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Some General Relations in Relativistic Magnetohydrodynamics

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A method is outlined for obtaining general relations governing the behavior of magnetofluids in general relativity. Several such relations are obtained for the case of infinite conductivity, and their possible relevance to galactic cosmogony, gravitational collapse, and pulsar theory is briefly discussed.

I. INTRODUCTION

In view of the frequent occurrence of magnetic fields in astronomical systems (for instance, stars, galaxies, and pulsars), it is clear that the study of relativistic magnetohydrodynamics will be important for the development of relativistic astrophysics. While classical magnetohydrodynamics is rather well developed, not a great deal is known about relativistic magnetohydrodynamics, and (partly because of the usual subtleties in defining physically meaningful frames of reference in general relativity) one must be wary of applying the classical results to systems with intense gravitational fields.

In his electrodynamics of moving bodies, Minkowski[1] has given a covariant decomposition of the electromagnetic field which, if we can be permitted to extend it to nonuniform motions in curved space-times, is very well suited to the study of magnetofluids in general relativity. This decomposition has, in fact, already been used in relativistic magnetohydrodynamics by Pham Mau Quan and Lichnerowicz.[2]

We show below that the decomposition leads very directly to three useful *differential identities* for magnetofluids; from these identities one can obtain a variety of general relations which govern the behavior of relativistic magnetofluids.

Since it appears that relativistic generalizations of magnetohydrodynamics are likely to be of physical interest in the realm of astrophysics, and since it is usually an excellent approximation to treat cosmic magnetofluids as infinitely conducting, we apply the differential identities to this case, and find (1) an expression for the proper electrical charge density in terms of the magnetic field and the angular-velocity vector, (2) a relation between the absolute derivative of the magnitude of the magnetic field and the absolute derivative of the nonmagnetic proper-energy density, and (3) an expression for the derivative along a magnetic field line of the magnitude of the angular velocity in terms of the curvature tensor.

In the course of the discussion, we make a number of remarks (some rather speculative) about the possible relevance of these results to various astrophysical problems. While our method is too general to permit very definite conclusions to be drawn from it, we do get good indications of the possible significance of magnetic effects in galactic cosmogony and gravitational collapse, and of a gravitational effect in pulsar theory. The method should be useful for estimating the significance of other possible magnetohydrodynamic effects as well.

Also, a knowledge of some general properties of solutions should be helpful in attempts to find explicit solutions of the complete magnetohydrodynamic equations.

Our space-time metric has signature +2. Latin indices take the values 1, 2, 3, 4; Greek ones the values 1, 2, 3. Covariant differentiation is denoted by a subscript preceded by a vertical line.

II. ELECTRODYNAMICS

For a body moving uniformly (that is, with a uniform translation) in flat space-time, one can generalize Maxwell's electrodynamics as follows.[1,3] Let u be the four-velocity of the body. (For a uniform translation in flat space-time, u is the same at all points of the body.) We describe the electromagnetic field by two skew-symmetric tensors F^{ab}, H^{ab}, with their associated dual tensors denoted by $^*F^{ab}$, $^*H^{ab}$. Let

$$\begin{aligned} E_a &\equiv F_{ab}u^b, \\ B_a &\equiv {}^*F_{ab}u^b, \\ D_a &\equiv H_{ab}u^b, \\ H_a &\equiv {}^*H_{ab}u^b. \end{aligned} \tag{1}$$

These satisfy

$$u\cdot E = u\cdot B = u\cdot D = u\cdot H = 0,$$

and they fix F^{ab} and H^{ab} through the decomposition

$$F^{ab}=u^aE^b-u^bE^a-\eta^{abcd}u_cB_d\,,$$
$$^*F^{ab}=u^aB^b-u^bB^a+\eta^{abcd}u_cE_d\,,$$
$$H^{ab}=u^aD^b-u^bD^a-\eta^{abcd}u_cH_d\,, \qquad (2)$$
$$^*H^{ab}=u^aH^b-u^bH^a+\eta^{abcd}u_cD_d\,,$$

where η is the permutation tensor.

In the rest frame of the body, that is a coordinate system in which $u^a \overset{*}{=} \delta^a_4$ (the symbol $\overset{*}{=}$ means that the equality holds only in a special coordinate system), we have $E_4 \overset{*}{=} B_4 \overset{*}{=} D_4 \overset{*}{=} H_4 \overset{*}{=} 0$. If, in the rest frame, we identify E_μ with the components of the electric field, B_μ with the components of the magnetic induction, D_μ with the components of the electric displacement, and H_μ with the components of the magnetic field, we can write Maxwell's equations for the rest frame as

$$^*F^{ab}{}_{|b} \overset{*}{=} 0, \quad H^{ab}{}_{|b} \overset{*}{=} J^b,$$

where J is the electric current. But these are tensor equations, so we have in any coordinate system

$$^*F^{ab}{}_{|b}=0, \quad H^{ab}{}_{|b}=J^a\,. \qquad (3)$$

For a medium with conductivity σ, the constitutive relation $\vec{J}=\sigma\vec{E}$ of Maxwellian electrodynamics becomes

$$J^a=\epsilon u^a+\sigma E^a, \qquad (4)$$

where ϵ is the proper electric-charge density. We must also specify the constitutive relations $D^a=\lambda E^a$ and $B^a=\mu H^a$, where λ is the permittivity and μ is the permeability. Note that while σ, λ, and μ transform as scalars under coordinate transformations, they are nevertheless frame-dependent in that the constitutive relations which define them contain (implicitly) the four-velocity u.

Minkowski takes for the electromagnetic energy-momentum tensor

$$E^{ab}=F^{ac}H^b{}_c-\tfrac{1}{4}g^{ab}F^{cd}H_{cd}\,.$$

Straightforward algebra, using Eqs. (2), yields

$$E^{ab}=(u^au^b+\tfrac{1}{2}g^{ab})(E\cdot D+B\cdot H)$$
$$-(E^aD^b+H^aB^b)+(u^aS^b+u^bP^a), \qquad (5)$$

where

$$S^a=\eta^{abcd}E_bH_cu_d, \quad P^a=\eta^{abcd}D_bB_cu_d.$$

If one looks at S and P in the rest frame where $u^a \overset{*}{=} \delta^a_4$, one finds that S is the Poynting vector and P is the electromagnetic momentum-density vector.

Minkowski's energy-momentum tensor [Eq. (5)] is, in general, asymmetric. This has given rise to a long discussion in the literature, the upshot of which seems to be that Minkowski's E^{ab} is generally regarded as being "correct," despite its asymmetry.[3]

The formalism that we have been describing, let us recall, applies to bodies which are moving uniformly in a flat space-time. We shall want to consider electromagnetic fields in fluids which are moving nonuniformly in curved space-times. Let $u(x)$ be the four-velocity of the fluid element at the point x (we define u more precisely below). We shall *assume* that the above formalism still applies, with u in all the above equations replaced by the fluid four-velocity field $u(x)$.[4]

In a curved space-time, the quantities defined in Eqs. (1) are to be interpreted as follows. Let $\lambda^a_{(i)}$ be an orthonormal tetrad with $\lambda^a_{(4)}=u^a$. Then, for instance, the physical components of E, defined by $E_{(i)}=\lambda^a_{(i)}E_a$, are the physical components of the electric field, as seen by a hypothetical observer with local Minkowski frame $\lambda^a_{(i)}$.[5] It should be remembered that with respect to whatever coordinate system we express the tensor components E^a of E, it always represents the field "seen" by an observer with four-velocity u. One interprets B, D, H, S, and P similarly.

In classical magnetohydrodynamics, there is never any question about the meaning of one's electromagnetic field variables. One simply works always in a fixed global inertial frame. In general relativity, this is a luxury we must do without[6]; the components of F^{ab} and H^{ab} have in general no direct physical meaning. By writing things down in terms of the variables defined in Eqs. (1), we can retain the use of an arbitrary coordinate system without losing sight of the physical meaning of our electromagnetic field variables.

III. MAGNETOHYDRODYNAMICS

We consider charged fluids (which may be plasmas that we are treating in the magnetohydrodynamic approximation). We shall assume that the energy-momentum tensor of the system has the form $T^{ab}=M^{ab}+E^{ab}$, where E^{ab} is the above electromagnetic energy-momentum tensor, and where the "material" energy-momentum tensor M^{ab} has the form

$$M^{ab}=\rho u^au^b-S^{ab}\,,$$

with $u_aS^{ab}=S^{ab}u_b=0$. The four-vector u, which is taken to be a timelike, future-oriented unit vector, is an eigenvector of M^{ab}. This eigenvector we define to be "the four-velocity field of the fluid," referred to above. Its eigenvalue ρ is the proper energy density of the matter. Since E^{ab} is in general asymmetric, so is the material stress tensor S^{ab}. The energy-momentum tensor is required to satisfy $T_a{}^b{}_{|b}=0$, which can be written as

$$\rho_{|b}u_a u^b + \rho u_{a|b}u^b + \rho u_a u^b{}_{|b} - S_a{}^b{}_{|b} - \epsilon E_a - E\cdot E\sigma u_a + \eta_{abcd}E^b u^c B^d - \tfrac{1}{2}(E\cdot E\lambda_{|a} + H\cdot H\mu_{|a}) + (P^b - S^b)u_{b|a} = 0. \quad (6)$$

The field equations for the system are then Einstein's field equations

$$G^{ab} = -\kappa T^{ab}, \quad (7)$$

along with the Maxwell equations (3), which can be written in terms of the decomposition (2) as

$$u^a{}_{|b}B^b + u^a B^b{}_{|b} - u^b{}_{|b}B^a - u^b B^a{}_{|b} + \eta^{abcd}(u_{c|b}E_d + u_c E_{d|b}) = 0, \quad (8)$$

$$u^a{}_{|b}D^b + u^a D^b{}_{|b} - u^b{}_{|b}D^a - u^b D^a{}_{|b} - \eta^{abcd}(u_{c|b}H_d + u_c H_{d|b}) = \epsilon u^a + \sigma E^a. \quad (9)$$

Plainly, to actually solve the field equations (7)–(9) for any realistic problem (even in the case of infinite conductivity, which is much simpler than finite conductivity) is a formidable task. One might wonder, however, what sort of general properties might be possessed by solutions of these equations. Certainly, if we had a solution before us it would satisfy Eqs. (6), (8), and (9) identically – and this would be so, of course, for *any* solution. We may, therefore, regard Eqs. (6), (8), and (9) as *differential identities* which embody certain general properties of relativistic magnetofluids.

We shall not make any explicit use of Eq. (7). It is, however, always lurking in the background, in that the covariant derivative that we use is assumed to be taken with respect to a metric for which (7) is satisfied.

IV. INFINITE CONDUCTIVITY

We shall demonstrate this use of Eqs. (6), (8), and (9) for the case of infinite conductivity, which is the most interesting case from the point of view of astrophysics. By infinite conductivity one means, of course, a conductivity which is so large that it can be regarded as infinite.[7] Then Eq. (4) implies that E^a is negligible, *unless* ρ is also extremely large. Leaving aside for the moment the possibility of infinite ρ, we find that Eqs. (6), (8), and (9) reduce to

$$u^a{}_{|b}B^b + u^a B^b{}_{|b} - u^b{}_{|b}B^a - u^b B^a{}_{|b} = 0, \quad (10)$$

$$\eta^{abcd}(u_{c|b}H_d + u_c H_{d|b}) = -\epsilon u^a, \quad (11)$$

$$\rho_{|b}u^b u_a + \rho u_{a|b}u^b + \rho u^b{}_{|b}u_a - S_a{}^b{}_{|b} - \tfrac{1}{2}H\cdot H\mu_{|a} = 0. \quad (12)$$

From these we can, by taking various contractions or by covariantly differentiating and then taking contractions, generate many identities, some of which have a useful physical interpretation and some of which do not.

The interpretation of these identities is facilitated by the use of the following kinematical parameters[8] for a timelike unit vector field u:

$$\omega_{ab} \equiv u_{[a|b]} + \alpha_{[a}u_{b]},$$

$$\sigma_{ab} \equiv u_{(a|b)} + \alpha_{(a}u_{b)} - \tfrac{1}{3}\theta(g_{ab} + u_a u_b),$$

$$\theta \equiv u^a{}_{|a},$$

where the parentheses on indices denote symmetrization, square brackets denote antisymmetrization, and $\alpha_a \equiv u_{a|b}u^b$ is the acceleration vector. We have

$$\omega_{(ab)} = 0, \quad u^a\omega_{ab} = 0,$$

$$\sigma_{[ab]} = 0, \quad u^a\sigma_{ab} = 0, \quad \sigma^a{}_a = 0.$$

The tensor σ_{ab} gives the shear of the fluid world-line congruence, and θ gives its expansion. Associated with the angular-velocity tensor ω_{ab} we have the angular-velocity vector $\omega^a \equiv \frac{1}{2}\eta^{abcd}u_b u_{c|d}$; it points along the axis of rotation. Let us recall exactly which angular velocity ω represents: If an orthonormal frame is Fermi-propagated along a fluid-element world line, the physical components of ω give the angular velocity of neighboring fluid elements relative to the Fermi-propagated frame.[5] That is, ω gives the local rotation of the fluid relative to a local inertial frame. It is well to bear in mind that this may be quite different from one's intuitive notion of the rotation of a fluid, which is based upon the Newtonian concept of global inertial frame.[6]

Now we are ready to find some consequences of (10)–(12). If we contract (11) with u_a, we obtain

$$\epsilon = 2\omega\cdot H.$$

In a region where the charge density is zero, the magnetic field lines are orthogonal to the rotation vector, and in any region where these two vectors are not orthogonal, there must be a nonzero charge density.

A partially ionized protogalactic gas cloud would have, to a very good approximation, infinite conductivity, and would be unlikely to have any large concentrations of charge. Thus, in a rotating disk, the field lines would lie in the (proto-) galactic plane. If the present field in our galaxy were of primordial origin, one would then expect the field lines to be oriented parallel to the galactic plane. There are some indications that this is the case.[9] This lends some support to the hypothesis of a primordial origin for the galactic field.[10] It is not, however, any sort of compelling argument for this hypothesis. For one thing, the empirical data on the orientation of the field lines is rather sparse,[9] and even if the field lines really are oriented parallel to the galactic plane, there could be other ways of explaining it. Furthermore, it could be that at some point in the contraction of the cloud and formation of condensations (protostars), the assumption of infinite conductivity might fail, so that the magnetic field might not remain "frozen in" (though this may not have much effect on the large-scale, general configuration of the field).

Now contract Eq. (10) with B_a, yielding

$$B^aB^b\sigma_{ab} - \tfrac{2}{3}\theta B\cdot B - \tfrac{1}{2}\mathfrak{D}_u(B\cdot B) = 0,$$

where $\mathfrak{D}_u$ denotes the absolute derivative with respect to u. If we contract (12) with u_a, we find

$$\mathfrak{D}_u\rho + \theta\rho - S^{ab}\sigma_{ab} - \tfrac{1}{3}\theta S^a{}_a + \tfrac{1}{2}H\cdot H\mathfrak{D}_u\mu = 0.$$

Combining these last two equations,

$$\mathfrak{D}_u\rho - S^{ab}\sigma_{ab} + \tfrac{1}{2}H\cdot H\mathfrak{D}_u\mu + \tfrac{3}{2}(\rho - \tfrac{1}{3}S^a{}_a)[B^aB^b\sigma_{ab} - \tfrac{1}{2}\mathfrak{D}_u(B\cdot B)]/B\cdot B = 0. \tag{13}$$

Since for infinite conductivity, E^{ab}, hence S^{ab}, is symmetric, we may consider the possibility that S^{ab} is well approximated by the perfect-fluid form

$$S^{ab} = -p(u^au^b + g^{ab}).$$

Also, for the sake of discussion, it is entirely reasonable to take μ constant. Then we can write (13) as

$$\mathfrak{D}_u\rho = \tfrac{3}{2}(\rho + p)[\tfrac{1}{2}\mathfrak{D}_u(B\cdot B) - B^aB^b\sigma_{ab}]/B\cdot B.$$

This indicates that the growth of small density inhomogeneities is strongly influenced by the magnetic field (even if that field is initially rather weak). The density will tend to grow faster in regions where the field is growing faster, and a shear along the field lines (as opposed to one orthogonal to the field lines) will affect the growth of ρ. This may be relevant to the "density-wave" theory of spiral structure in galaxies.[11] That theory seems quite successful in describing the structure of spiral arms, but it leaves open the question of how the spiral arms, that is, the density waves, originate. According to the preceding discussion, magnetic fields should play an important role here.

There has been some discussion of the possibility that, in gravitational collapse, an initially weak magnetic field may ultimately come to dominate all other forms of energy.[12] Since the total proper energy density is

$$u_au_bT^{ab} = \rho + \tfrac{1}{2}B\cdot H,$$

the ratio $R = \frac{1}{2}(B\cdot H)/\rho$ provides a measure of the relative importance of magnetic and nonmagnetic effects. We can study the evolution of R by the use of Eq. (13), which can be written in the form

$$\mathfrak{D}_uR = \tfrac{1}{6}[B\cdot H(\rho + S^a{}_a)\mathfrak{D}_u\rho]/[\rho^2(\rho - \tfrac{1}{3}S^a{}_a)] - \tfrac{2}{3}B\cdot HS^{ab}\sigma_{ab}/[\rho(\rho - \tfrac{1}{3}S^a{}_a)] + B^aH^b\sigma_{ab}/\rho + (2\rho)^{-1}[\tfrac{2}{3}B\cdot H/(\rho - \tfrac{1}{3}S^a{}_a) - 1]H\cdot H\mathfrak{D}_u\mu.$$

Again it is reasonable to set $\mathfrak{D}_u\mu = 0$. Taking the perfect-fluid form for the material stress tensor, we have

$$\mathfrak{D}_uR = \tfrac{1}{6}[B\cdot H(\rho - 3p)/\rho^2(\rho + p)]\mathfrak{D}_u\rho + B^aH^b\sigma_{ab}/\rho.$$

Suppose that the fluid is collapsing, so that $\mathfrak{D}_u\rho$ is always positive. Consider first an isotropic collapse: $\sigma_{ab} = 0$. Certainly in the early stages of collapse, $\rho > 3p$, so that R increases. Whether this tendency is halted or not depends entirely upon the equation of state for very large densities. Most people would have been inclined to suppose until recently that it would always be true that $\rho > 3p$. In this case, R would never cease growing, so that eventually the magnetic field would dominate all other considerations. However, it now appears that one could have $\rho \leqslant 3p$ in extremely dense matter,[13] in which case ρ might level off at some constant (though still possibly large) value, or even decrease in the latter stages of collapse, so that the nonmagnetic energy would finally dominate.

For a nonisotropic collapse, we see that a shear in the direction of the magnetic field lines could augment the dominance of magnetic energy, but this still could, of course, be overcome by an extremely large asymptotic pressure.

Thus an initial magnetic field, *no matter how weak,* could become the dominant factor in the final stages of gravitational collapse, *if* the fluid may be considered as infinitely conducting throughout the collapse, as concluded also by Cocke.[12] However, recall our *caveat* at the beginning of this section. Even extremely large values of σ cannot be considered infinite if ρ is of the order of σ. In a gravitational collapse there should come a point (at which R may already be quite large) when electric fields, too, are important. Since magnetic fields are rather pervasive in nature (it is commonly speculated, for example, that all stars must have some magnetic field), this question clearly deserves further study.

Finally, we consider the behavior of the angular velocity in a steady-state magnetofluid. By "steady-state" we mean that $\theta = 0$, $\sigma_{ab} = 0$, $\mathfrak{D}_uB^a = 0$, $\mathfrak{D}_u\omega_{ab} = 0$. First, contract Eq. (10) with u_a, which yields

$$B^a{}_{|a} = 2\alpha\cdot B = -2u^a\mathfrak{D}_uB_a = 0. \tag{14}$$

Now covariantly differentiate (10) with respect to x^c, and contract the resulting equation with ω^{ac}, taking (14) into account. After some manipulation, one finds

$$(\omega^2)_{|a}B^a = \tfrac{1}{4}u^a\,\omega^{bc}B^dR_{abcd}\,,$$

where $\omega^2 = \omega^{ab}\omega_{ab}$ is the magnitude of the angular velocity and R_{abcd} is the curvature tensor. This is a generalization of Ferraro's theorem of isorotation[14]; when gravitational effects are negligible, it says that the magnitude of the angular velocity is constant along a field line. Ferraro's theorem is used, for example, in discussing the corotation of a plasma surrounding a magnetic star. We see that if the star has an intense gravitational field, this corotation phenomenon could be significantly altered. This could be important for understanding pulsars, which are thought by some to be rapidly rotating, magnetic neutron stars, with the pulses originating in a corotating plasma "magnetosphere."[15]

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