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Long-Range Neutrino Forces Exerted by Kerr Black Holes*

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In flat space there is a long-range $1/r^5$ potential between two collections of matter arising from the exchange of neutrino pairs between them. It is shown that a Kerr black hole cannot exert a weak-interaction force of this kind. A Kerr black hole has no exterior neutrino field with classical effects.

One electron can exert a long-range force on another electron by the exchange of zero-rest-mass quanta such as gravitons or photons. In general relativity a black hole can also exert long-range gravitational and electrical forces. The gravitational and electrical multipoles which black holes can have, however, are remarkably restricted.[1] An electron can also exert a force on another electron through the exchange of a neutrino pair.[2] The question naturally arises as to whether a black hole can also have a long-range neutrino field with classical effects. Recently this question has been answered negatively for Schwarzschild black holes.[3] It is the purpose of this paper to show that a Kerr black hole cannot exert long-range forces through its coupling to the weak interactions.

A theory of the classical effects of a neutrino field interacting with classical electrons and gravity has been outlined elsewhere.[3,4] To lowest order in the weak interactions, the force on an electron is determined by the curl of a potential B^μ. The potential of a stationary source is calculable from bilinear vacuum expectation values of the neutrino field with the weak interactions turned on, $\psi(x)$, and the neutrino field with the weak interactions turned off, $\psi^0(x)$, as follows:

$$B^\mu(x) = \langle\bar{\psi}(x)\gamma^\mu(1+\gamma^5)\psi(x)\rangle - \langle\bar{\psi}^0(x)\gamma^\mu(1+\gamma^5)\psi^0(x)\rangle. \tag{1}$$

The subtraction eliminates the infinite bare-vacuum effects. The neutrino field is coupled to the classical number current of electrons $N^\mu(x)$ through the weak-interaction coupling constant G_W, and solves the Dirac equation[5]

$$i\gamma^\mu[\nabla_\mu - 2^{-1/2}G_W N_\mu(x)(1+\gamma^5)]\psi(x) = 0. \tag{2}$$

In general relativity the Dirac matrices satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{3}$$

and $\nabla_\mu = \partial/\partial x^\mu - \Gamma_\mu$ denotes a covariant spinor derivative.[6] The solutions to the Dirac equation are normalized by the canonical anticommutation relations on a spacelike hypersurface. If this hypersurface coincides with a t = constant hypersurface for some coordinate t, and if the normal to the surface is n^μ, then these anticommutation relations have the form

$$\{\psi(\vec{x}), \psi^\dagger(\vec{x}')\} = n_t(-g)^{-1/2}\,\delta^{(3)}(\vec{x}-\vec{x}'). \tag{4}$$

Let us now consider the Kerr metric. Written in the Boyer-Lindquist coordinates, it has the form[7]

$$ds^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{B}{\rho^2}\sin^2\theta\left(d\varphi - \frac{2amr}{B}dt\right)^2 - \frac{\Delta\rho^2}{B}dt^2, \tag{5}$$

where

$$\Delta = r^2 - 2mr + a^2,\quad \rho^2 = r^2 + a^2\cos^2\theta,\quad B = (r^2+a^2)^2 - a^2\sin^2\theta. \tag{6}$$

The outermost event horizon is located at $r_+ = m + (m^2 - a^2)^{1/2}$.

In order to evaluate the possible potentials $B^\mu(x)$ which can exist outside a Kerr black hole in the absence of any sources, we begin by considering black holes whose exterior neutrino fields are so weak that they do not significantly perturb the Kerr geometry. Our program will be to evaluate Eq. (1) in terms of c-number solutions to the sourceless neutrino Dirac equation [Eq. (2) with $N^\mu = 0$]. To do this we will expand ψ and ψ^0 in annihilation and creation operators multiplied by these c-number solutions. Next, we will examine the *possible* ways in which the c-number solutions with the weak interactions turned off can differ from the corresponding solutions with the weak interactions turned on. When these solutions are used to evaluate $B^\mu(x)$, this will give all possible exterior weak-interaction potentials a Kerr black hole can have.

A Hilbert space of c-number solutions may be defined by introducing the scalar product between two elements U and V,

$$(U, V) = \int d^3x(-g)^{1/2}[\bar{U}(\vec{x})\gamma^t V(\vec{x})], \tag{7}$$

the integral being taken over a constant-t, spacelike hypersurface outside the event horizon at r_+. Members of this Hilbert space are normalizable with a finite value of (U, U).

Let us consider the generator of translations along the time coordinate t. Denoting this operator by H, we have

$$i\frac{\partial U}{\partial t} = HU = [-i(\gamma^t)^{-1}\gamma^i\nabla_i + i\Gamma_t]U. \quad (8)$$

We now show that H is Hermitian and acts on the Hilbert space of normalizable spinors U. To show this, one writes out the condition $(HU, U) = (U, HU)$, integrates by parts, and reduces it to

$$\int_\infty d^2x\,(-g)^{1/2}(\bar{U}\gamma^r U) - \int_{r_+} d^2x\,(-g)^{1/2}(\bar{U}\gamma^r U) = 0. \quad (9)$$

Here, the surface integrals are taken over the event horizon $r = r_+$ and the two-surface at spatial infinity. What we now need to show is that a finite value of (U, U) implies Eq. (9).

For the two-surface at spatial infinity, this is not difficult because the geometry becomes flat. The normalization condition implies that $U^\dagger U$ decreases faster than r^{-3}, and this is enough to guarantee that the associated surface integral [Eq. (9)] vanishes.

To study the second surface integral at the event horizon, we introduce an orthogonal tetrad $\lambda^\mu{}_{(\alpha)}$. Writing

$$\lambda^\mu{}_{(\alpha)} = (\lambda^t{}_{(\alpha)}, \lambda^\varphi{}_{(\alpha)}, \lambda^r{}_{(\alpha)}, \lambda^\theta{}_{(\alpha)}),$$

we choose

$$\begin{aligned} &\lambda^\mu{}_{(t)} = (-(B/\Delta\rho^2)^{1/2},\ -2amr/(\Delta B\rho^2)^{1/2},\ 0,\ 0),\\ &\lambda^\mu{}_{(\varphi)} = (0,\ \rho/B^{1/2},\ 0,\ 0),\\ &\lambda^\mu{}_{(r)} = (0,\ 0,\ \Delta^{1/2}/\rho,\ 0),\\ &\lambda^\mu{}_{(\theta)} = (0,\ 0,\ 0,\ 1/\rho). \end{aligned} \quad (10)$$

Dirac matrices which satisfy the anticommutation relations [Eq. (3)] may then be written

$$\gamma^\mu = \lambda^\mu{}_{(\alpha)}\gamma^{(\alpha)}, \quad (11)$$

where $\gamma^{(\alpha)}$ are numerical matrices satisfying the flat-space anticommutation relations. The normalization condition [Eq. (7)] then implies that near $r = r_+$, $\psi^\dagger\psi$ behaves like

$$(-g)^{1/2}(\psi^\dagger\psi)(B/\rho^2\Delta)^{1/2} \sim (r - r_+)^{-1+\epsilon} \quad (12)$$

and in no more singular fashion. In turn, this implies that the argument in the surface integral of Eq. (9) vanishes like $(r - r_+)^\epsilon$. The operator H is therefore Hermitian.

Since the operator H is Hermitian, it will have a complete set of orthogonal eigenfunctions. Let us denote one such set by $U_{\omega\kappa}(x)$, where ω is the eigenvalue of H and κ any other quantum numbers needed. These eigenfunctions are unique up to a normalization and a unitary transformation. The quantum field $\psi(x)$ may be expanded in these c-number solutions multiplied by annihilation and creation operators,

$$\psi(x) = \sum_\kappa \int_0^\infty d\omega\,[U_{\omega\kappa}(\vec{x})e^{-i\omega t}b_{\omega\kappa} + U_{-\omega\kappa}(\vec{x})e^{i\omega t}d^\dagger_{\omega\kappa}]. \quad (13)$$

Here, $b_{\omega\kappa}$ and $d^\dagger_{\omega\kappa}$ satisfy the usual anticommutation relations, and the normalization of the $U_{\omega\kappa}(\vec{x})$ is determined by the anticommutation relations for $\psi(x)$ [Eq. (3)]. A similar expansion holds for the exterior neutrino field when the weak interactions are turned off. The c-number orthogonal eigenfunctions will be denoted by $U^0_{\omega\kappa}(\vec{x})$ in that case.

Evaluating the expression (1) for the classical potential $B^\mu(\vec{x})$, one finds the stationary result

$$B^\mu(\vec{x}) = \int_0^\infty d\omega\left[\sum_\kappa \bar{U}_{\omega\kappa}(\vec{x})\gamma^\mu(1+\gamma^5)U_{\omega\kappa}(\vec{x}) - \sum_\tau \bar{U}^0_{\omega\tau}(\vec{x})\gamma^\mu(1+\gamma^5)U^0_{\omega\tau}(\vec{x})\right]. \quad (14)$$

The eigenfunctions $U_{\omega\kappa}(\vec{x})$ with the weak interactions turned on can differ at most by unitary transformation from the eigenfunctions $U^0_{\omega\kappa}(\vec{x})$ with the weak interactions turned off, both sets being orthogonal and normalized by the anticommutation relations [Eq. (3)]. Each term in Eq. (14) is invariant, however, under such transformations and we conclude that $B^\mu(\vec{x})$ *vanishes identically*. A Kerr black hole can therefore have no long-range weak-interaction potential $B^\mu(\vec{x})$.

The restriction that the neutrino field of the black hole does not significantly perturb the Kerr geometry may now be removed because the argument may be stated self-consistently as follows: Assume without restriction on the weak-interaction parameters of the Kerr black hole that the stress-energy from the neutrino field will vanish. Calculate the stress-energy from an arbitrary Kerr black hole under this assumption. Find, in analogy with the argument above, that it vanishes. This justifies the assumption.

The result that a Kerr black hole can exert no long-range weak-interaction forces has been obtained so simply that it deserves some comment. The essential point in the proof was that the complete set of eigenfunctions of H with the weak interactions turned on ($U_{\omega\kappa}$) could differ at most by a unitary transformation from the corresponding eigenfunctions with the weak interactions turned off ($U^0_{\omega\kappa}$). At first sight this might seem obvious, since the operator H with the weak interactions turned on has the same form as the operator with the weak interactions turned off, because N^μ vanishes everywhere outside the event horizon. To see that it is not obvious, and to see the essential role which the event horizon plays in the demonstration, let us consider the same argument where the two-surface at $r=r_+$ is replaced by another at $r=r_0>r_+$. The operator H is not Hermitian on the space defined by Eq. (7) with r_+ replaced by r_0, because the surface integral corresponding to Eq. (9) does not automatically vanish. It is, therefore, possible to find complete sets of eigenfunctions which are not unitarily equivalent. A quick way to see this is to imagine H being made Hermitian on the region $r>r_0$ by the imposition of some suitable boundary condition at r_0. This boundary condition, however, could depend on G_W, and thus the eigenfunctions with the weak interactions turned on could be not unitarily equivalent to those with the weak interactions turned off. Indeed, this will be precisely the situation if there is a nonvanishing N^μ between r_+ and r_0 (see Ref. 3 for a more detailed illustration).

The reason that H is Hermitian on the space of functions normalizable according to Eq. (7), without any further boundary conditions at the event horizon, may be stated in physical terms as follows: The operator H generates displacements along a particular time coordinate t. A neutrino cannot escape the region bounded by the event horizon and spatial infinity in a finite value of this coordinate time t.[7] This means, physically, that the neutrino number, which can be defined as (U, U), is conserved in t, and this is equivalent to a Hermitian time-development operator H. This result will not be true for any other two-surface outside the event horizon.

It is currently believed[1] that the Kerr metric describes asymptotically the geometry outside any collection of matter with zero total charge[8] in the last stages of gravitational collapse. The proof given here that the Kerr geometry has no stationary long-range weak-interaction potential suggests that the long-range neutrino potential of any chargeless collapsing star is all radiated away.[9] There are no conserved surface integrals as in the electromagnetic case to prevent this.

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[1]V. Ginzburg and L. Ozernoi, Zh. Eksperim. i Teor. Fiz. 47, 1030 (1964) [Soviet Phys. JETP 20, 689 (1965)]. A. Doroskevich, Ya. Zel'dovich, and I. Novikov, Zh. Eksperim. i Teor. Fiz. 49, 170 (1965) [Soviet Phys. JETP 22, 122 (1966)]. W. Israel, Phys. Rev. 164, 1776 (1967); Commun. Math. Phys. 8, 245 (1968). R. Price (unpublished). B. Carter, Phys. Rev. Letters 26, 331 (1971). For a review see also K. S. Thorne (unpublished).

[2]G. Feinberg and J. Sucher, Phys. Rev. 166, 1638 (1968), who give references to earlier calculations of this force.

[3]J. B. Hartle (unpublished).

[4]J. B. Hartle, Phys. Rev. D 1, 394 (1970).

[5]We use units in which $\hbar=c=1$, and a metric with signature (+ − − −). Greek indices range over space and time, and Latin indices only over space.

[6]For the theory of neutrinos interacting with gravity, see, e.g., D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. 29, 465 (1957); V. Bargmann, Sitzber. Preuss. Akad. Wiss., Phys.–Math. Kl. 346 (1932).

[7]R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).

[8]See, e.g., W. Israel, Phys. Rev. D 2, 641 (1970), B. Carter, Ref. 1, and J. Bardeen, Nature 226, 64 (1970) for a discussion of this idea.

[9]The considerations given here for the Kerr metric are also easily extendable to the charged metrics of E. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. 6, 918 (1965).