In fact, in the present case the limit of $J_0(f_R)$ as $R\to\infty$ does not exist even between local states. Hence a global dilation operator does not exist at all.

Denoting hereafter $J_0(f_R)$ by D_R, we will now prove that even when dilation is spontaneously broken ($c\neq 0$), the following commutation relation,

$$\lim_{R\to\infty}[D_R,P_\mu]=iP_\mu, \tag{52}$$

is true, in the sense of equality of matrix elements evaluated between quasilocal states. The limit $R\to\infty$ on the left-hand side of Eq. (52) is to be taken after taking the commutator. To derive Eq. (52) we proceed as follows. Because of the displacement property of $\Theta_{0\nu}(x)$,

$$[\Theta_{0\nu}(x),P_\mu]=i\partial_\mu\Theta_{0\nu}(x), \tag{53}$$

it follows that

$$\begin{aligned}[x^\nu\Theta_{0\nu}(x),P_\mu]&=ix^\nu\partial_\mu\Theta_{0\nu}(x)\\&=i\partial_\mu(x^\nu\Theta_{0\nu}(x))-i\Theta_{0\mu}(x).\end{aligned} \tag{54}$$

From Eq. (54), we get the double commutator

$$\int_{|\vec x|<R}d^3x\,[[x^\nu\Theta_{0\nu}(x),P_\mu],A]=i\int_{|\vec x|<R}d^3x\,[\partial_\mu(x^\nu\Theta_{0\nu}(x)),A]-i\int_{|\vec x|<R}d^3x\,[\Theta_{0\mu}(x),A]; \tag{55}$$

but because $x^{\nu}\Theta_{0\nu}(x)$ is the 0th component of a conserved vector current and $\Theta_{0\nu}(x)$ is local relative to the basic fields, the first term on the right-hand side of Eq. (55) is zero, for R greater than (some) R_0. Thus Eq. (55) reduces to

$$[\,[D_R, P_\mu], A\,] = -i[\Theta_{0\mu}(f_R), A], \quad R > R_0. \tag{56}$$

From Eq. (56) we obtain, for arbitrary local polynomials B and C,

$$\begin{aligned}\langle 0|B[\,[D_R, P_\mu], C\,]|0\rangle &= -i\langle 0|B[\Theta_{0\mu}(f_R), C\,]|0\rangle \\ &= i\langle 0|BP_\mu C|0\rangle .\end{aligned} \tag{57}$$

Hence

$$\langle 0|B[D_R, P_\mu]C|0\rangle = i\langle 0|BP_\mu C|0\rangle + \langle 0|BC[D_R, P_\mu]|0\rangle, \quad R > R_0. \tag{58}$$

Thus Eq. (52) will be true in the sense of equality of matrix elements between local states, obtained by the application of B and C on vacuum, provided the second term on the right-hand side of Eq. (58) vanishes in the limit $R\to\infty$. Thus, to complete the proof of Eq. (52) it remains for us to show that

$$\lim_{R\to\infty} \langle 0|[D_R, P_\mu]A|0\rangle = 0. \tag{59}$$

We will prove Eq. (58) by direct calculation. From Eqs. (2), (6), (14), (45), (47), and (53) we get

$$\begin{aligned}\lim_{R\to\infty} \langle 0|[D_R, P_\mu]A|0\rangle = {}& i\int d^3x\, x^\nu \int_0^\infty d\mu^2 \int d^3y \frac{\partial}{\partial x_\mu}\Delta^{(+)}(x-y, x_0, \mu^2)\frac{\partial}{\partial y_i}\sigma^i_{1\nu}(\mu^2, y) \\ &+ i\int d^3x\, x^\nu \int_0^\infty d\mu^2 \int d^3y \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_0}\Delta^{(+)}(x-y, x_0, \mu^2)\frac{\partial}{\partial y_i}\sigma^i_{2\nu}(\mu^2, y).\end{aligned} \tag{60}$$

By partial integration and using the explicit representation of $\Delta^{(+)}(x, x_0, \mu^2)$, we cast Eq. (60) into the form

$$\begin{aligned}\lim_{R\to\infty} \langle 0|[D_R, P_\mu]A|0\rangle = {}& x_0(2\pi)^3\Big[\int_0^\infty d\mu^2 \int d^4k\, k_\mu k_i \tilde\sigma^i_{10}(\mu^2, k)e^{ix_0k_0}\theta(k_0)\delta^3(k)\delta(k_0^{\,2} - |\vec{k}|^2 - \mu^2) \\ &+ i\int_0^\infty d\mu^2 \int d^4k\, k_\mu k_i k_0 \tilde\sigma^i_{20}(\mu^2, k)e^{ix_0k_0}\theta(k_0)\delta^3(k)\delta(k_0^{\,2} - |\vec{k}|^2 - \mu^2)\Big] \\ &+ i(2\pi)^3\Big[\int_0^\infty d\mu^2 \int d^4k\, k_\mu k_i \tilde\sigma^i_{1j}(\mu^2, k)e^{ix_0k_0}\theta(k_0)\delta(k_0^{\,2} - |\vec{k}|^2 - \mu^2)\frac{\partial}{\partial k^j}\delta^3(k) \\ &+ i\int_0^\infty d\mu^2 \int d^4k\, k_\mu k_i k_0 \tilde\sigma^i_{2j}(\mu^2, k)e^{ix_0k_0}\theta(k_0)\delta(k_0^{\,2} - |\vec{k}|^2 - \mu^2)\frac{\partial}{\partial k^j}\delta^3(k)\Big].\end{aligned} \tag{61}$$

The term multiplying x_0 in Eq. (61) vanishes without further ado, as it behaves, at worst, as

$$\int_0^\infty d\mu^2 \int d^3k\, k_i k_j \tilde\sigma^i_{10}(\mu^2, k)\delta^3(k)\frac{1}{(|\vec{k}|^2 + \mu^2)^{1/2}}\exp[ix_0(|\vec{k}|^2 + \mu^2)^{1/2}].$$

The vanishing of the remaining terms on the right-hand side of Eq. (61) is guaranteed by Eqs. (50) and (20a), which is a consequence of Eq. (10). Consider, for instance, the case in which the space-time index $\mu = 0$. The right-hand side of Eq. (61) now behaves as

$$\begin{aligned}&\int_0^\infty d\mu^2 \frac{\partial}{\partial k^j}\{k_i \tilde\sigma^i_{1j}(\mu^2, k)\exp[ix_0(|\vec{k}|^2 + \mu^2)^{1/2}]\}_{\vec{k}=0} + i\int_0^\infty d\mu^2 \frac{\partial}{\partial k_j}\{k_i(|\vec{k}|^2 + \mu^2)^{1/2}\tilde\sigma^i_{2j}(\mu^2, k)\exp[ix_0(|\vec{k}|^2 + \mu^2)^{1/2}]\}_{\vec{k}=0} \\ &\qquad = \int_0^\infty \tilde\sigma^i_{1i}(\mu^2, 0)e^{ix_0\mu}d\mu^2 + i\int_0^\infty \mu\tilde\sigma^i_{2i}(\mu^2, 0)e^{ix_0\mu}d\mu^2.\end{aligned} \tag{62}$$

The above vanishes as a result of Eqs. (20a) and (50). For the case $\mu = n = 1, 2, 3$, the corresponding term behaves as

$$\int_0^\infty d\mu^2 \frac{\partial}{\partial k^j}\left\{k_i k_n \frac{1}{(|\vec{k}|^2+\mu^2)^{1/2}} \exp[ix_0(|\vec{k}|^2+\mu^2)^{1/2}]\,\tilde{\sigma}^i_{1j}(\mu^2,k)\right\}_{\vec{k}=0}$$

$$+i\int_0^\infty d\mu^2 \frac{\partial}{\partial k^j}\{k_i k_n \exp[ix_0(|\vec{k}|^2+\mu^2)^{1/2}]\,\tilde{\sigma}^i_{2j}(\mu^2,k)\}_{\vec{k}=0}$$

$$=\int_0^\infty d\mu^2 e^{ix_0\mu}\left(\frac{\tilde{\sigma}^i_{1i}(\mu^2,k)k_n+\tilde{\sigma}^i_{1n}(\mu^2,k)k_i}{(|\vec{k}|^2+\mu^2)^{1/2}}-\frac{\tilde{\sigma}^i_{1j}(\mu^2,k)k_ik_jk_n}{(|\vec{k}|^2+\mu^2)^{3/2}}\right)_{\vec{k}=0}. \tag{63}$$

The above vanishes due to Eq. (20a). Proof of Eq. (59) is now complete. Hence, Eq. (52) is established for local states. Proceeding exactly as in Sec. II, it is now straightforward to extend this result to quasilocal states.

We now discuss a possible implication of Eq. (52). Although proved for a dense set of states, it is not clear whether Eq. (52) is valid in the sense of equality of matrix elements evaluated between single-particle states. This is because only in theories with a mass gap is it known that single-particle states are quasilocal.[17] However, if Eq. (52) is true also for single-particle states, then there cannot exist single-particle states of finite mass. To see this, denote by $|M^2\rangle$ a normalized single-particle state of mass M and consider the matrix element

$$\lim_{R\to\infty}\langle M^2|[D_R, P_\mu{}^2]|M^2\rangle=\lim_{R\to\infty}\left(\langle M^2|[D_R,P_\mu]P^\mu|M^2\rangle+\langle M^2|P^\mu[D_R,P_\mu]|M^2\rangle\right). \tag{64}$$

The left-hand side of the above is clearly zero, while the right-hand side is $2iM^2$. This is possible only if M is zero. Thus spontaneous breakdown of dilation symmetry does not admit the possibility in which some particle states remain massive. Although not rigorously proved, there are strong reasons to believe that this result is indeed true.[18] We have seen that there is a gap in the argument only because we are not sure if Eq. (52) is valid between single-particle states. But exactly the same problem is present even when dilation is not spontaneously broken. Looking back, we notice that for the vanishing of Eq. (61) it is irrelevant whether the constant c [in Eq. (50)] is zero or not. But then how does one understand the result that in a theory in which dilation symmetry is good and the (global) dilation operator is built as $\lim_{R\to\infty} D_R$ all single-particle masses must be zero, unless the same is true even when the symmetry is spontaneously broken?

We end this section with the following technical remark. We found that Eq. (10) – or what amounts to the same thing, Eqs. (14) and (20a) – was essential in proving Eq. (52). However, it is interesting to note that if we do not assume Eq. (20a) but only Eq. (14), then Eq. (45) yields, in addition to Eq. (50), also the relation $\tilde{\sigma}^i_{1i}(\mu^2,0)=0$. This last relation, however, is not sufficient to prove Eq. (52).

IV. CONCLUDING REMARKS

In the preceding sections we have studied two distinct but related questions. First, we considered the problem of constructing the infinitesimal generators of the Poincaré group as space integrals of corresponding densities built out of a local energy-momentum tensor. We have seen that this is possible over a dense set of states. If zero-mass neutral particles are present in the spectrum, then certain conditions, in addition to the assumed properties of the energy-momentum tensor, must be fulfilled. These conditions have been derived. Secondly, we have studied dilation symmetry and its spontaneous breakdown. The Goldstone theorem for this case has been proved, and arguments have been presented that a spontaneous breaking of dilation does not admit single-particle states of finite mass. The reader will notice that we have refrained from discussing spontaneous breakdown of Poincaré invariance. Although there seems to be some interest in this subject,[19,20] it is not clear to us how to treat it within the framework of Wightman's field theory, which includes relativistic invariance as one of its basic postulates. In particular, the vacuum state is defined as the invariant state under the Poincaré group. An important problem not discussed in this paper is the commutation rules of Poincaré-group generators constructed as space integrals of corresponding densities. Does the requirement that the generators so constructed have the correct commutation properties lead to any further restrictions? This question is being investigated. While this paper was being written, we received a preprint by Reeh[21] where a perhaps more rigorous discussion of dilation breaking is given. The Goldstone theorem for dilation is proved and an example given of spontaneous breaking of dilation in a relativistic field theory.

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Propagator Gauge Transformations for Non-Abelian Gauge Fields*

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An explicit demonstration is given of the invariance under propagator gauge transformations of the Feynman-graph prescription obtained by several authors for a non-Abelian gauge field; the role of the anomalous closed-loop vertices of that prescription is thereby clarified. This invariance property permits a derivation of the covariant prescription starting with canonical quantization in a noncovariant gauge without unphysical degrees of freedom.

I. INTRODUCTION

A variety of different quantization procedures[1-6] has been used to derive the Feynman-graph prescription for the massless non-Abelian gauge field. It is the purpose of this note to investigate in graphical terms the explicit equivalence of the prescriptions corresponding to different gauges; that is, the formal invariance of the prescription under propagator gauge transformations. This demonstrates in explicit terms the general invariance shown by DeWitt,[2] and permits a derivation of the covariant prescription which starts with canonical quantization in a noncovariant gauge, without unphysical degrees of freedom, in close analogy with the corresponding procedure in quantum electrodynamics. The anomalous closed-loop vertices characteristic of the non-Abelian gauge field prescription are found, not surprisingly, to be essential for the gauge invariance of the prescription.

We restrict our attention to transformations which leave the prescription explicitly translation-invariant, since these are considerably easier to handle, and are the only ones normally encountered. The general case is treated by DeWitt, who makes use of a functional integral formulation.

II. PRELIMINARIES

We denote the gauge field by $b_{r\alpha}$, with r the group index and α the space-time index ($\alpha = 0, 1, 2, 3$; $g_{00}=1$; $g_{11}=g_{22}=g_{33}=-1$; x_α and p_α are the usual contravariant coordinate and momentum components). We deal with the case in which the gauge field is coupled to a spin-$\frac{1}{2}$ field $\psi_{i\kappa}$ (κ is the spinor index) belonging to a particular representation, $(T_r)_{ij}$, of the Lie algebra:

$$[T_r, T_s] = ic_{rst}T_t, \tag{1}$$

where c_{rst} are the structure constants, taken as