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[13]The ρ mass could have been treated as a parameter to be determined from the data in the same way as the ρ width. Its value is less controversial, however, so it was not varied throughout the calculation.

[14]Unfortunately, both results are not new. $C=30\mu^2 \simeq m_\rho{}^2$ corresponds to the ρ-meson dominance together with an ordinary-type Breit-Wigner formula for $F_\pi(s)$. $\Gamma_r = 110$ MeV is the best fit from both Orsay and Novosibirsk, Ref. 5.

[15]Our treatment of the timelike and spacelike regions is different for two reasons: (1) Both the right-hand and left-hand (whether present or not) cuts are independent in the dispersion approach, and (2) an analytic formula like the Veneziano one does not permit a satisfactory continuation from the forward to backward direction, say.

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Exact Sum Rule for Transition Amplitudes and $K_L - K_S$ Decay

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It is pointed out that there is just one exact form of the sum rule holding for specific transition amplitudes. The assumptions and approximations implicit in the various usual formulations are analyzed on the basis of this result. The adequate description of the time dependence is discussed.

The "unitarity sum rule" of Bell and Steinberger[1] plays an important role in the analysis of the K^0 system. It is related to the Weisskopf-Wigner treatment[2] of this decay problem. Starting from a phenomenological S matrix, further sum rules have been obtained stating that the result is different[3] or that the phase information agrees.[4] Subsequently, several authors[5-8] have come to the conclusion that the rules should agree completely. The recent effort spent on getting a satisfactory description of the K^0 system also includes additional scattering treatments[9-11] and an investigation[12] of the deviations from the Weisskopf-Wigner equations caused by postulating the Bell-Steinberger sum rule.

In the following, it is shown that the difficulties indicated above disappear by using recent results of the present author on specific time-dependent probabilities[13] and on the detailed structure of related amplitudes.[14] In particular, the exact form of the sum rule can be written down. Since the various amplitudes of interest are properly specified and given explicitly, the assumptions implicit in the usual scattering treatments as well as the nature of the conventional Weisskopf-Wigner approach can be analyzed. The adequate description of the time dependence is derived from first principles.

First we review some general relations.[14] Defining

$$P=\sum_\nu |\chi_\nu\rangle\langle\chi_\nu| ,$$

with $\langle\chi_\mu|\chi_\nu\rangle = \delta_{\mu\nu}$, $Q=1-P$, and $g(z)=(QHQ-z)^{-1}$, a biorthogonal basis with

$$\langle v_m(z)|u_n(z)\rangle = \delta_{mn}$$

and

$$P=\sum_n |u_n(z)\rangle\langle v_n(z)|$$

is introduced which diagonalizes

$$P(H-HQg(z)H)P=\sum_n \lambda_n(z)|u_n(z)\rangle\langle v_n(z)| .$$

If the χ_ν are (proper) eigenvectors of an "unperturbed Hamiltonian" H_r, writing the total Hamiltonian $H=H_r+V_r$, one has

$$P(H-HQgH)P=PH_r+P(V_r-V_rQgV_r)P .$$

We now put $z=E+i0$ and introduce

$$E_n(E)=\mathrm{Re}\lambda_n(E+i0), \quad \Gamma_n(E)=-2\,\mathrm{Im}\,\lambda_n(E+i0) .$$

Then the amplitude which turns out to be essential for the description of decays, involving an outgoing scattering state $\psi^-_{E,\beta}$ (improper eigenvector of H), gets the form

$$\langle\psi^-_{E,\beta}|u_n\rangle = A_{\beta,n}(E-E_n+\tfrac{1}{2}i\Gamma_n)^{-1} , \qquad (1)$$

where, for the transition amplitudes $A_{\beta,n}(E)$, the exact sum rule

$$2\pi\sum_{\beta}\bar{A}_{\beta,m}A_{\beta,n}=\langle u_m|u_n\rangle\,[-i(E_m-E_n)+\tfrac{1}{2}(\Gamma_m+\Gamma_n)] \quad (2)$$

holds. Similarly, the scattering amplitude can be written

$$T_{\beta\alpha}(E)=T^{(0)}_{\beta\alpha}+\sum_n C_{\beta,n}(E-E_n+\tfrac{1}{2}i\Gamma_n)^{-1}B_{\alpha,n}\,. \quad (3)$$

The further quantities in Eqs. (1)–(3) are given by

$$A_{\beta,n}=\langle\chi^-_{E,\beta}|[V_r-V_rQg(E+i0)V_r]u_n\rangle\,, \quad (4)$$

$$T^{(0)}_{\beta\alpha}=\langle\varphi^{(b)}_{E,\hat\beta}|[V_a-V_bQg(E+i0)V_a]\varphi^{(a)}_{E,\hat\alpha}\rangle\,, \quad (5)$$

$$B_{\alpha,n}=\langle v_n|[V_a-V_rQg(E+i0)V_a]\varphi^{(a)}_{E,\hat\alpha}\rangle\,, \quad (6)$$

$$C_{\beta,n}=\langle\varphi^{(b)}_{E,\hat\beta}|[V_b-V_bQg(E+i0)V_r]u_n\rangle\,. \quad (7)$$

The state $\chi^-_{E,\beta}$ occurring in (4) is introduced by the isometric mapping

$$\chi^-_{E,\beta}=[1+G_r(E-i0)V_r]\psi^-_{E,\beta}\,, \quad (8)$$

where $G_r(z)=(H_r-z)^{-1}$. This can be done since so far only the discrete part of H_r has been involved. By (8), the physical meaning of E and β, originally defined for $\psi^-_{E,\beta}$, is carried over also to $\chi^-_{E,\beta}$. It is to be noted that the use of H_r may also be avoided in deriving and applying Eqs. (1)–(3). To achieve this, one has to replace V_r by H in (6) and (7), and to use the mapping

$$\phi^-_{E,\beta}=[1+g(E-i0)(H-QHQ)]\psi^-_{E,\beta}\,,$$

which, instead of (4), gives the form $A_{\beta,n}=\langle\phi^-_{E,\beta}|Hu_n\rangle$. The operators V_a and V_b occurring in (5)–(7) represent the scattering interactions which need not be the same initially and finally for arbitrary arrangement channel situations. $\varphi^{(a)}_{E,\hat\alpha}$ and $\varphi^{(b)}_{E,\hat\beta}$ are the corresponding free states (the notation is such that α includes a and $\hat\alpha$).

In the case of K^0 decay, with a two-dimensional P, the χ_ν represent the states of K^0 and $\bar{K}^0$, and the u_n are the K_L and K_S states. Incidentally, we note that the properties of the mass matrix with respect to discrete transformations of V_r, which usually are derived in special approximations, can easily be obtained in a general way from $\langle\chi_\mu|(V_r-V_rQgV_r)\chi_\nu\rangle$. For the description of the decay and for the discussion of different approaches, it will be of interest to consider the relations which one gets from (1)–(3) by replacing the E-dependent quantities [except E itself in the resonance denominators of (1) and (3)] by their values at a fixed energy of about the kaon mass.

Let us now consider the S-matrix treatments which arrive at the Bell-Steinberger rule or which are at least consistent with it. Comparing with the general form (3) of resonance scattering amplitudes and with the general sum rule (2), it turns out that none of them distinguishes between the quantities which correspond to (4) and (7) in the present formulation. The explicit form, which is given in some of these approaches,[5,7-9] in our notation reads

$$\langle\chi^-_{E,\beta}|V_ru_n\rangle\,. \quad (9)$$

Thus, we have to look for the assumptions by which (4) and (7) get the form (9). For (4), this holds if

$$QV_rQ=0\,, \quad (10a)$$

or also, of course, if only the lowest-order approximation is considered. For (7), in addition to (10a), the more serious assumptions

$$P(V_b-V_r)Q=0,\quad P\varphi^{(b)}_{E,\hat\beta}=0 \quad (10b)$$

are to be made to arrive at the form (9), which then is obtained by using

$$\chi^-_{E,\beta}=[1-G_r(E-i0)(V_b-V_r)]\varphi^{(b)}_{E,\hat\beta} \quad (11)$$

and by noting that from (10a) one gets $QG_r=Qg$. In the language of radiation theory, (10) means that V_r does not interconnect unperturbed continuum states (of either sort) while V_b-V_r does not connect these states with the discrete ones. If no distinction is made between the interactions related to scattering and to decay, as occurs in some approaches,[7,11] (10b) is trivially satisfied. However, V_b and V_r are clearly different for the present application, and they are different things in general.[13,14] A further consequence of (10) is that (5) gets the form $\langle\chi^-_{E,\beta}|V_a\varphi^{(a)}_{E,\hat\alpha}\rangle$. This amplitude would describe scattering if V_r could be switched off; this agrees with the meaning ascribed to the background term by perturbation-theoretic arguments[5,9] as well as in a scattering treatment[8] based on the assumptions defined here by (10) and on $V_a=V_b$.

Actually, one cannot expect (10b) to be generally realizable, since the free states of scattering ordinarily span the total Hilbert space and even show overlap in general multichannel cases, quite apart from further difficulties related to field theory. Therefore, we now consider the difference between (4) and (7) without making any assumptions. After some calculations, using (11), $Q\chi^-_{E,\beta}=\chi^-_{E,\beta}$, the second resolvent equation for G_r and g, and the definitions of $E_n(E)$ and $\Gamma_n(E)$, one obtains

$$C_{\beta,n}-A_{\beta,n}=(E_n-\tfrac{1}{2}i\Gamma_n-E)\langle\varphi^{(b)}_{E,\hat\beta}|u_n\rangle\,. \quad (12)$$

From (12), it is seen that for small widths the difference may become negligible at the resonance energy. Thus, within that approximation the S-matrix approaches can be justified without further assumptions. In reality, however, one is not confronted with all these S-matrix problems since the sum rule (2) holds for the amplitudes which are actually involved in the K^0 decay probabilities as is considered in detail below.

If a physical system is described by a (pure) state ψ and the question asked in an experiment is described by a projection onto a vector ϕ, one gets the time-dependent probability $|\langle\phi|e^{-iHt}\psi\rangle|^2$ with ψ and ϕ both normalized to unity. According to the specific situation, the amplitude $\langle\phi|e^{-iHt}\psi\rangle$ gets,[13] for example, for a scattering reaction the form

$$\int dE \sum_{\beta,\alpha} \bar{c}_\beta^{(b)}(E)[\delta_{\beta\alpha} - 2\pi i T_{\beta\alpha}(E)]c_\alpha^{(a)}(E)e^{-iEt}, \tag{13}$$

while for a decay process one has

$$\int dE \sum_\beta \bar{c}_\beta(E)\langle\psi_{E,\beta}^-|\chi_\nu\rangle e^{-iEt}. \tag{14}$$

Here $c_\alpha^{(a)}$ describes the "initial wave packet" and, similarly, $c_\beta^{(b)}$ and c_β depend on details of the experimental question. It is seen that for special conditions, the time-independent probability of ordinary scattering or the (approximately) exponentially decreasing one of ordinary decays may occur. In general, however, (13) and (14) can lead to time dependences which are considerably different from each other as well as from the standard behaviors.[13] In the case of K^0 experiments, the events are selected in such a way that (14) applies. Since variations in the relative energy of the final particles are noticed at best in the MeV region, $c_\beta(E)$ can certainly be regarded as slowly varying near the resonances with widths smaller than 10^{-5} eV. This holds for stable as well as for unstable decay products because it is brought about by the properties of the relative motion [further details of the structure of $c_\beta(E)$ due to giving up the sharp energies of the fragments can be worked out; however, they do not matter in the present context]. Then, introducing (1), (14) can be evaluated within the usual type of approximations, taking the E-dependent quantities (except E itself) at a fixed energy of about the kaon mass and integrating E from $-\infty$ to $+\infty$, which gives, for positive t,

$$-2\pi i \sum_\beta \bar{c}_\beta \sum_n A_{\beta,n}\langle v_n|\chi_\nu\rangle \exp(-iE_n t - \tfrac{1}{2}\Gamma_n t). \tag{15}$$

On the other hand, conventional perturbation approaches effectively start from probability densities of the type $|\langle\tilde{\chi}_{E',\lambda}|e^{-iHt}\chi_\nu\rangle|^2$, where $\tilde{\chi}_{E',\lambda}$ belongs to some set of improper eigenvectors of H_r. Since they ask for orthogonal states instead of asking directly for those of the decay products, the time derivative is needed to arrive at what is measured. Instead of the usual Weisskopf-Wigner procedures, we can use our formalism[14] to get a general evaluation. In doing this, we employ the set of $\chi_{E,\beta}^-$, whose physical meaning is specified by (8). From the relations given previously,[14] for $t>0$ one obtains

$$\begin{aligned}2\pi i\langle\chi_{E',\beta}^-|e^{-iHt}u_n\rangle &= \int dE\,(E'-E-i0)^{-1} \\ &\times\langle\chi_{E',\beta}^-|[V_r - V_r Q g(E+i0)V_r]u_n\rangle \\ &\times(E-E_n+\tfrac{1}{2}i\Gamma_n)^{-1}e^{-iEt}.\end{aligned}$$

This can be used to calculate

$$\frac{d}{dt}\int dE'\,|\langle\chi_{E',\beta}^-|e^{-iHt}\chi_\nu\rangle|^2,$$

which, in the same type of approximations applied to get (15), for $t>0$ gives

$$2\pi\Big|\sum_n A_{\beta,n}\langle v_n|\chi_\nu\rangle\exp(-iE_n t - \tfrac{1}{2}\Gamma_n t)\Big|^2. \tag{16}$$

Thus, for the case considered, one has the same probability as one gets from (14) with special conditions.

Now, from (15) as well as from (16), it is seen that the time-dependent amplitude

$$\sum_n A_{\beta,n}\langle v_n|\chi_\nu\rangle\exp(-iE_n t - \tfrac{1}{2}\Gamma_n t) \tag{17}$$

is the one occurring in the decay probability. The quantities appearing in (17) are those obtained by taking the corresponding E-dependent ones at a fixed energy of about the kaon mass (i.e., intermediate between the K_L and K_S masses which are related to the E_n). From Eq. (2) at this energy, one has the sum rule of practical interest. It seems worthwhile to emphasize that, according to (4), the structure of $A_{\beta,n}$ is rather different from that of a T-matrix element in scattering. It is further to be noted that (17) may also be regarded as being related to the Weisskopf-Wigner-type state vector

$$\sum_n u_n\langle v_n|\chi_\nu\rangle\exp(-iE_n t - \tfrac{1}{2}\Gamma_n t).$$

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Singlet-Octet Mixing of Pseudoscalar Mesons in Broken Chiral Symmetry

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The results of Pande for η-X^0 mixing in broken chiral $SU(3)\otimes SU(3)$ symmetry are examined. Since there is strong evidence for three $I=Y=0$ pseudoscalar mesons, the model is extended to describe mixing of the pure octet member and two $SU(3)$ singlets. Here, there are two parameters, whose values are determined by using the $X^0(958)$ and $E(1422)$ masses as input. One then obtains a prediction for the mass of the η, in very good agreement with the experiment. The mixing parameters, both magnitude and sign, are also obtained.

Recently, the dynamical model of Gell-Mann, Oakes, and Renner[1] for breaking of chiral $SU(3)\otimes SU(3)$ symmetry has been used to describe η-X^0 mixing.[2] The Hamiltonian of the model is

$$H=H_0+H'(c)\,, \tag{1}$$

where H_0 is invariant under $SU(3)\otimes SU(3)$, while the symmetry-breaking part takes the simple form

$$H'(c)=-U_0-cU_8\,. \tag{2}$$

U_k and V_k form a scalar and a pseudoscalar nonet, respectively, belonging to the representation $(3^*,3)\oplus(3,3^*)$, and c is a parameter whose physical value is approximately $-\sqrt{2}$. From this symmetry-breaking Hamiltonian follows the partial conservation of the vector and axial-vector current octets V^i_μ and A^i_μ, associated with the symmetry, according to[1]

$$\begin{aligned}\partial_\mu V^i_\mu &= cf_{i8k}U_k\,,\\ \partial_\mu A^i_\mu &= -w_i(c)\delta_{ik}V_k-(\tfrac{2}{3})^{1/2}c\delta_{i8}V_0\,,\end{aligned} \tag{3}$$

where $i=1,\ldots,8$, $k=0,1,\ldots,8$, and

$$w_i(c)=\begin{cases}(\sqrt{2}+c)/\sqrt{3}\,, & i=1,2,3\\ (\sqrt{2}-\frac{1}{2}c)/\sqrt{3}\,, & i=4,5,6,7\\ (\sqrt{2}-c)/\sqrt{3}\,, & i=8\,.\end{cases} \tag{4}$$

Pande uses the Goldstone character of some of the pseudoscalar mesons in certain symmetry limits, corresponding to certain values of the parameter c. For $c=-\sqrt{2}$, one has $SU(2)\otimes SU(2)$ symmetry and mass-0 pions, and mass-0 kaons appear at $c=2\sqrt{2}$. Also, at $c=0$, one has $SU(3)$ symmetry. From this, one obtains the physical value of c,[1-3]

$$c=-2\sqrt{2}(m_K{}^2-m_\pi{}^2)/(m_\pi{}^2+2m_K{}^2)\,. \tag{5}$$

Similar considerations at $c=\sqrt{2}$, combined with further investigation of the symmetries at $c=0$ and $c=2\sqrt{2}$, lead to the following results[2]:

$$\begin{aligned}\langle\eta_0|U_0|\eta_0\rangle &= -\tfrac{5}{2}m_8{}^2\,,\\ \langle\eta_0|U_8|\eta_0\rangle &= 0\,,\\ m_{\eta_8}{}^2 &= m_8{}^2(1-c/2\sqrt{2})\,,\\ m_{\eta_0}{}^2 &= M^2+\tfrac{5}{2}m_8{}^2\,,\end{aligned} \tag{6}$$

where η_8 and η_0 are the two unmixed $I=Y=0$ pseudoscalar mesons, m_8 is the degenerate mass of the pseudoscalar meson octet in the $SU(3)$ symmetry limit, and M is the nonzero mass of the η_0 in the limit of chiral $SU(3)\otimes SU(3)$. The mass matrix is diagonalized to give the observed masses of the X^0 and η. Using the experimental mass of the η, one obtains the value of the unknown parameter M, as well as a prediction for M_{X^0} and the mixing angle θ.

The predicted value for M_{X^0} is very sensitive to the values chosen for m_π and m_K, which are the degenerate masses in the limit of $SU(2)$ invariance. For neutral masses for m_π and m_K, the results are

$$\begin{aligned}M &= 620\text{ MeV}\,,\\ M_{X^0} &= 914\text{ MeV}\,,\\ \theta &= 12.0^\circ\,.\end{aligned} \tag{7}$$

If charged masses for m_π and m_K are used, we obtain